Linear Systems

Last time we saw that the solutions a homogeneous system of linear equation Homogeneous linear systems

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

satisfy the principle of superposition, because the operations of differentiation and multiplication by a matrix are both linear operations on the vector space of pairs of differentiable functions. Now consider the system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Here are two solution pairs:

$$\begin{pmatrix} 2e^{-t} \\ e^{-t} \end{pmatrix} \quad \begin{pmatrix} e^{-6t} \\ -2e^{-6t} \end{pmatrix}$$

$$\begin{aligned} \begin{pmatrix} 2e^{-t} \\ e^{-t} \end{pmatrix}' &= \begin{pmatrix} -2e^{-t} \\ -e^{-t} \end{pmatrix} \\ \begin{pmatrix} -2 & 2 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} 2e^{-t} \\ e^{-t} \end{pmatrix} &= \begin{pmatrix} -2e^{-t} \\ -e^{-t} \end{pmatrix} \end{aligned}$$

$$\begin{aligned} (e^{-t} \begin{pmatrix} 2 \\ 1 \end{pmatrix})' &= -e^{-t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = (e^{-t} \begin{pmatrix} -2 \\ -1 \end{pmatrix}) \\ \begin{pmatrix} -2 & 2 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t} &= \begin{pmatrix} -2 \\ -1 \end{pmatrix} e^{-t} \end{aligned}$$

We now have two solutions

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \quad \text{with initial values} \quad \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Find solutions satisfying the following sets of initial conditions:

1. $(x(0), y(0)) = (1, 0)$
2. $x(0) = 0$, $y(0) = 1$
3. $x(0) = 3$, $y(0) = -2$

So we want to find C_1 and C_2 so that

$$\left[C_1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + C_2 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right] \Bigg|_{t=0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$C_1\begin{pmatrix}2\\1\end{pmatrix}+C_2\begin{pmatrix}1\\-2\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}$$

$$\begin{aligned}2C_1+C_2&=1\\C_1-2C_2&=0\end{aligned}$$

$$C_1=2/5, C_2=1/5$$

So the answer to (1) is

$$\begin{aligned}x&=\frac{4}{5}e^{-t}+\frac{1}{5}e^{-6t}\\y&=\frac{2}{5}e^{-t}-\frac{2}{5}e^{-6t}\end{aligned}$$

So the answer to (2) is

$$\begin{aligned}x&=\frac{2}{5}e^{-t}-\frac{2}{5}e^{-6t}\\y&=\frac{1}{5}e^{-t}+\frac{4}{5}e^{-6t}\end{aligned}$$

If we want a solution of the form

$$C_1\begin{pmatrix}2e^{-t}\\e^{-t}\end{pmatrix}+C_2\begin{pmatrix}e^{-6t}\\-2e^{-6t}\end{pmatrix}$$

satisfying $x(0)=x_0$ and $y(0)=y_0$, then we need to solve the linear equation

$$C_1\begin{pmatrix}2\\1\end{pmatrix}+C_2\begin{pmatrix}1\\-2\end{pmatrix}=\begin{pmatrix}x_0\\y_0\end{pmatrix}$$

or

$$\begin{pmatrix}2&1\\1&-2\end{pmatrix}\begin{pmatrix}C_1\\C_2\end{pmatrix}=\begin{pmatrix}x_0\\y_0\end{pmatrix}$$

We can satisfy any initial conditions because the initial conditions of our two original solutions are linearly independent. On the other hand, once we have found a solution satisfying a given set of initial conditions, we know it is the only one. So linear combinations of our two solutions form the general solution.

How do we come up with solutions?