Existence Theorem

Just as in the case of ordinary first-order differential equations, we have an existence theorem for systems of first-order differential equations. It tells that the initial value problem

$$\begin{aligned} x' &= f(x, y, t), \quad x(t_0) = x_0 \\ y' &= g(x, y, t), \quad y(t_0) = y_0 \end{aligned}$$

has a unique solution pair on an interval around t_0 if f and g are half-way decent functions and defined at (x_0, y_0, t_0). More precisely, they and their partial derivatives with respect to x and y are continuous on a neighbourhood of (x_0, y_0, t_0). Last time we performed the mental exercise of stepping through the solution of a differential equation. The proof of the theorem turns this into a numerical exercise with a proof of convergence.

Linear Systems

A system is linear if f and g depend on x and y linearly, i.e.

$$\begin{aligned} x' &= a(t)x + b(t)y + h_1(t) \\ y' &= c(t)x + d(t)y + h_2(t) \end{aligned}$$

$$x' = a(t)x$$

Principle of superposition: If x_1 and x_2 are solutions, then $C_1x_1 + C_2x_2$ is a solution

$$\begin{aligned} x_1' &= a(t)x_1 \\ x_2' &= a(t)x_2 \end{aligned}$$

Take C_1 times equation 1 plust C_2 times equation 2

$$\begin{aligned} C_1x_1' + C_2x_2' &= C_1a(t)x_1 + C_2a(t)x_2 \\ (C_1x_1 + C_2x_2)' &= a(t)(C_1x_1 + C_2x_2) \end{aligned}$$

Key facts: differentiation is a linear operator on the space of differential functions, and multiplication by $a(t)$ is a linear operator Like linear first order differential equations, these come in homogeneous and inhomogenous varieties, and solutions to the homogeneous ones satify the principle of superposition.

$$x' = a(t)x$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Underlying vector space is the space of order pairs $(x(t), y(t))$ of differentiabel functions. Operator on the left is differentiation, which is still linear, operator on the right is matrix multiplication,. Since matrix multiplication is linear, we still have a principle of superposition. So if (x_1, y_1) is a solution and (x_2, y_2) is a solution then $(C_1x_1 + C_2x_2, C_1y_1 + C_2y_2)$ is also a solution.