Review Problems for Math 407

December, 2005

Algebra

In Problems 1–4, the solution to the equation depends on the constant a. Assuming a is positive, what is the effect of increasing a on the value of the solution? Does the solution increase, decrease, or remain unchanged? Give a reason for your answer that can be understood without solving the equation.

1. $x - a = 0$
2. $ax = 1$
3. $ax = a$
4. $\dfrac{x}{a} = 1$

5. If the tickets for a concert cost $\$p$ each, the number of people who will attend is $2500 - 80p$. Which of the following best describes the meaning of the 80 in this expression?
   (i) The price of an individual ticket.
   (ii) The slope of the graph of attendance against ticket price.
   (iii) The price at which no-one will go to the concert.
   (iv) The number of people who will decide not to go if the price is raised by one dollar.

6. A gas company charges residential customers $\$8$ per month even if they use no gas, plus 82¢ per therm used. (A therm is a quantity of gas.) In addition, the company is authorized to add a rate adjustment, or surcharge, per therm. The total cost of g therms of gas is given by the expression
$$\text{Total cost } = 8 + 0.82g + 0.109g.$$
   (a) Which term represents the rate adjustment? What is the rate adjustment in cents per therm?
   (b) Is the expression for the total cost linear?

7. A car trip costs $\$1.50$ per fifteen miles for gas and 30¢ per mile for other expenses, plus $\$20$ for car rental. The total cost for a trip of d miles is given by the expression
$$\text{Total cost } = 1.5\left(\frac{d}{15}\right) + 0.3d + 20.$$
   (a) Explain what each of the three terms in the expression represents in terms of the trip.
   (b) What units for cost and distance are being used?
   (c) Is the expression for cost linear?

8. A street vendor of t-shirts finds that if the price of a t-shirt is set at $\$p$, the revenue from a days sales is $p(900 - 60p)$. The best form of this expression for figuring what price to set is
   A. $p(900 - 60p)$
   B. $-60(p - 7.5)^2 + 3375$
   C. $-60p(p - 15)$
   D. $900p - 60p^2$

Without solving them, say whether the equations in Problems 9–16 have two solutions, one solution, or no solution. Give a reason for your answer.

9. $3(x - 3)(x + 2) = 0$
10. $(x - 2)(x - 2) = 0$
11. $(x+5)(x+5) = -10$
12. $(x + 2)^2 = 17$
13. $(x - 3)^2 = 0$
14. $3(x + 2)^2 + 5 = 1$
15. $-2(x - 1)^2 + 7 = 5$
16. $2(x - 3)^2 + 10 = 10$

In Problems 17–20, decide for what values of the constant A (if any) the equation has no solution. Give a reason for your answer.

17. $3(x - 2)^2 = A$
18. $(x - A)^2 = 10$
19. $A(x - 2)^2 + 5 = 0$
20. $5(x - 3)^2 + A = 10$

21. Match the statement (a)–(b) with the solutions to one or more of the equations (I)–(VI).

   I. $10(1.2)^t = 5$ II. $10 = 5(1.2)^t$
   III. $10 + 5(1.2)^t = 0$ IV. $5 + 10(1.2)^t = 0$
   V. $10(0.8)^t = 5$ VI. $5(0.8)^t = 10$

   (a) The time an exponentially growing quantity takes to grow from 5 to 10 grams.
   (b) The time an exponentially decaying quantity takes to drop from 10 to 5 grams.

22. Assume $0 < r < 1$. Without solving equations (I)–(IV) for x, decide which one has
   (a) The largest solution
   (b) The smallest solution
   (c) No solution

   I. $3(1 + r)^x = 7$ II. $3(1 + 2r)^x = 7$
   III. $3(1 + 0.01r)^x = 7$ IV. $3(1 - r)^x = 7$

Functions

In Problems 23–28, what is the exponent of the power function? Which of (I)–(IV) in Figure 1.1 best fits its graph? Assume all constants are positive.

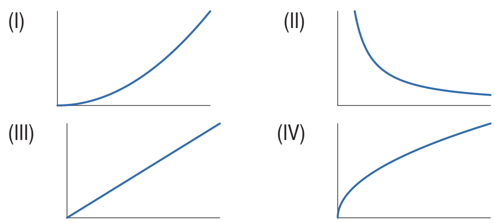

Figure 1.1

23. The number of species, N, on an island as a function of the area, A, of the island:

$$N = k\sqrt[3]{A}.$$

24. The surface area, S, of a mammal as a function of the body mass, B:

$$S = kB^{2/3}.$$

25. The number of animal species, N, of a certain body length as a function of the body length, L:

$$N = \frac{A}{L^2}.$$

26. The weight, W, of plaice (a type of fish) as a function of the length, L, of the fish:

$$W = \frac{a}{b} \cdot L^3.$$

27. The surface area, s, of a person with weight w and height h as a function of w, if height h is fixed:

$$s = 0.01w^{0.25}h^{0.75}.$$

28. The judged loudness, J, of a sound as a function of the actual loudness L:

$$J = aL^{0.3}.$$

29. Figure 1.2 gives the graph of the rational function

$$y = \frac{k(x-p)(x-q)}{(x-r)(x-s)}.$$

Based on the graph, find values of k, p, q, r, s given that $p < q$ and $r < s$.

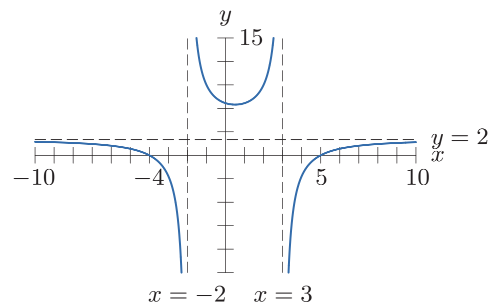

Figure 1.2

30. Formulas I–III all describe the growth of the same population, with time, t, in years: I. $P = 15(2)^{t/6}$
II. $P = 15(4)^{t/12}$ III. $P = 15(16)^{t/24}$
    (a) Show that the three formulas are equivalent.
    (b) What does formula I tell you about the doubling time of the population?
    (c) What do formulas II and III tell you about the growth of the population? Give answers similar to the statement which is the answer to part (b).

31. A container of ice cream is taken from the freezer and sits in a room for t minutes. Its temperature in degrees Fahrenheit is $a - b2^{-t} + b$, where a and b are positive constants. Write this expression in a form that
    (a) Shows that the temperature is always less than $a + b$
    (b) Shows that the temperature is always greater than a
    (c) What are reasonable values for a and b?

32. Let $R(x) = \dfrac{5}{(x-3)(x-2)}$.
    (a) Show that $R(x) = \dfrac{5}{x-3} - \dfrac{5}{x-2}$.
    (b) Which of the two forms of $R(x)$ helps to easily determine the zeros and the vertical asymptotes of $R(x)$? Find them.
    (c) If $\dfrac{k}{(x-a)(x-b)} = \dfrac{k}{x-a} - \dfrac{k}{x-b}$ is an identity in x, what can you say about k, a, and b?

Geometry and Proof

In Problems 33–36, the points (a, b) and (c, d) are on the unit circle $x^2 + y^2 = 1$.

33. Show that $(ac - bd, ad + bc)$ is on the unit circle.
34. Show that there is an angle θ such that $(a, b) = (\cos\theta, \sin\theta)$.
35. If $(a, b) = (\cos\theta, \sin\theta)$, then describe the relationship between (c, d) and $(ac - bd, ad + bc)$.
36. If $(a, b) = (\cos\theta, \sin\theta)$, then what is $(a^2 - b^2, 2ab)$?
37. True or false? Given triangle $\triangle ABC$ and $\triangle DEF$, if $AB = DE$, $BC = EF$, and $\angle ACB = \angle DFE$, then the two triangles are congruent. If false, give a counterexample.
38. Prove that if n is an odd integer, then $n^2 - 1$ is divisible by 8.
39. Prove that if the sum of the squares of two integers is divisible by 4, then both of the integers must be even.

In Problems 40–45, say whether the given matrix is a rotation matrix, and if it is, give the angle of rotation.

40. $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

41. $\begin{pmatrix} 1/4 & -3/4 \\ 3/4 & 1/4 \end{pmatrix}$

42. $\begin{pmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3} \end{pmatrix}$

43. $\begin{pmatrix} 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 \end{pmatrix}$

44. $\begin{pmatrix} 5/13 & -12/13 \\ 12/13 & 5/13 \end{pmatrix}$

45. $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

46. Define what it means for a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ to be linear, and give an example of both a linear and a nonlinear transformation.
47. Define what it means for a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ to be an isometry, and give an example of a linear transformation which is not an isometry.

In Problems 48–53, say whether the function is (a) linear (b) an isometry.

In each case, either justify your answer by showing that it satisfies the definition, or give a counterexample to the definition.

48. $f(x, y) = x + y$
49. $f(x, y) = (x + 1, y - 3)$
50. $f(x, y) = (x + y, x - y)$
51. $f(x, y) = (\alpha x - \beta y, \beta x + \alpha y)$, where $\alpha^2 + \beta^2 = 1$.
52. $f(x, y) = (y + 2, x - 3)$
53. $f(x, y) = (x^2 + y^2, 2xy)$

Trigonometry

54. Do the following problems from the textbook: page 439, nos. 6 and 7; page 443, nos. 1, 2, 6, and 7; page 462, nos. 4, 9, 10