**Variation of coefficients**

To solve

$$x' = 4x - 3e^{\pi t}$$

try a solution of the form

$$x = ke^{\pi t}.$$

Sometimes this doesn't work, if the solution you want to try is also a solution of the corresponding homogeneous equation:

$$x' = 3x - 15e^{3t}.$$

In this case if we tried $x = ke^{3t}$, we would not find a solution, because it is a solution of

$$x' = 3x.$$

So we remember the product rule and try

$$x = kte^{3t}.$$

This gives

$$\begin{aligned} (kte^{3t})' &= 3kte^{3t} - 15e^{3t} \\ ke^{3t} + 3kte^{3t} &= 3kte^{3t} - 15e^{3t} \\ ke^{3t} &= -15e^{3t} \\ k &= -15 \\ x &= -15te^{3t} \end{aligned}$$

So the general solution to this equation is

$$x = Ce^{3t} - 15te^{3t}.$$

For an equation like

$$x' = -x + 2e^t - 3\sin t \tag{1}$$

we could try

$$x = k_1 e^t + k_2 \sin t + k_3 \cos t.$$

Alternatively, we can break this up into

$$x' = -x + 2e^t \tag{2}$$

$$x' = -x - 3\sin t. \tag{3}$$

To solve (3) we try

$$x = k_1 \sin t + k_2 \cos t.$$

This gives

$$\begin{aligned}
k_1 \cos t - k_2 \sin t &= -k_1 \sin t - k_2 \cos t - 3 \sin t \\
-k_2 \sin t &= -k_1 \sin t - 3 \sin t \\
k_1 \cos t &= -k_2 \cos t \\
-k_2 &= -k_1 - 3 \\
k_1 &= -k_2 \\
k_1 &= -3/2 \\
k_2 &= 3/2
\end{aligned}$$

Adding this to the solution of (3), we get a particular solution to (1):

$$x_p = e^t - \frac{3}{2} \sin t + \frac{3}{2} \cos t$$

**Newton's Law of Cooling**

Let $x(t)$ be the temperature of a cup of coffee $t$ minutes after it was made. The rate of cooling decreases as the $x$ gets closer to room temperature. Let room temperature be $b$. We postulate that the rate of cooling is proportional to $x - b$.

$$x' = -a(x - b), \quad x(0) = x_0, \quad a \text{ positive}$$

$$x' = -ax + ab$$

The general solution to the corresponding homogeneous equation

$$x' = -ax$$

is

$$x = Ce^{-at}$$

The inhomogeneous equation has the particular solution $x = b$, so the general solution is

$$x = b + Ce^{-at}.$$

Finally, setting $x(0) = x_0$ and solving for $C$, we get

$$x = b + (x_0 - b)e^{-at}.$$

We can also use the general formula for the solution of a linear equation to get the same result:

$$\begin{aligned}
x &= x_0 e^{-at} + e^{-at} \int_0^t e^{au} ab \, du \\
x &= x_0 e^{-at} + e^{-at} \frac{1}{a} [e^{at} ab - ab] \\
x &= (x_0 - b)e^{-at} + b.
\end{aligned}$$

What if the room temperature is changing, so $b$ is replaced by $b(t)$? For example, suppose that

$$b(t) = b_{av} + \alpha \sin(\frac{2\pi}{T} t).$$

Then we want to solve

$$x' = -a(x - b_{av} - \alpha \sin(\frac{2\pi}{T} t)),$$

or equivalently

$$x' = -ax + ab_{av} + a\alpha \sin(\frac{2\pi}{T} t).$$