

Great Issues of Our Time: The Quadratic Formula

William McCallum

Tucson Teachers' Circle, March 26, 2008

A proposal to eliminate quadratic equations



BBC NEWS WORLD EDITION

21 April 2003

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“pupils should be numerate . . . but numeracy can be divorced from mathematics. . . . How often do the majority of people need or use mathematical concepts once they have left school?”

[He advocated] allowing them to drop advanced concepts such as **quadratic equations** and trigonometry at the age of 14.

The proposal is debated in parliament



The UNITED KINGDOM PARLIAMENT

26 June 2003

Tony McWalter, Labour MP



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“A quadratic equation is not like a bleak room, devoid of furniture, in which one is asked to squat. It is a door to a room full of the unparalleled riches of human intellectual achievement. If you do not go through that door . . . much that passes for human wisdom will be forever denied you.”

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“Hear, hear”—Eleanor Laing, Conservative MP

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“Hear, hear”—Eleanor Laing, Conservative MP



“Oh dear. I would like to have support from elsewhere as well.”

The minister replies



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Alan Johnson, Minister





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- The 16-year-old thought that quadratic equations were logical and fairly straightforward because ‘you substitute stuff into a formula’. . . .



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- The 16-year-old thought that quadratic equations were logical and fairly straightforward because ‘you substitute stuff into a formula’. . . .
- The head of maths said that quadratic equations formed an important step in students’ ability to solve equations, . . .



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“ In preparing for this debate, the DFES conducted a straw poll involving a 16-year-old who had just sat maths GCSE, a head of maths and an experienced chemical engineer.”

- The 16-year-old thought that quadratic equations were logical and fairly straightforward because ‘you substitute stuff into a formula’. . . .
- The head of maths said that quadratic equations formed an important step in students’ ability to solve equations, . . .
- The engineer said that he did not use quadratic equations now, but had in the past . . . ”

Al-Khwarizmi, Hisab al-jabr w'al-muqabala, 830

... what is the square which combined with ten of its roots will give a sum total of 39? The manner of solving this type of equation is to take one-half of the roots just mentioned. ... Therefore take 5, which multiplied by itself gives 25, an amount which you add to 39 giving 64. Having taken then the square root of this which is 8, subtract from it half the roots, 5 leaving 3.

▶ Exercise



على تسعة وثلاثين قيم السطح الاصطناعي هو سطح ذو مربع
 ذلك انه مربع وسطي ماضيا جديدا وهو انما هو جرد
 اصناف الاصطناعي الاصطناعي بالاضافة منه مثل ما زادنا منه وهو
 خمسة هي تلكه وهو سطح اصطناعي آت القاي هو اثنان وهو جرد
 دليل تسعة وهذه هي



بدا ما اثنان واحد وستين هربعا يعادل عشرة اضعافه انا
 تبطل اثنان مضافا مريها سبعون الاصطناعي وهو سطح آت تم قسم
 اليه سطحا سداسي الاصطناعي مريها مثل احد السطح الاصطناعي آت وهو
 سطح من الاصطناعي آت فصار طول السطحين جميعا سطح ٣٥
 وقد علمنا ان طول عشرة من العدد من السطح مربع
 مضافا الاصطناعي الاصطناعي اثنان احد اقله مضافا الي واحد جرد
 فذلك السطح هو الذي جردوا فلما اثنان مثل واحد وستين
 يعادل عشرة اضعافه فلما ان طول سطح واحد عشرة اضعافه
 سطح واحد جرد اثنان مضافا سطح ٣٥ مضافا على التسعة

... *what is the square which combined with ten of its roots will give a sum total of 39?* The manner of solving this type of equation is to take one-half of the roots just mentioned. ... Therefore take 5, which multiplied by itself gives 25, an amount which you add to 39 giving 64. Having taken then the square root of this which is 8, subtract from it half the roots, 5 leaving 3.



على تسعة وثلاثين فيم المثلج المثلج الذي هو مثلج ذو مربع
 ذلكت انه واحد ومثلج واحدًا جبرًا وهو المثلج ذو مربع
 المثلج المثلج المثلج الذي هو مثلج ذو مربع
 خمسة في ثلثة وهو مثلج ذو المثلج هو ثلثة وهو جبر
 ولذا لسهة وهذه مبرهه



وبذا بال واحد وعشرون هربعا يعادل عشرة اعدادة ثلثة
 تبطل لاني مثلجًا مربعًا سهيلًا المثلج وهو مثلج ذو مربع
 اليه مثلجًا مربعًا المثلج مربعًا مثلج المثلج المثلج ذو
 المثلج ذو المثلج المثلج المثلج المثلج المثلج المثلج
 وقد مثلج ان ثلثة عشرة من العدد في ثلثة مثلج مربع
 مصلحي المثلج المثلج الذي هو المثلج المثلج المثلج
 فثلث المثلج ذو المثلج جبرًا لثلاثة مثلج واحد ومثلج
 يعادل عشرة اعدادة ثلثة ان ثلثة مثلج واحد عشرة اعدادة
 مثلج واحد جبرًا لثلاثة مثلج ذو ثلثة مثلج مثلج

$$x^2 + 10x = 39$$

... what is the square which combined with ten of its roots will give a sum total of 39? **The manner of solving this type of equation is to take one-half of the roots just mentioned. ... Therefore take 5, which multiplied by itself gives 25, an amount which you add to 39 giving 64. Having taken then the square root of this which is 8, subtract from it half the roots, 5 leaving 3.**



على تسعة وثلاثين قيم المثلث المثلث المثلث المثلث
 كانت تلك المثلث مربعة ماضيا جديدا وهو المثلث المثلث
 المثلث المثلث المثلث المثلث المثلث المثلث المثلث
 ماضيا هي تلك وهو ماضيا ماضيا ماضيا ماضيا
 وذلك تسعة وعشرون



وإذا كان واحد وعشرون ماضيا ماضيا ماضيا ماضيا
 متصل إلى ماضيا ماضيا ماضيا ماضيا ماضيا ماضيا
 إليه ماضيا ماضيا ماضيا ماضيا ماضيا ماضيا ماضيا
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Al-Khwarizmi, Hisab al-jabr w'al-muqabala, 830

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على تسعة وثلاثين قيم المثلج المثلج المثلج المثلج المثلج
كانت تلك أربعة وسبعين مائة واحد عشرة أجزاء مائة
التي هي المثلج المثلج المثلج المثلج المثلج المثلج المثلج
مقسمة على ثلثة وهو مبلغ مثلج آت القابل هو ثلثة وهو جدر
وذلك تسعة وعشرون مائة



إذا كان واحد وعشرون مائة واحد عشرة أجزاء مائة
تصل إلى مائة واحد وعشرون مائة واحد عشرة أجزاء مائة
إلى مائة واحد وعشرون مائة واحد عشرة أجزاء مائة
مبلغ من المثلج المثلج المثلج المثلج المثلج المثلج المثلج
وهو مائة من ثلثة مائة من ثلثة مائة من ثلثة مائة من ثلثة مائة
مقابل المثلج المثلج المثلج المثلج المثلج المثلج المثلج المثلج
فذلك المثلج المثلج المثلج المثلج المثلج المثلج المثلج المثلج
يصل مائة واحد وعشرون مائة واحد عشرة أجزاء مائة
مبلغ من المثلج المثلج المثلج المثلج المثلج المثلج المثلج المثلج

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على تسعة وثلاثين فيم المثلج المثلج الذي هو مثلج ذو مربع
كانت له أربعة وجوه فاحدا جدارها وهو المثلج وهو واحد
المثلج المثلج المثلج فاما من مثل ما زاد عليه وهو
مستطاب من ثلثة وهو مثلج آت القاب هو ثلث وهو جدار
وذلك تسعة وثلثة مربعه



وبما كان واحد وعشرون مربعها يبدل عشرة اجزاء فانا
نصل الى مثلج مربعها مربعة المثلج وهو مثلج آت تم قسم
اليه مثلج مربع المثلج مربعه مثل احد المثلج المثلج آت وهو
مثلج من المثلج آت فصار مثلج المثلج جميعه المثلج آت
وهو مثلج آت فيكون مثلج من المثلج من المثلج المثلج
مربعها المثلج المثلج فانا الى احد اقلها مربعة الى واحد جدار
فمثلج المثلج وله اثنين جدارها فاما ثلث مثلج واحد ومربعه
يبدل عشرة اجزاء فمثلج من ثلث مثلج واحد عشرة اجزاء
مثلج واحد جدارها المثلج المثلج فانا الى احد اقلها مربعة الى واحد

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على تسعة وثلاثين قيم السطح الاضلاع التي هو سطح زه ذليل
 ذلك انه واحد وسبعة وأصفا حذوا وهو اعماد وهو احد
 اشكال السطح الاضلاع فانا نقسمه على ما ذكرنا فله وهو
 خمسة من ثلثة وهو سطح آت الثاني هو الال وهو حذوا
 والال تسعة وهذه ميزان



وذا مال واحد وعشرون عريضا بعرض عشرة اخطار فانا
 تبصل الال سطحا مريحا سهول الاضلاع وهو سطح آت ثم نقسم
 الال سطحا مغربا الال مريحا مثل احد اضلاع سطح آت وهو
 سطح من السطح آت فصار طول السطحين جميعا سطح سه
 وقد قلنا ان طول عشرة من العدد من ثلثة سطح حذوا
 محاذين الاضلاع واخرها الال احد اطرافه مغربا الى واحد حذوا
 فلهذا السطح هو الصبي حذوا فانا مال واحد وستة
 يعدل عشرة اجزاء قلنا ان طول سطح الال عشرة اجزاء من
 سطح الال حذوا فانا نقسمه على خمسة فلهذا السطح على الثلثة

$$x^2 + 10x = 39$$

$$x^2 + 10x + 25 = 39 + 25 = 64$$

$$x + 5 = 8$$

$$x = 3$$

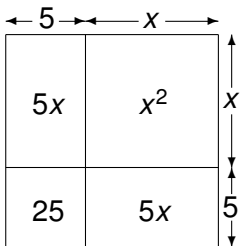
Al-Khwarizmi's geometric proof of his method

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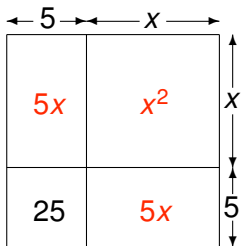
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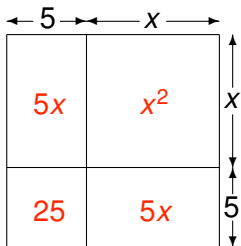
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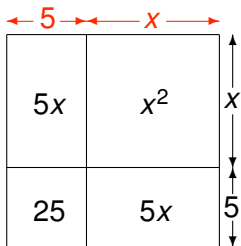
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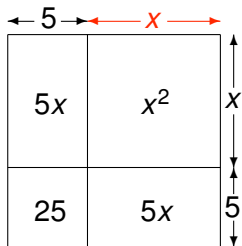
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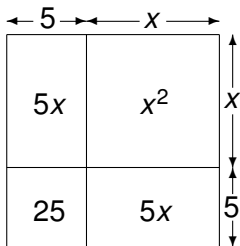
$$x = 3$$

▶ Exercise



Al-Khwarizmi's geometric proof of his method

$$\begin{aligned}x^2 + 10x &= 39 \\x^2 + 10x + 25 &= 39 + 25 = 64 \\x + 5 &= 8 \\x &= 3\end{aligned}$$



Exercise

Draw a diagram that illustrates the solution of the equation

$$x^2 = 39 + 10x.$$

▶ Answer

▶ Skip

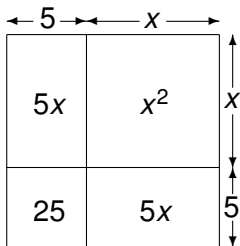
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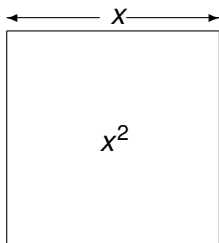
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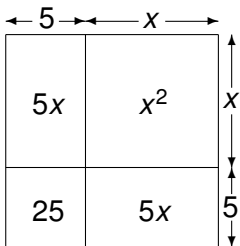


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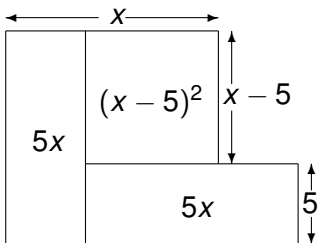


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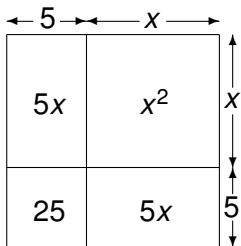
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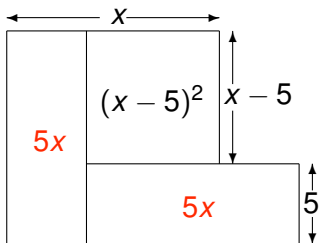
There is a better way.

Al-Khwarizmi's geometric proof of his method

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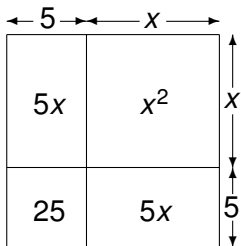
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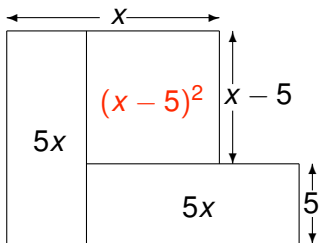
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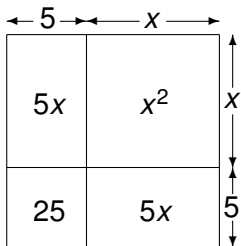
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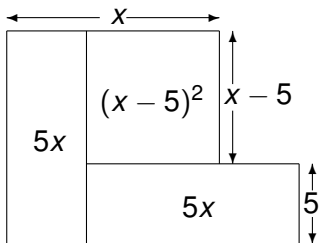
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Al-Khwarizmi's geometric proof of his method

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$$\begin{aligned}x^2 &= 39 + 10x \\x^2 + 25 &= 64 + 10x \\x - 5 &= 8 \\x &= 13\end{aligned}$$



There is a better way.

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- Problems were phrased in terms of areas, weights, money.

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$$\begin{aligned}x^2 + Cx &= N \\x^2 &= Cx + N\end{aligned}$$

where N and C are positive numbers.

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- So each equation must have a negative solution as well.
- Imaginary solutions were far in the future.

Completing the square the modern way

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$$\left(x + \frac{b}{2}\right)^2 = \frac{b^2 - 4c}{4}$$

$$x + \frac{b}{2} = \pm \frac{\sqrt{b^2 - 4c}}{2}$$

Completing the square the modern way

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$$\left(x + \frac{b}{2}\right)^2 = \frac{b^2 - 4c}{4}$$

$$x + \frac{b}{2} = \pm \frac{\sqrt{b^2 - 4c}}{2}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$$

Completing the square the modern way

$$ax^2 + bx + c = 0$$

$$x^2 + \frac{b}{a}x = \frac{-c}{a}$$

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \frac{-c}{a} + \left(\frac{b}{2a}\right)^2$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

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The quadratic formula

A number x satisfies

$$ax^2 + bx + c = 0, \quad a \neq 0,$$

if, and only if,

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

▶ Exercise

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Exercise

Give some simple conditions on the coefficients for a quadratic equation to have

- (a) two real roots
- (b) two positive roots.

▶ Answer

▶ Answer

▶ Examples

▶ Skip

The quadratic formula

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Answer

(a) Make a and c have opposite signs.

▶ Answer to (b)

▶ Examples

The quadratic formula

A number x satisfies

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Answer

- (a) Make a and c have opposite signs.
- (b) Make a and c have the same sign, and make b negative but large in magnitude.

▶ Examples

Another way of solving quadratic equations

For all numbers x ,

$$x^2 + 10x - 39 = (x - 3)(x + 13) = 0.$$

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The product of two numbers is zero if, and only if, one of them is zero, so either

$$x - 3 = 0 \quad \text{or} \quad x + 13 = 0.$$

That is, $x = 3$ or $x = -13$.

Viete's formulae

Theorem (Viete's formulae)

The numbers r and s are the solutions to

$$x^2 + bx + c = 0$$

if, and only if,

$$r + s = -b \quad \text{and} \quad rs = c.$$

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Proof of the "if" part.

If $r + s = -b$ and $rs = c$, then

$$x^2 + bx + c = (x - r)(x - s).$$

Then the argument goes as in the previous slide.

▶ Exercise



Viete's formulae

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Exercise

How do you prove the “only if” part? That is, if r and s are the solutions to $x^2 + bx + c$, then $r + s = -b$ and $rs = c$.

▶ Answer

▶ Skip

Viete's formulae

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The numbers r and s are the solutions to

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if, and only if,

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Proof of "only if" part

Divide $x - r$ into $x^2 + bx + c$, so

$$x^2 + bx + c = (x - r)q(x) + R.$$

Putting $x = r$ we get $R = 0$. Then $q(x) = x - t$ for some t , and the only possibility is $t = s$.

Viète's formulae and the quadratic formula

If

$$x^2 + bx + c = 0$$

then let

$$r = \frac{-b + \sqrt{b^2 - 4c}}{2} \quad \text{and} \quad s = \frac{-b - \sqrt{b^2 - 4c}}{2}$$

▶ Exercise

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Exercise

Give an explanation, purely in terms of the structure of the expressions, of why these two numbers satisfy

$$r + s = -b \quad \text{and} \quad rs = c.$$

▶ Answer

▶ Skip

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Answer

When you add r and s , the plus and minus signs cancel.

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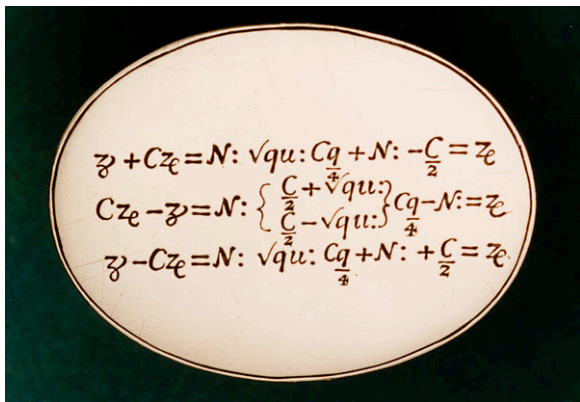
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Answer

When you add r and s , the plus and minus signs cancel.
When you multiply r and s , you get the difference of two squares in the numerator,

$$(-b)^2 - (\sqrt{b^2 - 4c})^2 = b^2 - (b^2 - 4c) = 4c.$$

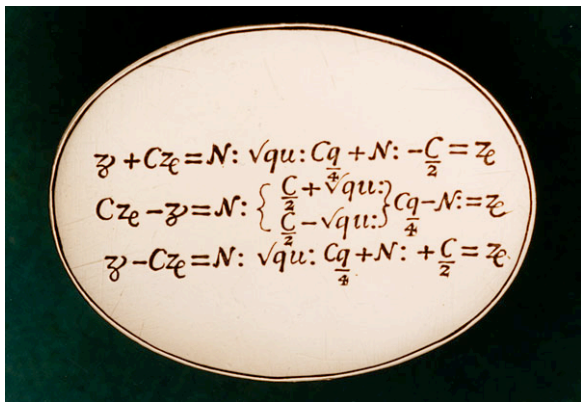
The quadratic formula in the 17th century



From the Oxford Museum of History of Science (Stephen Johnston, photo Bluebridge Farm Studio)

▶ Exercise

The quadratic formula in the 17th century



Exercise

What is going on here?

▶ Answer

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$$z + Cr = N : \sqrt{qu} : \frac{Cq}{4} + N : -\frac{C}{2} = r$$

$$Cr - z = N : \left\{ \begin{array}{l} \frac{C}{2} + \sqrt{qu} : \\ \frac{C}{2} - \sqrt{qu} : \end{array} \right\} \frac{Cq}{4} - N : = r$$

$$z - Cr = N : \sqrt{qu} : \frac{Cq}{4} + N : +\frac{C}{2} = r$$

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What is going on here?

$$x^2 + Cx = N, \quad \sqrt{\frac{C^2}{4} + N} - \frac{C}{2} = x$$

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$$x^2 - Cx = N, \quad \sqrt{\frac{C^2}{4} + N} + \frac{C}{2} = x$$

Example

If

$$x^2 + 10x - 39 = 0,$$

then

$$x = \frac{-10 \pm \sqrt{256}}{2} = -5 \pm \sqrt{64} = 3, -13.$$

Example

If

$$x^2 - 10x + 9 = 0$$

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[◀ Back to exercise](#)

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