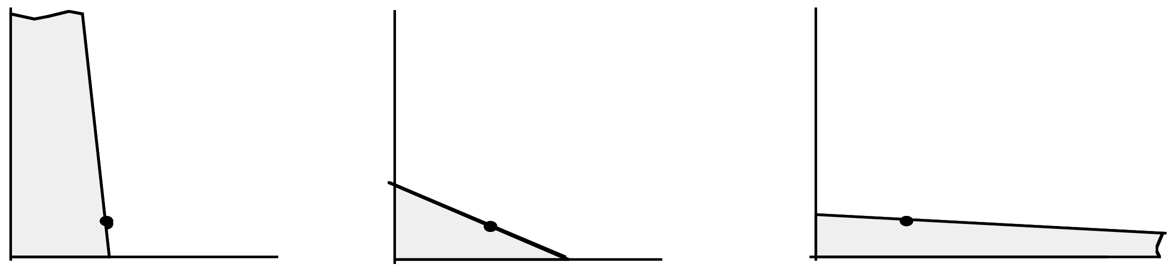
§1 discussion: Getting a feel for the problem

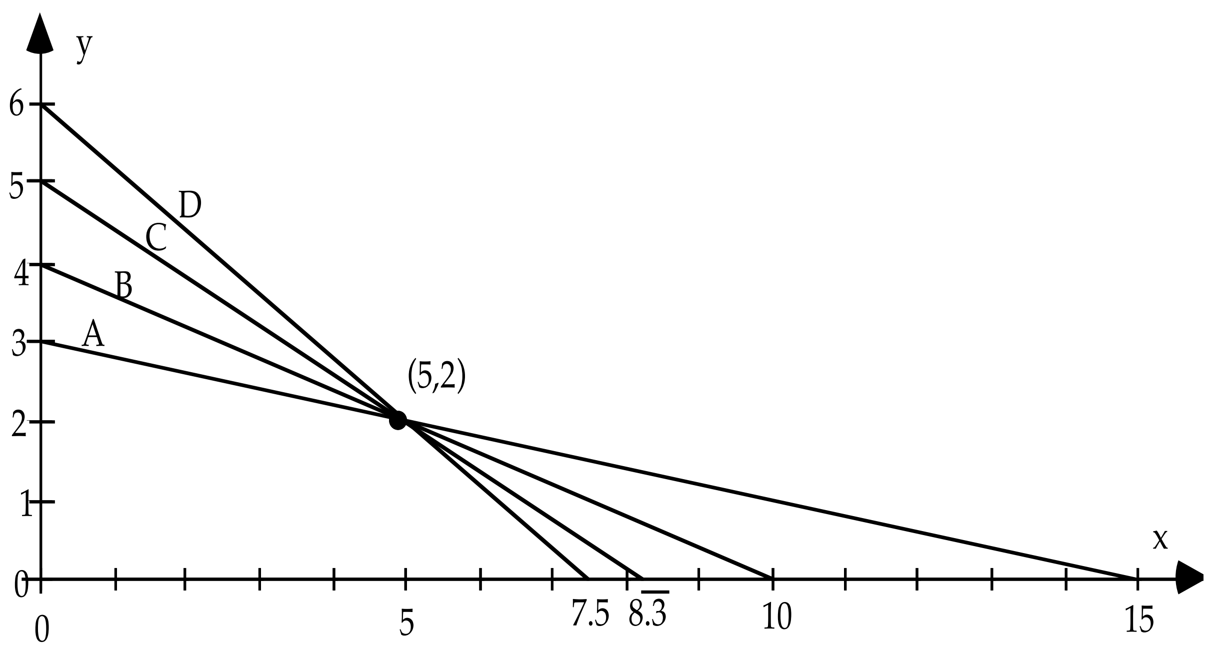
1a) We can get a rough feeling about this problem situation by sketching some different lines through the point (5,2). We can see from the diagram that a very steep line (left) will cut off a large area. Similarly, a very flat line (right) will also cut off a large area. (See Figure 1.) It makes sense that there is a line somewhere in the middle range of steepness that cuts off the smallest area.



Various lines through (5,2) cutting off triangles with different areas

1b) Below are four lines in the middle range of steepness labeled A, B, C and D drawn through the point (5,2). The areas are:

A 22.5 B 20 C ≈21 D 22.5



Four particular lines through the point (5,2).

Of these four lines, B creates the triangle with smallest area. But do other lines give even smaller areas? To answer this we need a more exact analysis, which we pursue in the following sections.

§2 discussion: Parameterizing the lines

2a) Parameters for a line are numbers that identify the line. Here are some parameters for the line through (5,2). See the diagram below.

b = y-intercept

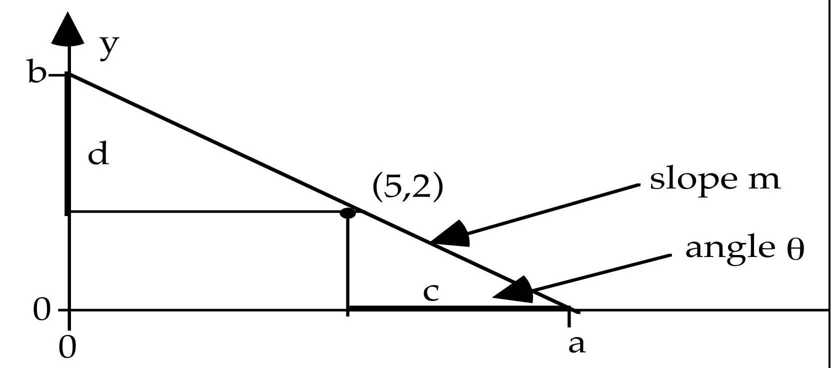
a = x-intercept

d = height of upper triangle

c = base of lower triangle

m = slope

θ = angle with x-axis



Some parameters for specifying a line through (5,2)

Giving a numerical value to *any one* of these six parameters will uniquely identify a line through (5,2). For example, if we specify that the y-intercept b is equal to 3, then we have two points on the line, (0,3) and (5,2), and thus have specified the line uniquely.

These six ways may be the most obvious ways of parameterizing a line through a given point, but they aren't the only parameters possible. Other ones to consider are:

- D = distance of the line from the origin
  (= length of the line segment from (0,0) to the line and perpendicular to it),
- L = length of the line,
- P = perimeter of the triangle through (0,0) and containing the point (5,2).

[Question: which of these parameters identify a unique line through (5,2)?]

2b) It is a straightforward matter to specify the relationship between pairs of the first six parameters in the list (b, a, d, c, m, and θ). For example, using basic trigonometry and properties of similar triangles we have:

$\tan\theta = -m$ $\qquad b = 2 \bullet \dfrac{a}{a - 5}$ $\qquad \dfrac{2}{c} = \dfrac{d}{5}$, $\qquad$ etc.

It is not hard to see that any one of the first six parameters can be found from any of the others. (It is a very different story for the proposed parameters D, L, and P. It is not at all obvious how to express these parameters in terms of the first six.)

2c) We have seen that a single number is sufficient to identify a unique line through (5,2). This means that the family of lines through (5,2) is a *one-parameter family*. Put differently, there is only one "degree of freedom" in varying the lines through (5,2).

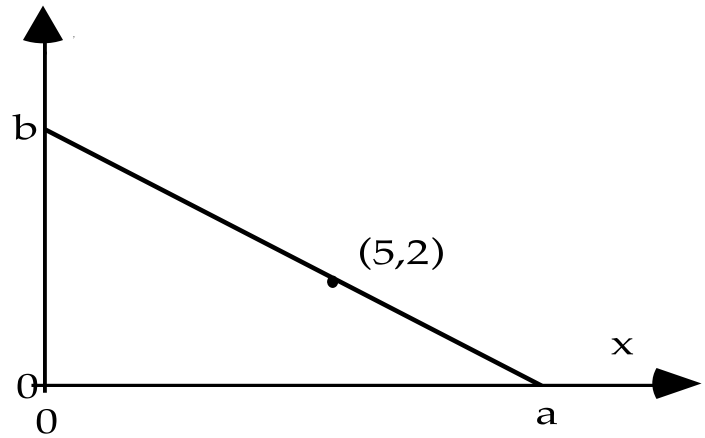
§3 discussion: Representing the area

3a) If the x and y intercepts are a and b, then the area of the triangle is $\frac{1}{2}ab$. (See the figure at the right.)

But notice that these parameters a and b are not independent. The fact that the line passes though (5,2) means that they are related by

(1) $$\frac{b-2}{5} = \frac{2}{a-5}$$

(Show why this is true.)



The intercepts *a* and *b* of a line through point (5,2)

Solving for a in terms of b gives

(2) $$a = \frac{5b}{b-2}$$

Hence the area as a function of b is

(3) $$\text{area} = f(b) = \frac{5\,b^2}{2\,(b-2)}$$

Below we express the area as a function of each of the six basic parameters of §2:

b: $\frac{1}{2} \bullet 5 \bullet \frac{b^2}{b-2}$ a: $\frac{1}{2} \bullet 2 \bullet \frac{a^2}{a-5}$ c: $(5\bullet 2) + \frac{1}{2} \bullet 2\,(c + \frac{5^2}{c})$

d: $(5\bullet 2) + \frac{1}{2} \bullet 5\,(d + \frac{2^2}{d})$ m: $(5\bullet 2) - \frac{1}{2} \bullet (5^2 \bullet m + \frac{2^2}{m})$ θ: messy

3b) The expression of the area in terms of c, d, or m is especially simple. (Each involves at its heart an expression of the form $(x + \frac{k}{x})$ for a variable x and a constant k.)

The expression of area in terms of b or a is not quite as simple. (Each involves at its heart an expression of the form $(\frac{x^2}{x-k})$ for a variable x and a constant k.)

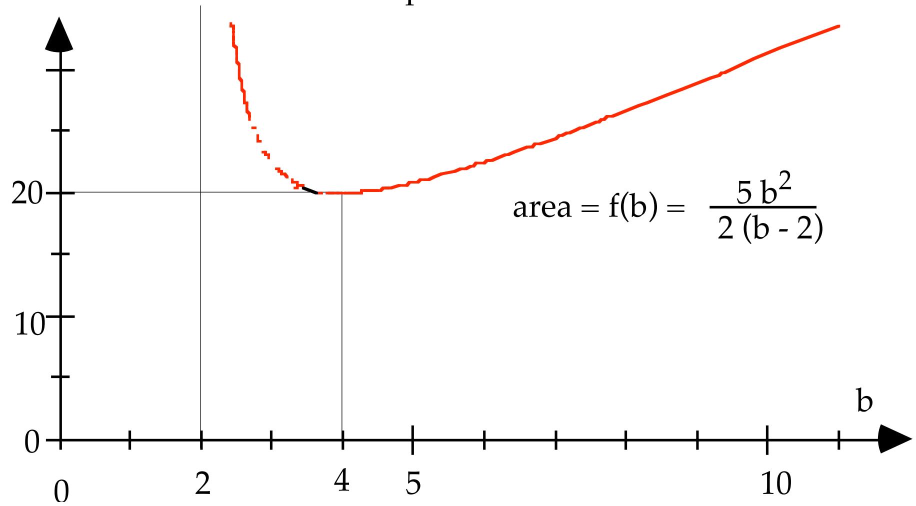
(Question: why do we say these representations in terms of c, d, and m are simpler than those in terms of b, a, or θ? This is something we will return to in the next sections.)

3c) It would be awkward to express the area directly in terms of θ (without using m = tanθ). It would be even more awkward to express the area in terms of the other possible parameters from §2 (such as D, L, or P).

§4 discussion: Finding the minimum area

4a) Using the parameter b: Below is a graph of the function (3) from §3: $f(b) = \dfrac{5\,b^2}{2\,(b-2)}$.

Estimating from the graph we find that the minimum area is 20 sq units and occurs at the value b = 4 of the vertical axis intercept.

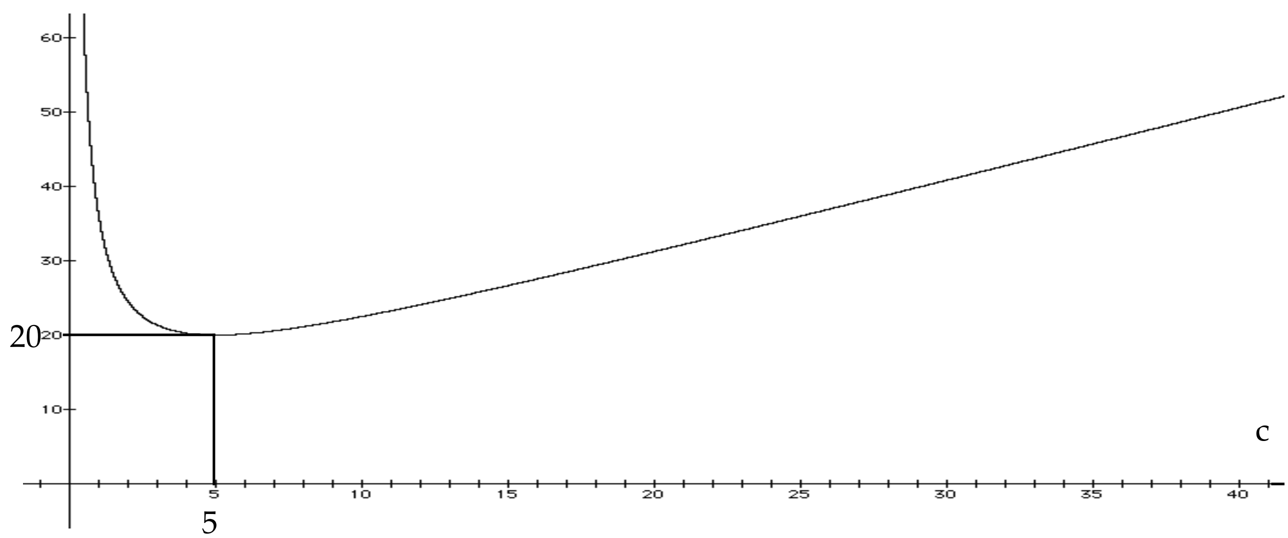


Graph of the area of the triangle as a function of the y-intercept b

Using the parameter c: Below is a graph of the function $g(c) = (5 \bullet 2) + \frac{1}{2} \bullet 2\,(c + \frac{5^2}{c})$.

Estimating from the graph we find that the minimum area is 20 sq units and occurs at the value c = 5 of the parameter c.

(You should convince yourself that c = 5 and b = 4 represent the same solution.)



Graph of the area of the triangle as a function of the parameter c

4b) The function f(b) from §4a can be analyzed using calculus. The derivative is

(4) $$f'(b) = \frac{5}{2}\,\frac{b^2 - 4b}{(b - 2)^2}$$

This derivative is 0 when b = 4. This verifies the approximation found from the graph. The minimum occurs precisely at b = 4.

The function g(c) from §4a can also be analyzed using calculus. This function is

iii) $$g(c) = (5 \bullet 2) + \frac{1}{2} \bullet 2\left(c + \frac{5^2}{c}\right)$$

The derivative of g(c) is somewhat easier to find than f'(b). It is

(6) $$g'(c) = 1 - \frac{5^2}{c^2}.$$

This equals 0 when c = 5, giving the same answer as in the analysis of the graph of g(c).

The same result can be found using calculus, with more or less ease, with any of the six parameters we have looked at. (The parameter θ would be the hardest to use.)
Note: In §10 we look at another way to find the minimum *without* using calculus.

We now have an answer to the original problem. The line through (5,2) cutting off minimal area has y-intercept b = 4. The area it cuts off is f(4) = 20. Here is a diagram to scale:

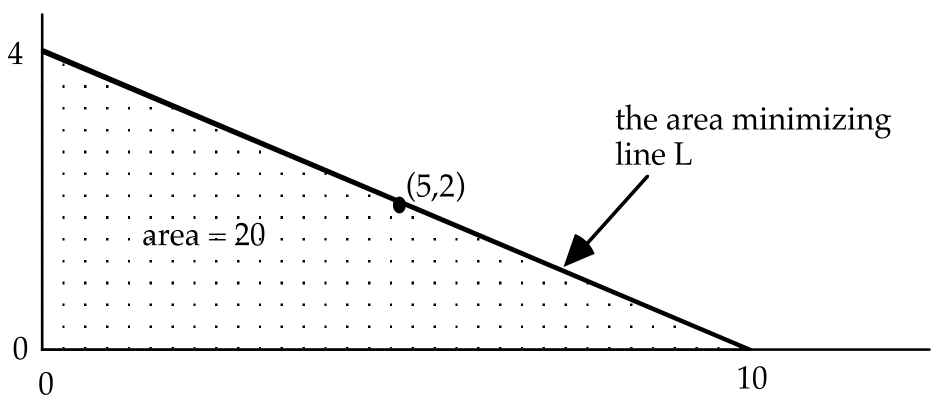


Diagram to scale of the area-minimizing line

4c) Notice that the rectangle with upper right vertex (5,2) is a fixed part of every triangle. Hence the only area that varies is that of the two small triangles. The sum of their areas is a function of c is a function we can call G(c):

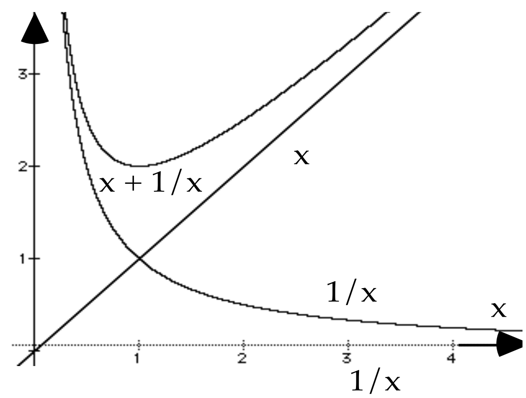
$$G(c) = \left(c + \frac{5^2}{c}\right) \qquad (7)$$

The function (7) is the sum of a variable x and a constant times its reciprocal 1/x.

So we have reduced the problem at hand to this simple question:
What is the minimum of a function of the form (8), where k is a constant?printed

$$x + \frac{k}{x} \qquad (8)$$

There is a simple argument showing that the minimum of this function occurs at the value of x where the graphs of x and k/x intersect. The argument uses the concavity of the graphs. These graphs intersect when $x = \sqrt{k}$. Hence (8) has a minimum where d = q. Alternatively, (7) can be easily differentiated to get the same result.

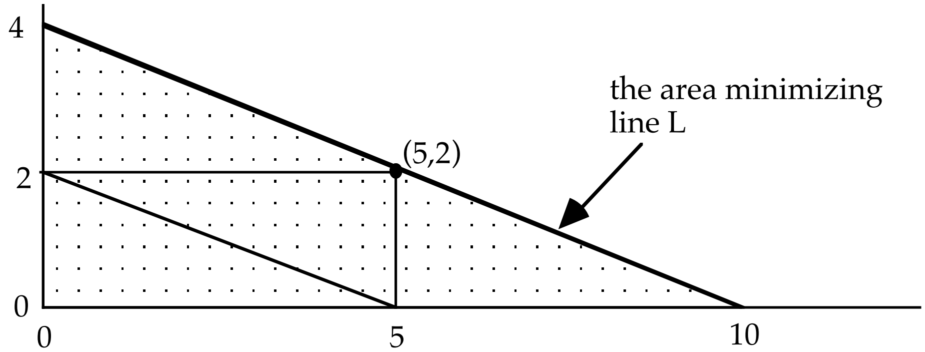


Graph of $x + \frac{k}{x}$

The purpose of these remarks is to show that by choosing parameters carefully, and by noticing that the functions that occur are common ones such as (8), the analysis can be shortened.

§5 discussion: Generalizing the initial result

5a) The figure in §4b shows the line through (5,2) that minimizes the area of the triangle. In the figure below we show the same line, but in a way that highlights its properties better:



5b) The area-minimizing line L is parallel to the diagonal of the rectangle which has the point (5,2) as its upper right vertex. The slope of that diagonal is $-\frac{5}{2}$. Hence, a reasonable conjecture generalizing this result to an arbitrary point (p,q) is:

(9) Of all lines through the point (p,q) in the first quadrant, the line that cuts off the triangle of smallest area has slope $-\frac{q}{p}$.

5c) We can verify this conjecture readily using calculus. Let us use the area function g(c) from §4b, generalized to a function we will call h(c) involving the general point (p,q):

(10) $$h(c) = (p \bullet q) + \frac{1}{2} \bullet q\left(c + \frac{p^2}{c}\right)$$

The area function (10) can be differentiated (with respect to *c*) as easily as f(b) or g(c) in §4b.

(11) $$h'(c) = q\left(1 - \frac{p^2}{c^2}\right)$$

This function has zero derivative when c = p. At this the area is 2pq. Further, the slope of the line is $-\frac{q}{p}$. This analysis shows immediately that the conjecture in §5b is correct. We can summarize the result so far:

(12) Of all lines through the point (p,q) in the first quadrant, the line that cuts off the triangle of smallest area has slope $-\frac{q}{p}$. The x intercept is 2p, the y-intercept is 2q, and the area is 2pq.

§6 discussion: Seeing this as purely a geometry problem

6a) At first this new problem seems very different from the original problem. The generalization in (12) is in terms of *slope* and doesn't seem to help. We seem to need to start from scratch.

However, if we look at Figure 2 and construct the parallelogram BFPG we see it is analogous to the rectangle of §5a. See Figure 2' below.

The solution to the original problem was that the line through (p,q) minimizing area was the line *parallel* to this diagonal. We have drawn this line segment, DPE, in Figure 2'. A natural conjecture is that this line segment DPE is also the area minimizing segment in the current situation.

Notice that there is a conspicuous geometrical property of this line segment DPE. Not only is it parallel to FG. More basically:

Point P is the *midpoint* of DPE.

(Show why this is true.)

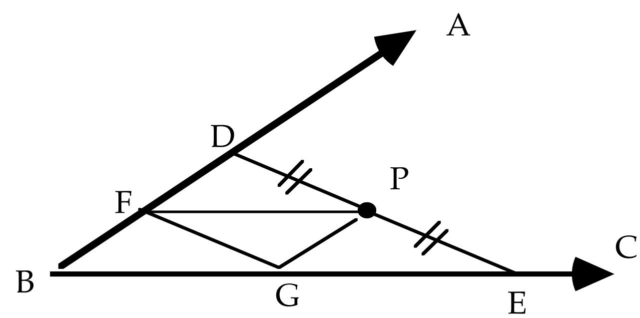


Figure 2'

Based on this reasoning, a reasonable conjecture is:

(13) **The line DE through a point P in an angle ABC that minimizes the area of triangle DPE is the line that has point P as its *bisector.***

6b) We now set out to prove the result (13) in terms of ideas from geometry alone, without using any of the tools of algebra, functions, graphing, or calculus.
Consider the diagram below. Line L is the line described in (13): it is bisected by point P. Consider any other line L' through P as shown. We will prove that L' cuts off more area than L. This will show that L cuts off less area than any line such as L'. (The line L' shown has greater slope than L. Below we consider the other case: a line with less slope than L.)

The method that we are using is a general and powerful technique for constructing proofs of results about optimization. The major step of the proof in this case is to show:

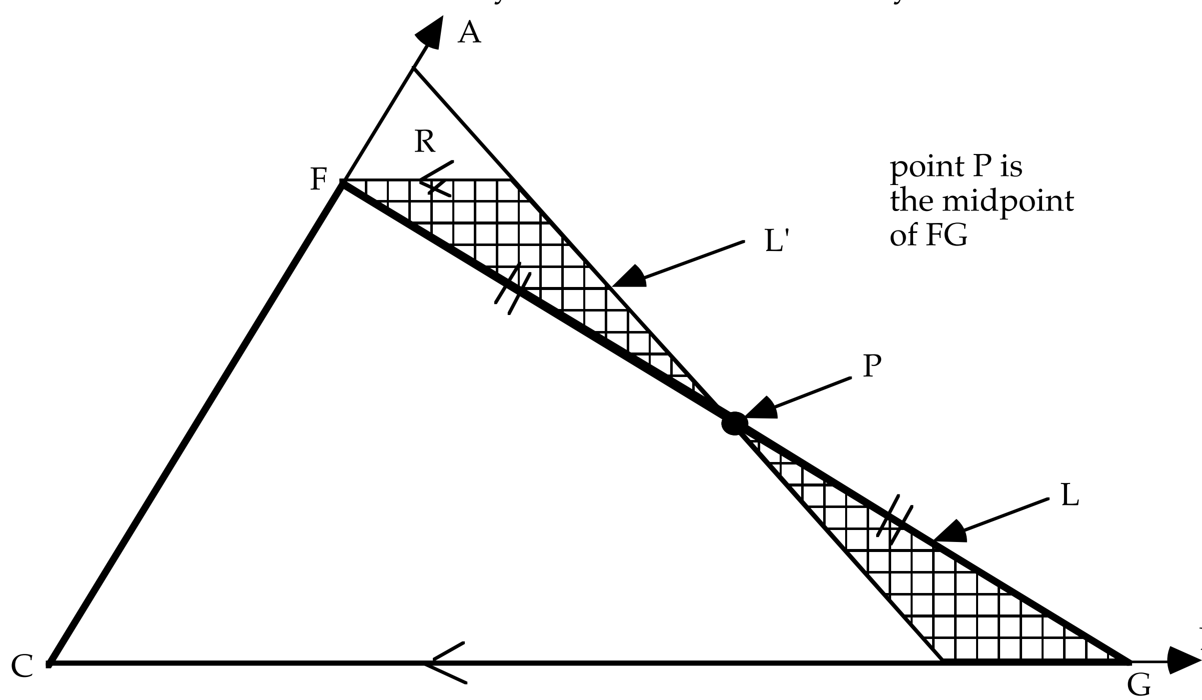
(14) The two triangles, FDP and GPE are congruent.

Exercise: Fill in the steps showing why (14) is true.

Note: This is a nice application of simple congruence theorems from high school geometry. For instance alternate angles are congruent as are vertical angles, etc.

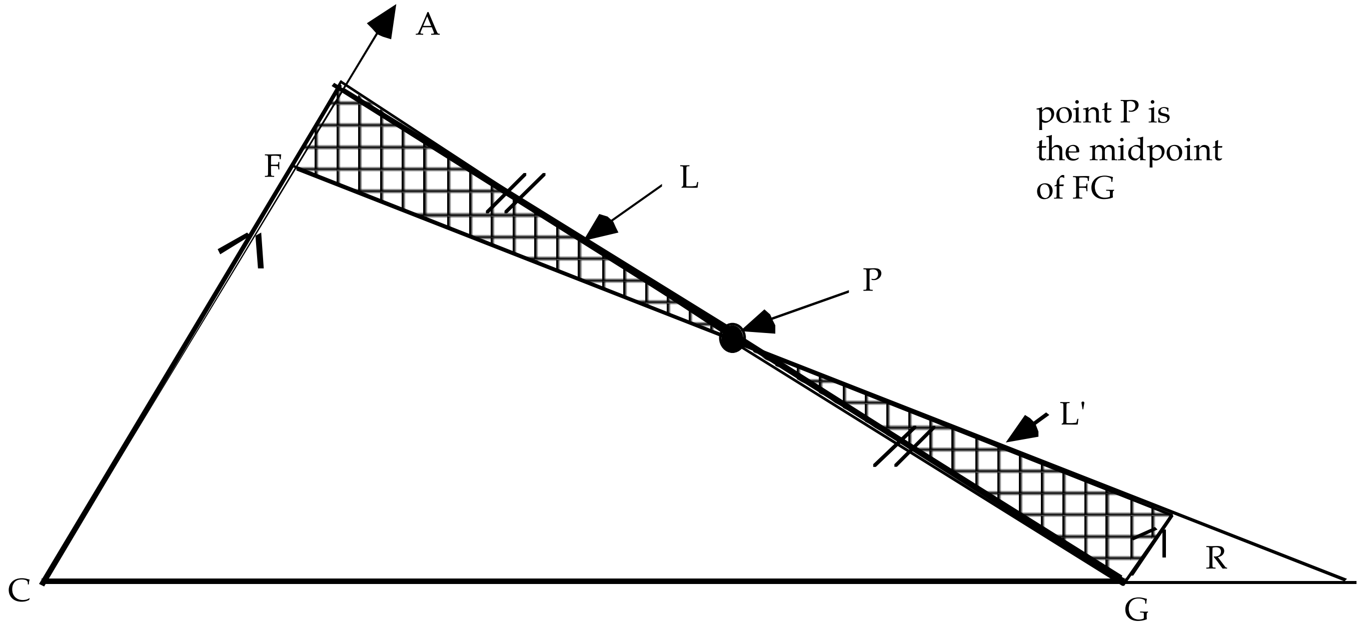
Thus we see that the area enclosed by line L' is greater than the area enclosed by line L. In fact, it is greater by exactly the area of region R.

This shows that the line L bisected by P cuts off less area than any other line such as L'.



Showing that the line L through P which P bisects is the line enclosing minimal area

The same argument holds if the line L' is oriented in the other way with respect to L. See the diagram below. The two shaded triangles are congruent. (Why?) Hence the area under line L’ is greater than the area under line L by the amount in region R.



Showing that the line L through P which P bisects is the line enclosing minimal area