Solutions of First Order Linear Differential Equations

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Introduction

Structure of solutions to linear first order differential equations

Here are some solutions to differential equations from last time.

$$\begin{aligned} x' &= 0.1x \\ x &= Ce^{0.1t} \\ x' &= 0.1x - 100 \\ x &= Ce^{0.1t} + 1000 \\ x' &= t^2x \\ x &= Ce^{t^3/3} \\ x' &= rx - h \\ x &= (x_0 - \frac{h}{r})e^{rt} + \frac{h}{r} \end{aligned}$$

These all have a similar structure.

It seems that the general solution to the homogeneous equation

$$x' = p(t)x$$

is

$$Cf(t).$$

Why? More precisely, the solutions form a one-dimensional vector space.

Principle of superposition

If x_1 and x_2 are any two solutions to

$$x' = p(t)x$$

then $\lambda x_1 + \mu x_2$ is also a solution for constants λ and μ. why? For example, why is $x_1 + x_2$ a solution?

$$(x_1 + x_2)' = x_1' + x_2' = p(t)x_1 + p(t)x_2 = p(t)(x_1 + x_2)$$

This phenomenon is false for non-homogeneous differential equations. Called the principle of superposition. Also true that if x is a solution, so is λx.

If x_p is a solution to

$$x' = p(t)x + q(t)$$

and x is any other solution, then $x - x_p$ is a solution to the corresponding homogeneous equation

$$x' = p(t)x.$$

Finding a particular solution

A lot of the differential equations we have been looking at consist of adding some non-homogeneous term to a simple homogeneous differential equation we already know how to solve. For example, we have been looking at population growth with various harvesting terms.

$$\begin{aligned} x' &= rx - h \\ x' &= rx - ae^{-bt} \\ x' &= rx + a \sin bt \end{aligned}$$

What is the corresponding homogeneous equation to all of these?

$$x' = rx$$

has general solution

$$x = Ce^{rt}$$

For all these equations, the general solution is

$$x = Ce^{rt} + x_p$$

where x_p is some particular solution. So the trick is to find particular solutions.

One way to do this is to use the method of integrating factors. It's important to realize that this gives us a completely general solution to any first order linear differential equation.

If $p(t)$ and $q(t)$ are continuous on $[a, b]$, and $t_0 \in [a, b]$, then the solution to

$$x' = p(t)x + q(t), \quad x(t_0) = x_0$$

is

$$x(t) = x_0 e^{P(t)} + e^{P(t)} \int_{t_0}^{t} e^{-P(s)} q(s)\, ds,$$

where

$$P(t) = \int_{t_0}^{t} p(s)\, ds.$$

In some sense this completely answers the question. However, if we want a closed formula for the solution, we have to do the integral. And sometimes there are shortcuts that will find a particular solution more quickly.

Method of undetermined coefficients

Consider

$$x' = 0.1x - 100$$

Suppose there is a constant solution $x = k$. Let's try it. Well $x' = 0$, so

$$0 = 0.1k - 100$$

so $k = 1000$.

Let's look at another equation,

$$x' = 0.1x - e^{-t}$$

Suppose there's a solution of the form

$$x = ke^{-t}$$

Try it.

$$\begin{aligned}
(ke^{-t})' &= 0.1(ke^{-t}) - e^{-t} \\
-ke^{-t} &= 0.1ke^{-t} - e^{-t} \\
-k &= 0.1k - 1 \\
k &= \frac{1}{1.1}
\end{aligned}$$

So a solution is

$$\frac{1}{1.1}e^{-t} + Ce^{0.1t}$$

What about

$$x' = 0.1x - \sin t$$

This time we try a solution of the form

$$x = k_1 \sin t + k_2 \cos t$$

Strategy: plug this in, get two equations in k_1 and k_2 (one for the sines and one for the cosines) then solve them simultaneously.

In general, this method works for

$$x' = p(t)x + q(t)$$

if $q(t)$ belongs to a finite-dimensional vector space of functions which is closed under differentiation. Then you try a solution which is a general linear combination, with undetermined coefficients, of a the elements of a basis of that vector space.