Separable Differential Equations

We have seen how to solve autonomous differential equations by rewrting the equation so that the left hand side is a function of x and the right hand side is 1. A similar method works for separable equations, except that can be any function of t on the right.

A separable differential equation has the form

$$x' = g(t)h(x)$$

For example, consider the equation

$$x' = tx$$

Dividing through by the x, we get

$$\begin{aligned} \frac{x'}{x} &= t \\ \int \frac{x'}{x}\,dt &= \int t\,dt \\ \ln|x| &= \frac{t^2}{2} + C \\ |x| &= e^{t^2/2}e^C \\ x &= \pm e^{t^2/2}e^C \end{aligned}$$

Notice that e^C is an arbitrary positive number. The $\pm$ sign then gives us solutions with either an arbitrary positive constant or an arbitrary constant out the front. This method misses the solution $x = 0$, because it started out dividing by x, which implicitly assumed that x was not the zero function. If we include the solution $x = 0$, then a simpler way of writing the general solution is

$$x = Ce^{t^2/2},$$

a solution we could have obtained more simply using the earlier methods we learned for solving linear differential equations.

The method of separation of variables really comes into it's own with non-linear equations like the following:

$$\begin{aligned} x' &= tx^2 \\ \int \frac{x'}{x^2}\,dt &= \int t\,dt \\ \frac{-1}{x} &= \frac{t^2}{2} + C \\ x &= \frac{-2}{t^2 + C} \end{aligned}$$

Here is another one:

$$x' = tx^2 + t.$$

Writing this as

$$x' = (x^2 + 1)t$$

we see that it can be solved using separation of variables to give

$$x = \tan(\frac{t^2}{2} + C)$$

On the other hand,

$$x' = tx^2 + t^2 x$$

cannot be solved using separation of variables, since the right-hand side cannot be expressed as the product of a function of x and a function of t.

The logistic equation

$$x' = rx\left(1 - \frac{x}{K}\right), \quad x(0) = x_0$$

has the general solution

$$x(t) = \frac{x_0 K}{x_0 + (K - x_0)e^{-rt}}$$

Given this, give the general solution of

$$x' = r(1 + \cos t)x\left(1 - \frac{x}{K}\right), \quad x(0) = x_0$$

is

$$x(t) = \frac{x_0 K}{x_0 + (K - x_0)e^{-r(t + \sin t)}}$$

Series solutions

Consider the differential equation

$$x' = tx^2 + t^2 x.$$

The equation tells us that the function on the left is the same as the function on the right. So their derivatives must also be equal. By repeatedly differentiating both sides of the equation, we get expressions for the higher order derivatives of x in terms of its lower order derivatives.

$$\begin{aligned}
x' &= tx^2 + t^2 x \\
x'' &= x^2 + 2txx' + 2tx + t^2 x' \\
x''' &= 2xx' + 2xx' + 2tx'^2 + 2txx'' + 2x + 2tx' + 2tx' + t^2 x'' \\
&= 2x + 4xx' + 4tx' + 2txx' + t^2 x'' + 2tx'^2 \\
x'''' &= 2x' + 4x'^2 + 4xx'' + 4x' + 4tx'' + 2xx' + 2tx'^2 + 2txx'' + t^2 x''' + 2tx'' + 2x'^2 + 4tx''x'
\end{aligned}$$

Evaluating both sides at $t = 0$ gives

$$\begin{aligned} x(0) &= 1 \\ x'(0) &= 0 \cdot x(0)^2 + 0^2 \cdot x(0) = 0 \\ x''(0) &= 1^2 + 0 + 0 + 0 = 1 \\ x'''(0) &= 2 \\ x''''(0) &= 4 \end{aligned}$$

So if $t = 0$ and $x(0) = 1$, then $x'(0) = 0$, $x''(0) = 1$, $x'''(0) = 2$, $x''''(0) = 4$. So

$$x(t) = 1 + \frac{t^2}{2} + \frac{t^3}{3} + \cdots$$