Lesson 2: Existence of Solutions

25 August 2005

Introduction

Our goal in this class is to look at a particular class of differential equations and understand why and when we know that they have solutions. We are also interested in what information about the solution is sufficient to pin down a unique solution.

Initial Value Problem

$$x' = f(x, t), \quad x(t_0) = x_0.$$

A solution is a function $x(t)$ such that

- $x'(t) = f(x(t), t)$
- $x(t_0) = x_0$

The domain of x should contain an open interval around t_0. We are not concerned at the moment about the largest interval, simply in finding a solution defined on any interval.

How do we know a solution exists?

Let's consider first the examples from calculus, where the function on the right is just a function of t.

$$x' = t^2, \quad x(t_0) = x_0$$

This is the special case where $f(x, t) = g(t)$ does not depend on x, and in calculus we learned how to solve problems like this using integration.

The general solution (without the initial value) is the indefinite integral:

$$x(t) = \frac{t^3}{3} + C.$$

Substitute in the initial value

$$\begin{aligned} x_0 &= \frac{t_0^3}{3} + C \\ C &= x_0 - \frac{t_0^3}{3} \end{aligned}$$

Thus

$$x(t) = \frac{t^3}{3} + x_0 - \frac{t_0^3}{3}.$$

We can also express this solution with a definite integral, using the Fundamental Theorem of Calculus.

The function

$$x(t) = x_0 + \int_{t_0}^{t} s^2\, ds$$

is a solution to the initial value problem, because by the Fundamental Theorem of Calculus,

$$x'(t) = \frac{d}{dt}\left(x_0 + \int_{t_0}^{t} s^2\, ds\right) = t^2,$$

and by the properties of definite integrals,

$$x(t_0) = x_0 + \int_{t_0}^{t_0} s^2\, ds = x_0.$$

We can also check directly that this is the same solution we obtained before by evaluating the integral.

We have

$$x_0 + \int_{t_0}^{t} s^2\, ds = x_0 + \frac{t^3}{3} - \frac{t_0^3}{3},$$

so this is the same answer we got before.

What if we don't know the indefinite integral?

Consider the differential equation

$$x' = \frac{1}{\sqrt{1+t^3}}, \quad x(t_0) = x_0.$$

This time we cannot find the antiderivative in closed form, so we use the integral representation

$$x(t) = x_0 + \int_{t_0}^{t} \frac{1}{\sqrt{1+s^3}}\, ds$$

More generally, if g is any function that is continuous in an interval around t_0, then we can solve

$$x'(t) = g(t), \quad x(t_0) = t_0$$

with

$$x(t) = x_0 + \int_{t_0}^{t} g(s)\, ds.$$

The definite integral can always be approximated numerically to arbitrary accuracy even if we don't know a formula for the indefinite integral, so in some sense it *is* a formula.

Equations like

$$x' = 0.001x(100 - x), \quad x(t_0) = x_0$$

cannot be solved by integrating both sides, because the function we want to find is on the right.

The slope field approach

We'd like to have a unified way of looking at all three types of equations that enables us to see whether they have solutions or not. This is provided by slope fields.

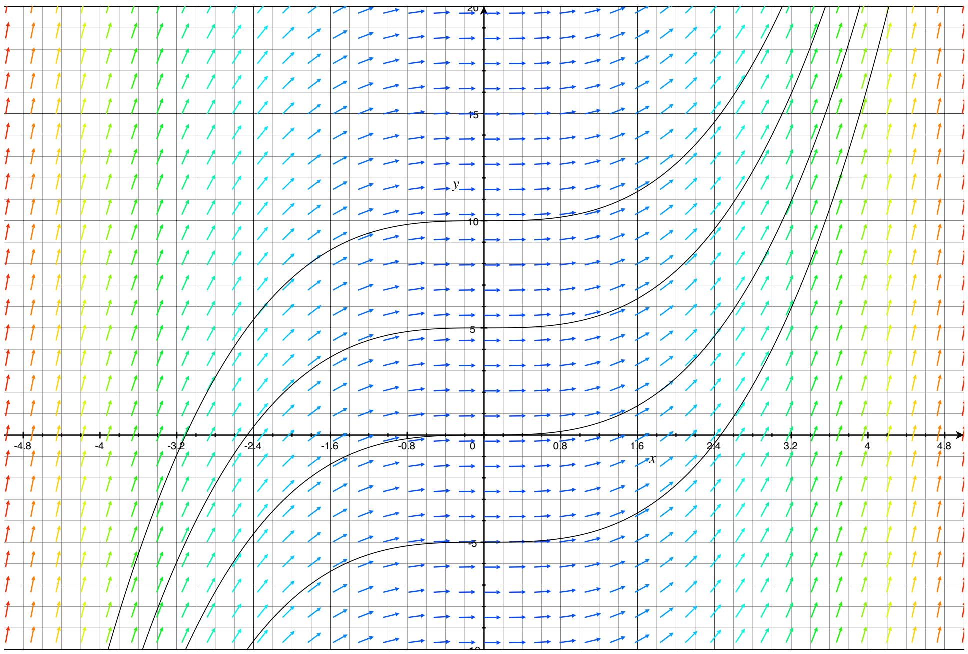

Figure 1: Slope field for and olutions to $x' = t^2$ with $t_0 = 3$ and $x_0 = -5, 0, 5, 10$

The process of stepping along through this slope field just gives the Riemann sum for the definite integral. But this process also works when we can't integrate both sides, as the example of the logistic equation from last time shows. Even when the function on the right depends on x, the slope field will show us a solution. (See Figure 2 in the first lecture.) However, that in this case the shapes of the graph can change radically when you have different initial values: there is no "$+C$".

We have seen three equations, one we know how to solve with a formula, one we know how to solve by expressing it as a definite integral, and one where we can just see the solution using the slope field. From the point of view of the slope field, they are all the same. So the slope field gives a very powerful tool.

The fundamental existence theorem

In the previous example we saw that the interval on which a solution is defined could be limited, and that there could be no solution at all if $g(t)$ is not continuous at t_0.

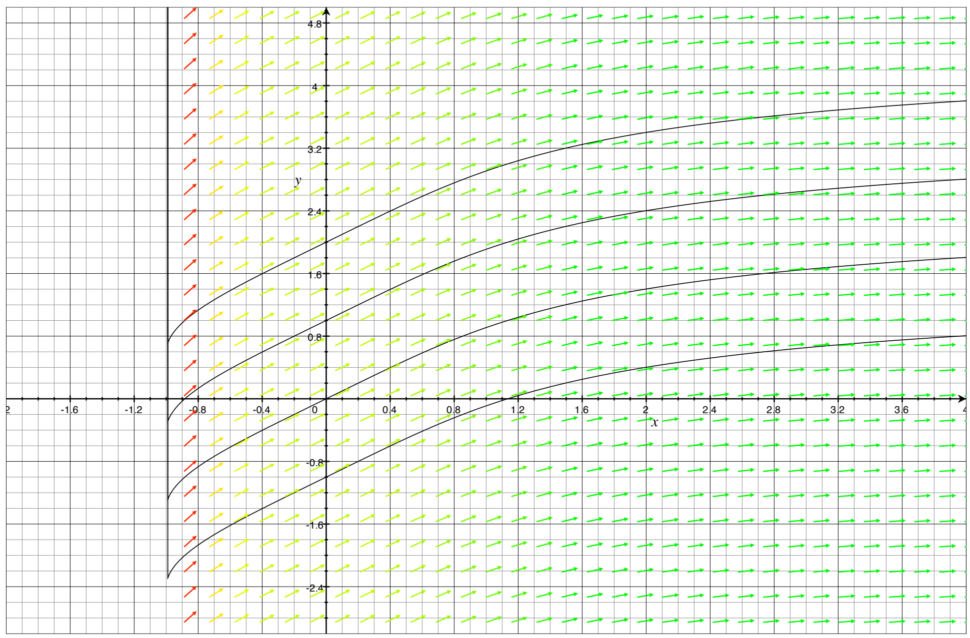

Figure 2: Slope field for and solutions to $x' = t^2$ with $t_0 = 1$ and $x_0 = -1, 0, 1, 2$

The initial value problem

$$x' = f(x, t), \quad x(t_0) = x_0$$

has a unique solution on an interval around t_0 if f and $\partial f / \partial x$ are continuous on a region containing (x_0, t_0).

The condition on the partial derivative is necessary, as the example

$$x' = x^{1/3}, \quad x(0) = 0$$

illustrates. (See book.)

The formal proof of this statement depends on the idea of making mathematically precise what the computer is doing when it draws a curve following a slope field, and proving that the approximations you get as you do this with finer and finer steps converge to a solution.

The behaviour of slope fields

Now we look at a more complicated slope field.

Last time we looked at the equation $x' = ax(L - x)$. If we factor out the L and rename the constants, we can write this as

$$x' = rx(1 - \frac{x}{K}).$$

Here K is still the carrying capacity, and r is now the initial growth rate. We modify this by supposing that there is an oscillating carrying capacity:

$$x' = rx(1 - \frac{x}{K + a\sin t}).$$

Here is a graph of the slope field with $a = 50$, $r = 0.1$, and $K = 100$, and a solution with initial value $x(0) = 50$.

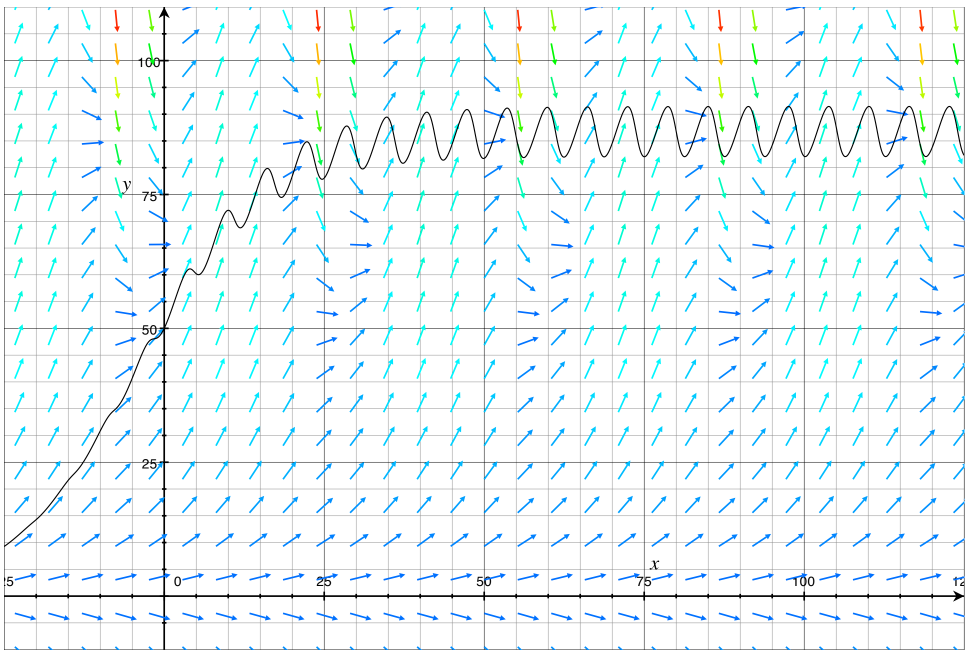

Figure 3: Slope field of oscillating logistic differential equation