System of differential equations

$$\begin{aligned} x' &= f(x, y, t) \\ y' &= g(x, y, t) \end{aligned}$$

For example, consider

$$\begin{aligned} x' &= y \\ y' &= x \end{aligned}$$

$$x(0) = 1, y(0) = 0$$

We can learn a lot about the graphs of x and y just from reasoning about this differential equation.

Notice that x starts out at $t = 0$ with a horizontal tangent line, since $x'(0) = y(0) = 0$, and that y starts out with slope 1, since $y'(0) = x(0) = 1$. As y increases, it become positive, so the derivative of x also becomes positive (since $x' = y$), and therefore x starts to increase as well. The interaction between x and y is mutually reinforcing, so x and y both increase more and more rapidly.

On the other hand, as we move in the negative direction, y becomes negative, and therefore x is decreasing for x negative. Thus x has a minimum at $t = 0$. Since $x(0) = 1$, the values of x are always positive, so so, since $y' = x$, the graph of y is always increasing. See Figure 1. Notice also that the pair of functions

$$x = e^t, y = e^t$$

satisfies the system. This is the solution satisfying

$$x(0) = 1, y(0) = 1.$$

Notice that solutions to systems of differentials equations come in pairs, and so do their initial conditions.

Are there other solutions like this? Try

$$x = e^{ct}, y = ce^{ct}$$

This satisfies

$$x' = y$$

What about

$$\begin{aligned} y' &= x \\ c^2 e^{ct} &= e^{ct} \end{aligned}$$

This is satisfied if $c = 1$ or $c = -1$. So this gives two solutions:

$$x = e^t, \qquad y = e^t$$

and

$$x = e^{-t} \qquad y = -e^{-t}$$

General solution is

$$x = C_1 e^t + C_2 e^{-t} \qquad y = C_1 e^t - C_2 e^{-t}.$$

Can we find choices of C_1 and C_2 such that

$$x(0) = 1 \quad y(0) = 0?$$

Yes,

$$C_1 = C_2 = 1/2$$

So the analytic solution to the original problem is

$$x(t) = \frac{1}{2}(e^t + e^{-t}) = \cosh t \qquad y = \frac{1}{2}(e^t - e^{-t}) = \sinh t$$

We can also transform

$$\begin{aligned} x' &= y \\ y' &= x \end{aligned}$$

into the single differential equation

$$x'' = x$$

Any second order differential equation

$$x'' = f(x, x', t)$$

can be converted into a system

$$\begin{aligned} x' &= y \\ y' &= f(x, y, t) \end{aligned}$$

Now consider the system:

$$\begin{aligned} x' &= y \\ y' &= -x \end{aligned}$$

$$x(0) = 1, y(0) = 0$$

A solution is

$$x(t) = \cos t, \quad y(t) = -\sin t$$

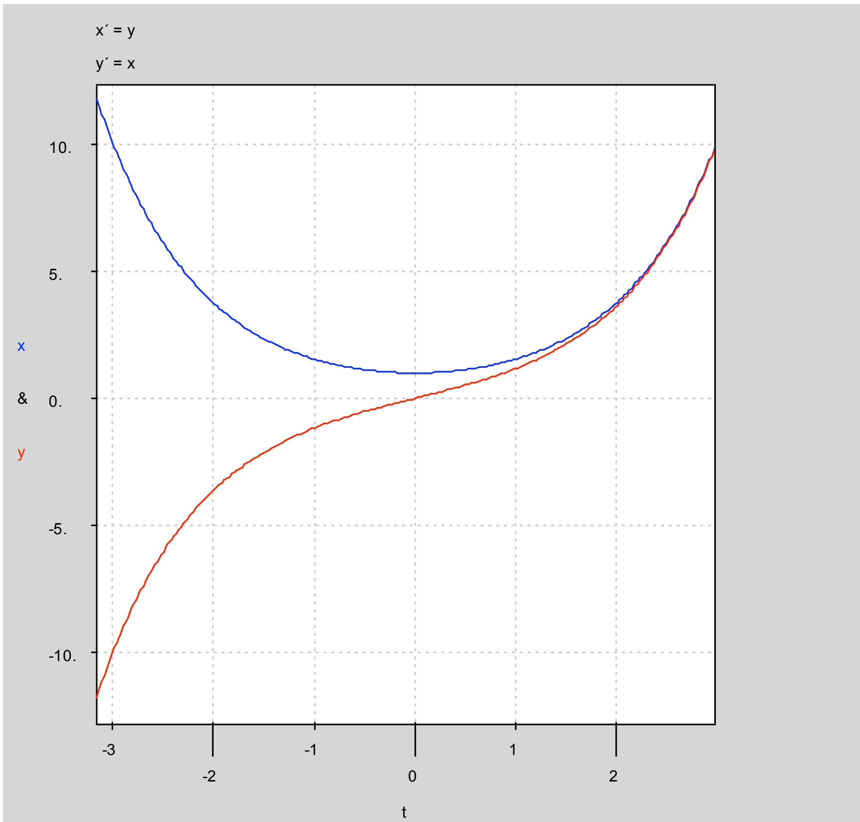

Figure 1: Solution to the system $x' = y$, $y' = x$, $x(0) = 1$, $y(0) = 0$