Lesson 5: First Order Linear Differential Equations

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Introduction

Having seen how to visualize and numerically approximate solutions to differential equations, we now look at how to find expressions for the solutions.

What is a Linear Equation?

We consider the simplest sort of differential equation—one which is linear in x and its derivatives. This means that the only terms in the equation involve 1, x, x', ... multiplied by functions of t.

A first order linear equation has the form

$$x' = p(t)x + q(t).$$

Which of the following are linear?

$$\begin{aligned}
x' &= t^2x + 3 \\
x' &= tx^2 + 3 \\
e^t x - tx' &= t^2 \\
e^t x - xx' &= 3 \\
x' &= rx \\
x' &= rx - h \\
x' &= rx\left(1 - \frac{x}{K}\right)
\end{aligned}$$

The equation is said to be *homogeneous* if the constant term, $q(t)$, is zero.

The differential equation

$$x' = 3x$$

is homogeneous, whereas

$$x' = 3x + \sin t$$

is inhomogeneous.

The Integrating Factor

Consider the homogeneous equation

$$x' = 0.1x.$$

How do we prove that the general solution is $x = Ce^{0.1}$? By proving that $xe^{-0.1t}$ is a constant.

Want to show that if

$$x' = 0.1x \tag{1}$$

then

$$\frac{d}{dt}(xe^{-0.1t}) = 0$$

So

$$\begin{aligned}
\frac{d}{dt}(xe^{-0.1t}) &= x'e^{-0.1t} + x(-0.1e^{-0.1t}) \\
&= e^{-0.1t}(x' - 0.1x) \\
&= 0.
\end{aligned}$$

By (1), we have

$$x' - 0.1x = 0$$

If we reverse the order of steps in this proof, we get a method of solving that works for inhomogeneous equations as well. For example,

$$\begin{aligned}
x' &= 0.1x - 100 \\
x' - 0.1x &= -100 \\
e^{-0.1t}(x' - 0.1x) &= -100e^{-0.1t} \\
x'e^{-0.1t} + (-0.1)e^{-0.1t}x &= -100e^{-0.1t} \\
\frac{d}{dt}(xe^{-0.1t}) &= -100e^{-0.1t} \\
xe^{-0.1t} &= \int (-100e^{-0.1t})\,dt = 1000e^{-0.1t} + C \\
x &= Ce^{0.1t} + 1000
\end{aligned}$$

The factor $e^{-0.1t}$ is called an integrating factor, because when we multiplied both sides of the equation by it, we were able to solve the differential equation by integration. Can we find integrating factors for more complicated equations?

We will try integrating factors of the form $e^{-P(t)}$, where $P(t)$ is some function.

$$\begin{aligned}
x' &= t^2 x \\
x' - t^2 x &= 0 \\
e^{-t^3/3}(x' - t^2 x) &= 0 \\
x' e^{-t^3/3} + (-t^2) e^{-t^3/3} x &= 0 \\
\frac{d}{dt}(x e^{-t^3/3}) &= 0 \\
x e^{-t^3/3} &= C \\
x &= C e^{t^3/3}
\end{aligned}$$

In general, the integrating factor for

$$x' = p(t)x + q(t)$$

is

$$e^{-P(t)}$$

where

$$P(t) = \int p(t)\, dt.$$

To solve the differential equation

- move the $p(t)$ term to the left
- multiply both sides by the integrating factor
- integrate both sides, using the product rule in reverse on the left
- isolate x

Integrating Factors for Initial Value Problems

For an initial value problem, we can use a definite integral in the integrating factor method.

Solve

$$x' = rx - h, \quad x(0) = x_0, \quad \text{where } r \text{ and } h \text{ are constant.}$$

$$
\begin{aligned}
x' &= rx - h \\
x' - rx &= -h \\
e^{-rt}(x' - rx) &= -he^{-rt} \\
x'e^{-rt} + (-r)e^{-rt}x &= -he^{-rt} \\
\frac{d}{dt}(xe^{-rt}) &= -he^{-rt} \\
\int_0^t \frac{d}{ds}(xe^{-rs})\,ds &= \int_0^t (-he^{-rs})\,ds \\
xe^{-rt} - x_0 &= \frac{h}{r}e^{-rs}\Big|_0^t = \frac{h}{r}e^{-rt} - \frac{h}{r} \\
xe^{-rt} &= \frac{h}{r}e^{-rt} + (x_0 - \frac{h}{r}) \\
x &= (x_0 - \frac{h}{r})e^{rt} + \frac{h}{r}
\end{aligned}
$$