Using Slope Fields to Analyze Differential Equations

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Doubling time

$$x' = rx$$

See Figures 1, 2 and 3.

Notice that the doubling time, T, doubles when the rate, r, halves, and halves when the rate doubles. This is because

$$T = \frac{\ln 2}{r}.$$

Carrying capacity

$$x' = rx(1 - \frac{x}{K})$$

On a scale that is small relative to the carrying capacity, the logistic differential equation looks very like the exponential. See Figure 4. As the scale increases, we begin to see a falloff from exponential growth. For example, in Figure 5, where the scale on the vertical axis is 1000, which is 1/10th of the carrying capacity, the doubling times are no longer all the same, it takes slightly longer for larger populations to double. Figure 6 shows the effect of the carrying capacity.

Exponential growth with harvesting

$$x' = rx - h$$

Notice that in Figure 7, initial populations above 1000 grow, and ones below 1000 die out. We say that the harvesting rate of 100 is the critical harvesting rate for an initial population of 1000. In general, given a starting population of x_0, the critical harvesting rate is

$$h_{cr} = \frac{x_0}{r},$$

because the constant function $x = x_0$ is the solution to

$$x' = rx - h_{cr}, \quad x(0) = x_0.$$

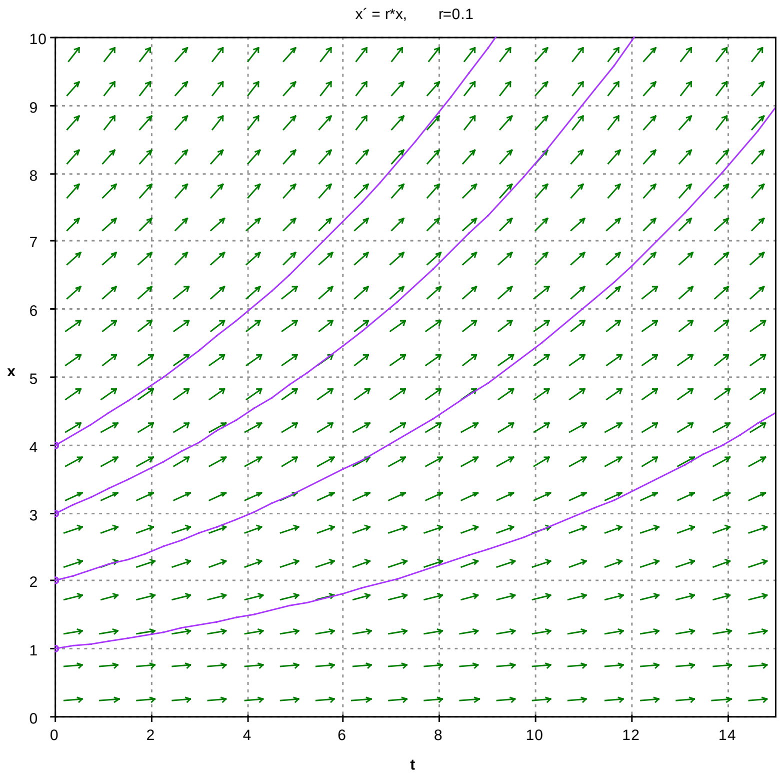

Figure 1: Doubling time is about 7 when $r = 0.1$

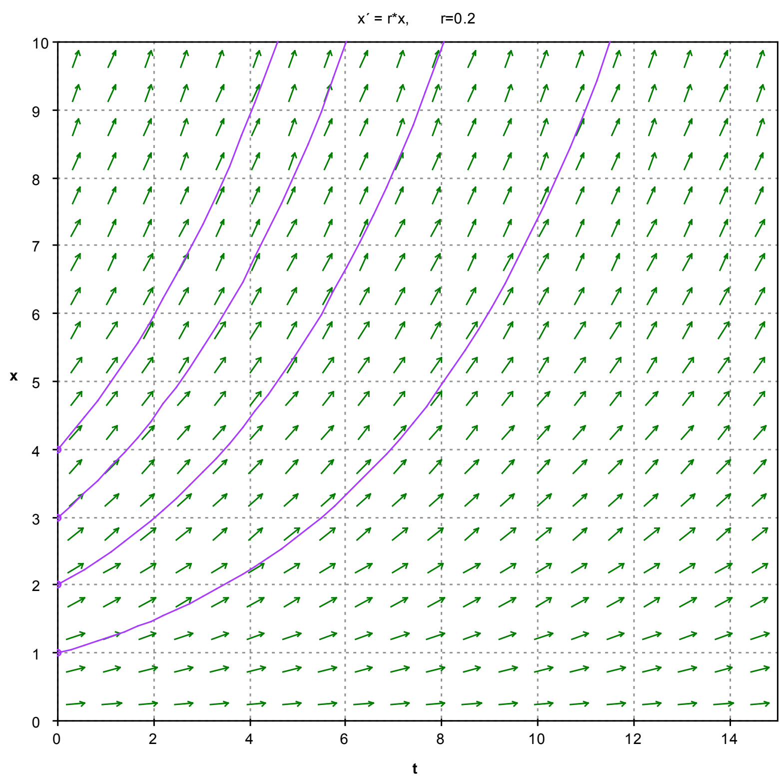

Figure 2: Doubling time is about 3.5 when $r = 0.2$

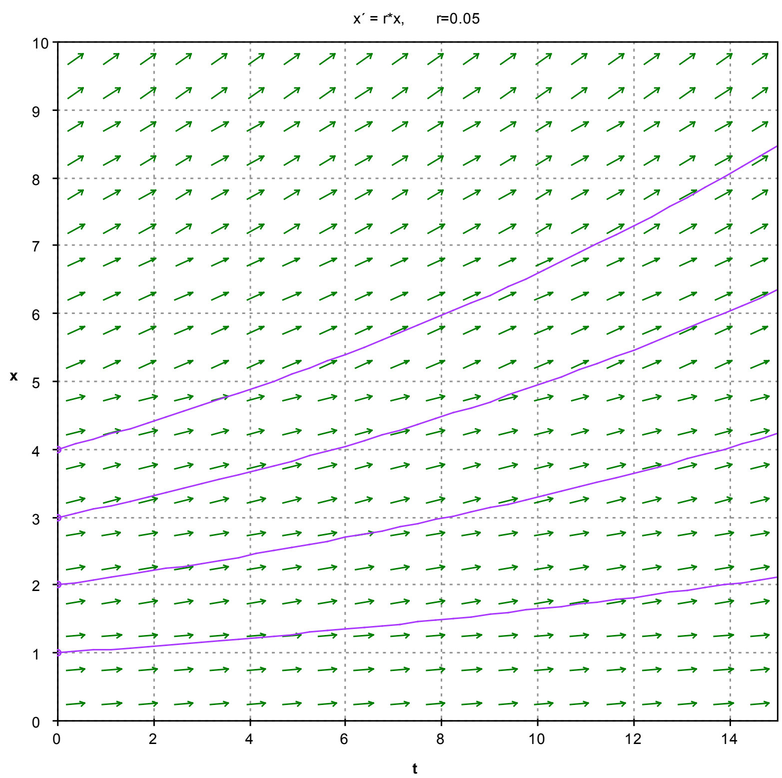

Figure 3: Doubling time is about 14 when $r = 0.05$

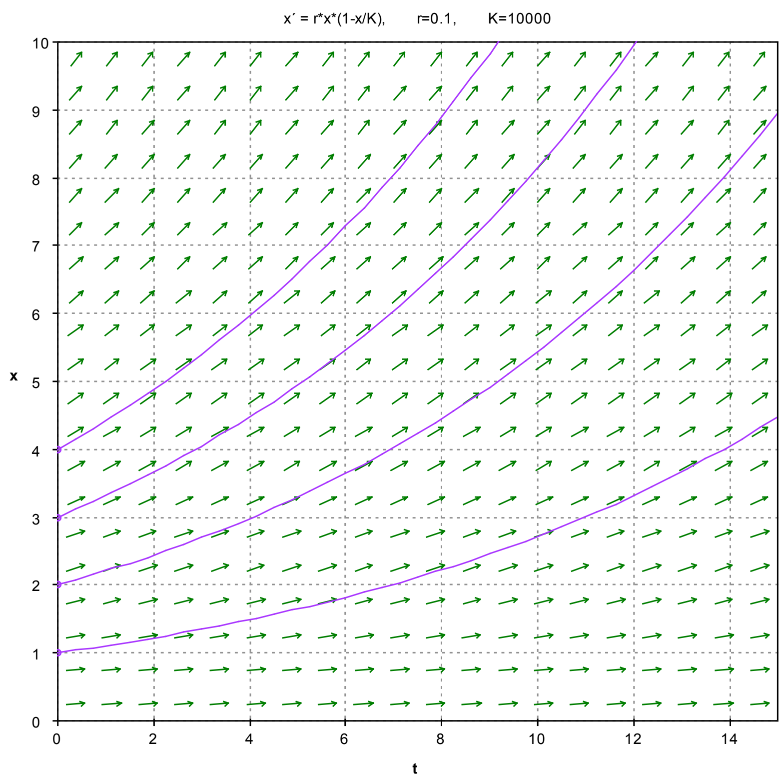

Figure 4: Logistic curves on a small scale

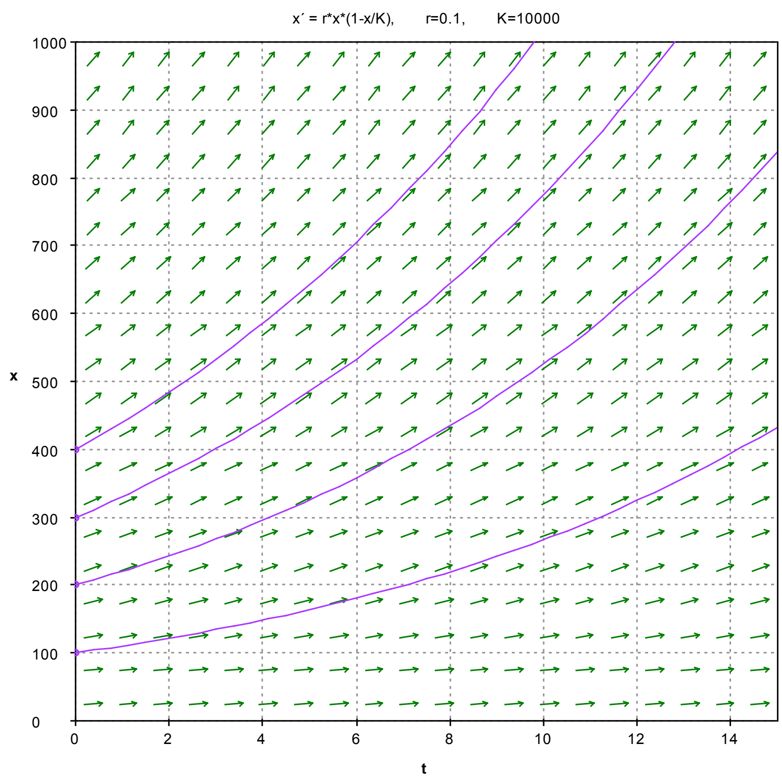

Figure 5: Logistic curves on a larger scale

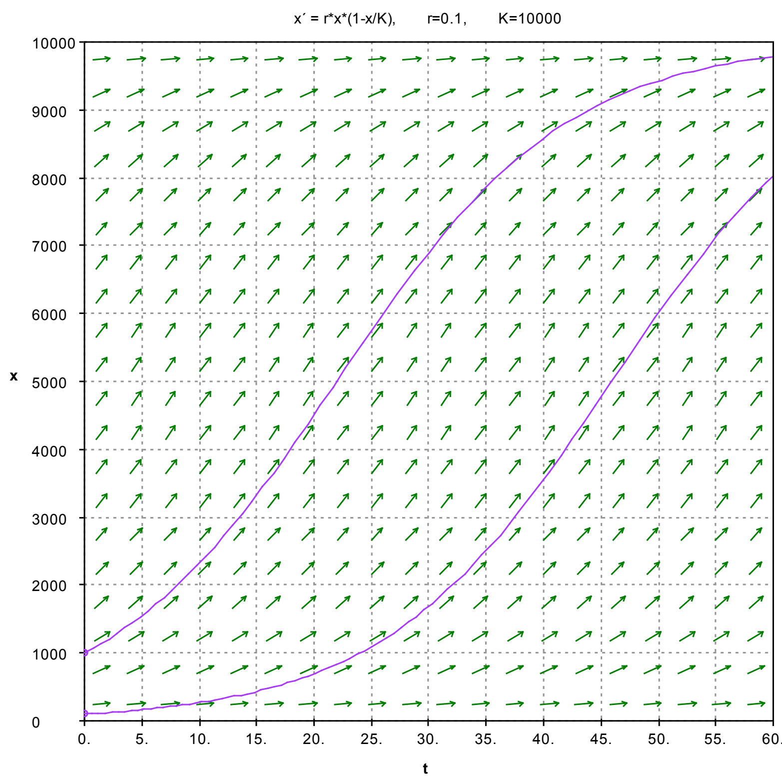

Figure 6: Logistic growth showing the carrying capacity

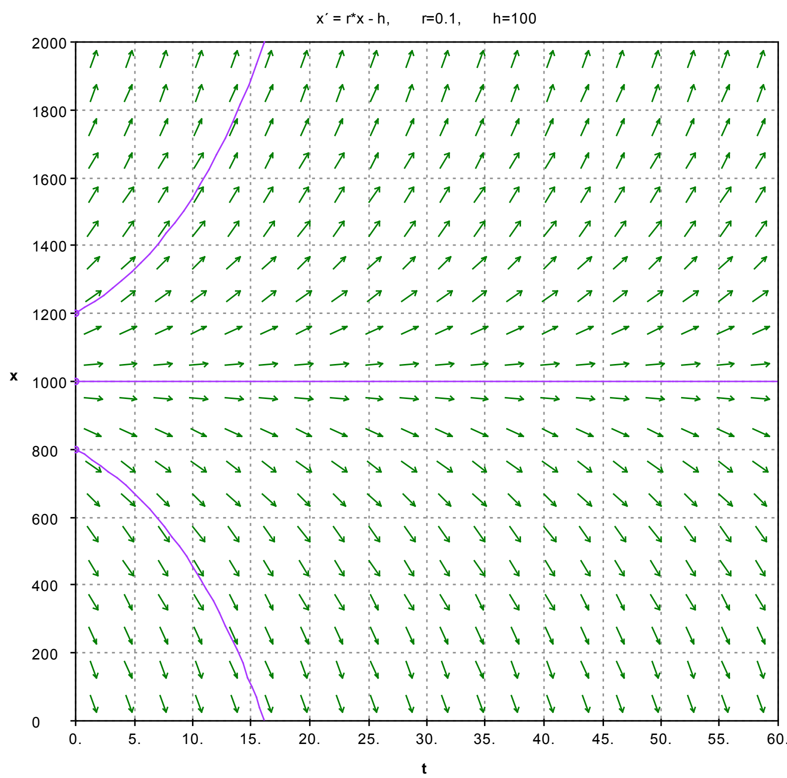

Figure 7: Exponential growth with harvesting at a rate of 100 per year