FOURIER–JACOBI PERIODS OF CLASSICAL SAITO–KUROKAWA LIFTS

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Abstract. Kohnen–Skuroppa [8] proved a formula for the ratio of the Petersson inner products of the half integral weight modular form and its Saito–Kurokawa lifting. We give an interpretation of this formula in the frame work of the refined Gan–Gross–Prasad conjecture. This provides us with an example of the refined Gan–Gross–Prasad conjecture for the nontempered representations.

Contents

1. Introduction

Let $\mathcal{H}$ be the upper half plane and k be an odd integer. Let $f \in S_{2k}(\Gamma_0(1))$ be a normalized Hecke eigenform of weight $2k$ and level one. Let $h \in S^+_{k+\frac{1}{2}}(\Gamma_0(4))$ be the Hecke eigenform of weight $k+\frac{1}{2}$ associated to f by the Shimura correspondences. Let $\mathcal{H}_2$ be the Siegel upper half space of genus two, i.e. the space of symmetric 2×2 matrices whose imaginary part is positive definite. Let $F \in S_{k+1}(\mathrm{Sp}_4(\mathbb{Z}))$ be the Saito–Kurokawa lifting of h.

Put

$$\langle h, h\rangle = \frac{1}{6}\int_{\Gamma_0(4)\backslash\mathcal{H}} |h(\tau)|^2 y^{k+\frac{1}{2}}\frac{\mathrm{d}x\mathrm{d}y}{y^2}, \quad \langle F, F\rangle = \int_{\mathrm{Sp}_4(\mathbb{Z})\backslash\mathcal{H}_2} |F(Z)|^2(\det \Im Z)^{k-2}\mathrm{d}Z.$$

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Let $\Gamma_{\mathbb{R}}(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})$ and $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$. Then Kohnen–Skoruppa [8] proved that (see also [6, Section 9] for some comments)

$$\frac{\langle F, F\rangle}{\langle h, h\rangle} = 2^{k-2}\pi^{-1}\xi(2)\Lambda(k+1, f), \tag{1.1}$$

where $\xi(s) = \zeta(s)\Gamma_{\mathbb{R}}(s)$ is the completed Riemann zeta function and $\Lambda(s, f) = L(s, f)\Gamma_{\mathbb{C}}(s)$ is the completed L-function of f. The goal of this note is to give an interpretation of (1.1) in the frame work of the refined Gan–Gross–Prasad conjecture for $\mathrm{Sp}_4 \times \widetilde{\mathrm{SL}}_2$.

To motivate the following discussion, we briefly recall the refined Gan–Gross–Prasad conjecture. We refer the readers to Section 2 for the precise formulation. Let π be an irreducible cuspidal automorphic representation of $\mathrm{Sp}_4(\mathbb{A}_{\mathbb{Q}})$ and σ be an irreducible cuspidal genuine automorphic representation of $\widetilde{\mathrm{SL}}_2(\mathbb{A}_{\mathbb{Q}})$. Here by genuine, we mean that σ is not the pullback of an automorphic representation of $\mathrm{SL}_2(\mathbb{A}_{\mathbb{Q}})$. Fix a nontrivial additive character $\psi : \mathbb{Q}\backslash\mathbb{A}_{\mathbb{Q}} \to \mathbb{C}^\times$. Let $F = \otimes F_v \in \pi$, $h = \otimes h_v \in \sigma$ be factorizable automorphic forms. Let $\phi = \otimes\phi_v \in \mathcal{S}(\mathbb{A}_{\mathbb{Q}})$ be a Schwartz function on $\mathbb{A}_{\mathbb{Q}}$. The refined Gan–Gross–Prasad conjecture proposed in [14], predicts that if π and σ are both tempered, then under suitable normalization, we have

$$\frac{|\mathcal{FJ}_\psi(F, h, \phi)|^2}{\langle F, F\rangle\langle h, h\rangle\langle \phi, \phi\rangle} = 2^{-\beta}\xi(2)^2\xi(4)\frac{L_\psi(\frac{1}{2}, \pi\times\sigma)}{L(1, \pi, \mathrm{Ad})L_\psi(1, \sigma, \mathrm{Ad})}\prod_v \alpha_v^\natural(F_v, h_v, \phi_v), \tag{1.2}$$

where

- $\langle -, -\rangle$ stands for the Petersson inner product;
- $\mathcal{FJ}_\psi(F, h, \phi)$ is the Fourier–Jacobi period of F, h and ϕ;
- $\alpha_v^\natural$ is a local linear form defined by integration of matrix coefficients;
- β is a positive integer such that $2^\beta = |S_\pi||S_\sigma|$ where S_π and S_σ are centralizer of the L-parameters of π and σ respectively.

This conjecture is motivated by the existing conjectures of this type [4, 6, 9]. The most subtle part of the conjecture is the integer β. It predicts strong connection between period integrals, special value of L-functions and the theory of endoscopy. The conjecture is proposed under the hypothesis that π and σ are both *tempered*. Therefore a natural question is:

Question 1.1. *What if π and σ are not tempered?*

At this moment, no precise conjecture has been formulated for the nontempered representations. The major problems are the following.

(1) It is not clear how to define the local linear forms $\alpha_v^\natural$ if π_v or σ_v is not tempered as the defining integral of $\alpha_v^\natural$ in the tempered case is not convergent when π_v or σ_v are not tempered. As in [6, Section 3], we expect that there is some way to do "analytically continuation", so that the local terms $\alpha_v^\natural$ are well-defined for all representations. This

problem has an analytic nature and usually can be solved based on the concrete examples at hand.

(2) With a suitable definition of $\alpha_v^\natural$, we expect that the same identity (1.2) should hold, but in general, the integer β might not measure the size of the centralizer. It should contain information on the global endoscopy, as well as some local data attached to π and σ. We do not have an explanation of this integer β yet. This problem is much more serious. One of the main goals of this note is to compute the integer β in the case π is the Saito–Kurokawa lifting as described at the beginning of the Introduction.

Ichino–Ikeda [6] provides several examples of the refined Gan–Gross–Prasad conjecture for nontempred representations of $\mathrm{SO}(n+1) \times \mathrm{SO}(n)$. Our main theorem (Theorem 4.6), which gives an interpretation of (1.1), shows that the identity (1.2) holds with $\beta = 2$ if F is the Saito–Kurokawa lifting as described at the beginning of the Introduction. We note that in the case $2^\beta = 4$ while $|S_\pi| = 4$ and $|S_\sigma| = 2$, $2^\beta \neq |S_\pi||S_\sigma|$.

We list here the main ingredients of this paper.

– Choosing correct F, h and ϕ in (1.2). In classical language, the result of [15] already shows how the left hand side of (1.1) is related to the Fourier–Jacobi periods. Our work is to carefully interprete several constructions in [15] adelically, in particular the correspondence between the Jacobi forms of level one and half integral weight modular forms of level $\Gamma_0(4)$. This is done in Section 3. Once we have chosen F, h and ϕ, their local components F_v, h_v and ϕ_v are then fixed.

– Computing the local linear form $\alpha_\infty^\natural(F_\infty, h_\infty, \phi_\infty)$ for the above chosen F_∞, h_∞ and ϕ_∞. This is done in Section 5. It is the major innovation of this note. To the best of the author's knowledge, this is the first time that the explicit computation of this type has been done when the groups involved are not reductive. This computation might be of some independent interest. Therefore Section 5 is written in a self-contained way so that it can be read independently from the other sections.

– Defining and computing the term $\alpha_v^\natural(F_v, h_v, \phi_v)$ when $v \neq \infty$. The representation π_v is not tempered and regularization of the defining integral of α_v is needed. Our situation is particularly simple as the representations are unramified. Heuristically, our method of regularization can be described as "analytic continuation along the Satake parameters". The definition of $\alpha_v^\natural$ is proposed in Section 4. Most of the computation has been done in [14, Section 4]. In the case $v = 2$, some additional work is needed. This is contained in Section 6.

Remark 1.2. Strictly speaking, the statement of our main theorem does not require the actual regularization of $\alpha_v^\natural$ for all F_v, h_v, ϕ_v, but just for our specific choices of F_v, h_v, ϕ_v which correspond to the local components of the automorphic forms. The only point is that $\alpha_v^\natural(F_v, h_v, \phi_v)$ is analytic with respect to the Satake parameters if F_v, h_v, ϕ_v vary in flat sections. For our choices of F_v, h_v, ϕ_v, this is clear as it is a constant functions, c.f. Proposition 4.3.

Remark 1.3. Our regularization of $\alpha_v^\natural$ works only for the unramified representations. Nevertheless, it clearly generalizes to other groups that appear in the refined Gan–Gross–Prasad conjecture. In particular, with this regularization, the refined Gan–Gross–Prasad conjecture can be formulated for those cuspidal automorphic representations whose local components are either unramified or tempered.

Remark 1.4. There are two directions to extend the results of this note. First, one may consider Saito–Kurokawa liftings of arbitrary level. In this situation, it is possible to regularize the local terms $\alpha_v^\natural$ by introducing certain "height functions" in the defining integral of $\alpha_v^\natural$. This technique of regularization has been used in [11] in a different situation. In addition, by interpreting the Saito–Kurokawa lifting as suitable theta liftings, the formula by Kohnen–Skoruppa can be deduced from certain inner product formula for theta liftings. Second, Ikeda and Yamana [7] recently generalize the classical Saito–Kurokawa liftings to the case of Hilbert modular forms (of level one). Again by a suitable interpretation of their results in term of theta liftings, it is possible to extend our results to the case of Hilbert modular forms.

Remark 1.5. Let τ be the automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ generated by f. Our main theorem is an example of the refined Gan–Gross–Prasad conjecture (1.2) in the case that π (resp. σ) is the Saito–Kurokawa lifting of τ (resp. Shimura correspondence of τ). If σ is not the Shimura correspondence of τ, then it follows from [15] that the period integral $\mathcal{FJ}_\psi(F, h, \phi)$ vanishes identically. We will not use this fact in the paper.

Notation. Let X be a set and $Y \subset X$ be a subset, then we denote by $\mathbf{1}_Y$ the characteristic function of Y.

We fix an additive character $\psi = \otimes\psi_v : \mathbb{Q}\backslash\mathbb{A}_\mathbb{Q} \to \mathbb{C}^\times$ as follows: if $v = \infty$, then $\psi_\infty(x) = e^{2\pi\sqrt{-1}x}$ for $x \in \mathbb{R}$; if $v = p$, then $\psi_p(x) = e^{-2\pi\sqrt{-1}x}$ for $x \in \mathbb{Z}[p^{-1}]$. For any $a \in \mathbb{Q}^\times$, we define ψ_a to be the additive character of $\mathbb{Q}\backslash\mathbb{A}_\mathbb{Q}$ given by $\psi_a(x) = \psi(ax)$. For any place v and $a \in \mathbb{Q}_v^\times$, we define $\psi_{v,a}$ in a similar way.

For any place v of $\mathbb{Q}$, we denote by $(-,-)_v$ the Hilbert symbol on $\mathbb{Q}_v^\times$. Let $\gamma_{\mathbb{Q}_v}(\psi_v)$ is the Weil index which is an eighth root of unity. Put $\gamma_{\mathbb{Q}_v}(a, \psi_v) = \gamma_{\mathbb{Q}_v}(\psi_{v,a})\gamma_{\mathbb{Q}_v}(\psi_v)^{-1}$. We refer the readers to [5, Appendix A.1] for explicit formulae of $\gamma_{\mathbb{Q}_v}(\psi_v)$ and $\gamma_{\mathbb{Q}_v}(a, \psi_v)$.

We put $\Gamma_\mathbb{R}(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})$ and $\Gamma_\mathbb{C}(s) = 2(2\pi)^{-s}\Gamma(s)$. Let $\xi(s) = \zeta(s)\Gamma_\mathbb{R}(s)$ be the completed Riemann zeta function.

Let $1_r \in \mathrm{GL}_r$ be the $r \times r$ identity matrix.

We let $B = MN \subset \mathrm{SL}_2$ the upper triangular Borel subgroup of SL_2 where M is the diagonal torus and N is the unipotent radical. Let

$$\mathrm{SO}_2(\mathbb{R}) = \left\{ k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \;\middle|\; \theta \in \mathbb{R}/2\pi\mathbb{Z} \right\}.$$

It is a maximal compact subgroup of $\mathrm{SL}_2(\mathbb{R})$. We write

$$n(x) = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \in N, \; m(y) = \begin{pmatrix} y & \\ & y{-1} \end{pmatrix} \in M, \; w = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}.$$

Let

$$\mathrm{Sp}_4 = \left\{ g \in \mathrm{GL}_4 \;\middle|\; {}^t g \begin{pmatrix} & 1_2 \\ -1_2 & \end{pmatrix} g = \begin{pmatrix} & 1_2 \\ -1_2 & \end{pmatrix} \right\}.$$

The subgroup R consist of elements of the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (\lambda, \mu, \kappa) = \begin{pmatrix} a & & b & \\ & 1 & & \\ c & & d & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \mu \\ \lambda & 1 & \mu & \kappa \\ & & 1 & -\lambda \\ & & & 1 \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2 .$$

We let $H = \{(\lambda, \mu, \kappa)\} \subset R$, $Z = \{(0, 0, \kappa)\} \subset H$. Let B_R be the subgroup of R consisting of elements of the form

$$\begin{pmatrix} 1 & & & * \\ & 1 & * & * \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} * & & * & \\ & 1 & & \\ & & * & \\ & & & 1 \end{pmatrix}$$

and N_R be its unipotent radical. The group B_R is called the Borel subgroup of R. We have $B_R \simeq MN_R$.

Let

$$\mathrm{U}_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \;\middle|\; a, b \in \mathbb{C} \right\}.$$

It is a maximum compact subgroup of $\mathrm{Sp}_4(\mathbb{R})$:

$$\left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \;\middle|\; A + \sqrt{-1}B \in \mathrm{U}_2(\mathbb{R}) \right\}.$$

We often consider SL_2 as a subgroup of Sp_4 consisting of element of the form

$$\begin{pmatrix} * & & * & \\ & 1 & & \\ * & & * & \\ & & & 1 \end{pmatrix},$$

and view elements and subgroups of SL_2 as those of Sp_4.

Let v be a place of $\mathbb{Q}$. We denote by $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_v) = \mathrm{SL}_2(\mathbb{Q}_v) \ltimes \{\pm 1\}$ the metaplectic double cover of $\mathrm{SL}_2(\mathbb{Q}_v)$. We usually write an element in $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_v)$ as (g, ϵ), where $g \in \mathrm{SL}_2(\mathbb{Q}_v)$ and $\epsilon = \pm 1$. The map $N(\mathbb{Q}_v) \to \widetilde{\mathrm{SL}}_2(\mathbb{Q}_v)$, $n \to (n, 1)$ is a homomorphism. In general, $g \mapsto (g, 1)$ is not a homomorphism. By abuse of notation, we usually write g for the element $(g, 1) \in \widetilde{\mathrm{SL}}_2(\mathbb{Q}_v)$.

Let $\widetilde{M}(\mathbb{Q}_v)$ be the inverse image of $M(\mathbb{Q}_v)$ in $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_v)$. Then $\widetilde{M}(\mathbb{Q}_v) \simeq \mathbb{Q}_v^\times \ltimes \{\pm 1\}$ with the multiplication given by $(m_1, \epsilon_1)(m_2, \epsilon_2) = (m_1 m_2, \epsilon_1 \epsilon_2 (m_1, m_2)_v)$. We define a genuine character of $\widetilde{M}(\mathbb{Q}_v)$ by

$$\chi_{\psi_v}(m, \epsilon) = \epsilon \gamma_{\mathbb{Q}_v}(m, \psi_v)^{-1}.$$

Let $\widetilde{B}(\mathbb{Q}_v)$ be the inverse image of $B(\mathbb{Q}_v)$ in $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_v)$. Let χ_v be a character of $\mathbb{Q}_v^\times$ (not necessarily unitary). We define the induced representation

$$\mathrm{Ind}_{\widetilde{B}(\mathbb{Q}_v)}^{\widetilde{\mathrm{SL}}_2(\mathbb{Q}_v)} \chi_v = \{f : \widetilde{\mathrm{SL}}_2(\mathbb{Q}_v) \to \mathbb{C} \mid f((m(a), \epsilon) n(b) g) = \epsilon \chi_{\psi_v}(a) \chi_v(a) |a| f(g)\},$$

where $a \in \mathbb{Q}_v^\times$, $b \in \mathbb{Q}_v$ and $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_v)$ acts by right multiplication.

Let

$$K_0(4, \mathbb{Z}_p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, c \equiv 0 \bmod 4\mathbb{Z}_p \right\};$$

$$K_1(4, \mathbb{Z}_p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, c \equiv 0 \bmod 4\mathbb{Z}_p, \; d \equiv 1 \bmod 4\mathbb{Z}_p \right\}.$$

If $p \neq 2$, then $K_0(4, \mathbb{Z}_p) = K_1(4, \mathbb{Z}_p) = \mathrm{SL}_2(\mathbb{Z}_p)$. There is a homomorphism $K_1(4, \mathbb{Z}_p) \to \widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$ which splits the metaplectic double cover. We refer the readers to [5] for an explicit description of the splitting.

Let $\widetilde{\mathrm{SO}}_2(\mathbb{R})$ be the inverse image of $\mathrm{SO}_2(\mathbb{R})$ in $\widetilde{\mathrm{SL}}_2(\mathbb{R})$. Then

$$\mathbb{R}/4\pi\mathbb{R} \to \widetilde{\mathrm{SO}}_2(\mathbb{R}), \quad \theta \mapsto \widetilde{k}_\theta = (k_\theta, \epsilon), \quad \epsilon = \begin{cases} 1, & -\pi < \theta \leq \pi \\ -1, & \pi < \theta \leq 3\pi \end{cases}$$

is an isomorphism.

Let $\widetilde{\mathrm{SL}}_2(\mathbb{A}_\mathbb{Q})$ be the metaplectic double cover of $\mathrm{SL}_2(\mathbb{A}_\mathbb{Q})$. Then $\mathrm{SL}_2(\mathbb{Q})$ canonically splits the metaplectic double cover and we identify $\mathrm{SL}_2(\mathbb{Q})$ with its image in $\widetilde{\mathrm{SL}}_2(\mathbb{A}_\mathbb{Q})$. Let $K_1(4, \widehat{\mathbb{Z}}) = \prod_{p<\infty} K_1(4, \mathbb{Z}_p)$. The metaplectic double cover also splits canonically over $K_1(4, \widehat{\mathbb{Z}})$ and $N(\mathbb{A}_\mathbb{Q})$.

Measures. We let $\mathcal{H}$ be the upper half plane and give it the usual measure $y^{-2} \mathrm{d}x \mathrm{d}y$. We let $\mathcal{H}_2$ be the Siegel upper half plane of genus two consisting of 2×2 matrices whose imaginary part is positive definite and give it the measure $|F(Z)|^2 (\det \Im Z)^{-2} \mathrm{d}Z$.

For any unipotent group N, we give $N(\mathbb{A}_\mathbb{Q})$ the self-dual measure with respect to ψ.

On $\mathrm{SL}_2(\mathbb{Q}_p)$, let $\mathrm{d}g_p$ be the measure so that the volume of $\mathrm{SL}_2(\mathbb{Z}_p)$ equals one. On $\mathrm{SL}_2(\mathbb{R})$, let $\mathrm{d}g_\infty = y^{-2} \mathrm{d}x \mathrm{d}y \mathrm{d}k_\infty$, where $g_\infty = n(x) m(y) k_\infty$ is the Iwasawa decomposition, and $\mathrm{d}k_\infty$ is the measure on $\mathrm{SO}_2(\mathbb{R})$ so that the volume of it equals one. Then $\xi(2)^{-1}\prod \mathrm{d}g_v$ is the Tamagawa measure on $\mathrm{SL}_2(\mathbb{A}_\mathbb{Q})$.

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2. Refined Gan–Gross–Prasad conjecture

In this section, we review the refined Gan–Gross–Prasad conjecture for $\mathrm{Sp}_4 \times \widetilde{\mathrm{SL}}_2$.

Let ω_ψ be the oscillating representation of $\widetilde{R}(\mathbb{A}_\mathbb{Q}) = \widetilde{\mathrm{SL}}_2(\mathbb{A}_\mathbb{Q}) \ltimes H(\mathbb{A}_\mathbb{Q})$. It is realized on the Schwartz space $\mathcal{S}(\mathbb{A}_\mathbb{Q})$ and is given by the following formulae.

(1) $\omega_\psi((\lambda,\mu,\kappa))\phi(x) = \psi(\kappa + (2x+\lambda)\mu)\phi(x+\lambda)$;
(2) $\omega_\psi(m(a))\phi(x) = |a|^{\frac{1}{2}}\phi(xa)$;
(3) $\omega_\psi(n(b))\phi(x) = \phi(x)\psi(bx^2)$;
(4) $\omega_\psi(w)\phi(x) = \int_{\mathbb{A}_\mathbb{Q}} \phi(y)\psi(2xy)\mathrm{d}y$;
(5) $\omega_\psi(1,\epsilon)\phi(x) = \epsilon\phi(x)$,

where $\mathrm{d}y$ is the self-dual additive measure on $\mathbb{A}_\mathbb{Q}$ for the character ψ_2. We define the theta function

$$\Theta(r,\phi) = \sum_{x\in\mathbb{Q}} \omega_\psi(r)\phi(x),\ \phi \in \mathcal{S}(\mathbb{A}_\mathbb{Q}),\ r \in \widetilde{R}(\mathbb{A}_\mathbb{Q}).$$

There is a canonical inner product on ω_ψ given by

$$\langle \phi, \phi' \rangle = \int_{\mathbb{A}_\mathbb{Q}} \phi(x)\overline{\phi'(x)}\mathrm{d}x.$$

Let v be a place of $\mathbb{Q}$. We denote by ω_{ψ_v} the oscillator representation of $\widetilde{R}(\mathbb{Q}_v)$ that is realized on the Schwartz space $\mathcal{S}(\mathbb{Q}_v)$. It is given by the analogous formulae as the global oscillator representation, c.f. [5, Section 4]. There is a canonical inner products on ω_{ψ_v} given by

$$\langle \phi_v, \phi'_v \rangle_v = \int_{\mathbb{Q}_v} \phi_v(x)\overline{\phi'_v(x)}\mathrm{d}x.$$

Then $\langle -,- \rangle = \prod_v \langle -,- \rangle_v$.

Let π be an irreducible cuspidal automorphic representation of $\mathrm{Sp}_4(\mathbb{A}_\mathbb{Q})$ and σ be an irreducible cuspidal genuine automorphic representation of $\widetilde{\mathrm{SL}}_2(\mathbb{A}_\mathbb{Q})$. Let $\langle -,- \rangle$ be the Petersson inner product on π given by

$$\langle F, F' \rangle = \int_{\mathrm{Sp}_4(\mathbb{Q})\backslash \mathrm{Sp}_4(\mathbb{A}_\mathbb{Q})} F(g)\overline{F'(g)}\mathrm{d}g.$$

We fix an inner product $\langle -,- \rangle_v$ on π_v for each v. Similarly let $\langle -,- \rangle$ be the Petersson inner product on σ given by

$$\langle h, h' \rangle = \int_{\mathrm{SL}_2(\mathbb{Q})\backslash \mathrm{SL}_2(\mathbb{A}_\mathbb{Q})} h(g)\overline{h'(g)}\mathrm{d}g.$$

We fix an inner product $\langle -,-\rangle_v$ on σ_v for each v.

Let $F = \otimes F_v \in \pi$, $h = \otimes h_v \in \sigma$ be factorizable automorphic forms. Let $\phi = \otimes \phi_v \in \mathcal{S}(\mathbb{A}_\mathbb{Q})$ be a Schwartz function. We define the global Fourier–Jacobi period of (F, h, ϕ) by

$$\mathcal{FJ}_\psi(F,h,\phi) = \int_{\mathrm{SL}_2(\mathbb{Q})\backslash \mathrm{SL}_2(\mathbb{A}_\mathbb{Q})} \int_{H(\mathbb{Q})\backslash H(\mathbb{A}_\mathbb{Q})} F(ng)h(g)\overline{\Theta_\psi(ng,\phi)}\mathrm{d}n\mathrm{d}g.$$

We define the local linear form for each place v by

$$\alpha_v(F_v,h_v,\phi_v) = \int_{\mathrm{SL}_2(\mathbb{Q}_v)} \int_{H(\mathbb{Q}_v)} \langle \pi_v(ng)F_v, F_v\rangle \langle \sigma_v(g)h_v, h_v\rangle \overline{\langle \omega_{\psi_v}(ng)\phi_v, \phi_v\rangle}\mathrm{d}n\mathrm{d}g.$$

The following has been proved in [14].

(1) If π_v and σ_v are both tempered, then the defining integral of α_v is absolutely convergent.

(2) Suppose that $v \neq 2$ or ∞, F_v is $\mathrm{Sp}_4(\mathbb{Z}_2)$-fixed, h_v is $\mathrm{SL}_2(\mathbb{Z}_v)$-fixed, $\phi_v = \mathbf{1}_{\mathbb{Z}_v}$. If the defining integral of α_v is convergent, then

$$\frac{\alpha_v(F_v,h_v,\phi_v)}{\langle F_v,F_v\rangle \langle h_v,h_v\rangle \langle \phi_v,\phi_v\rangle} = \xi_v(2)\xi_v(4)\frac{L_{\psi_v}(\frac{1}{2},\pi_v\times\sigma_v)}{L(1,\pi_v,\mathrm{Ad})L_{\psi_v}(1,\sigma_v,\mathrm{Ad})}. \tag{2.1}$$

We expect that in the case π_v or σ_v is not tempered, the expression defining α_v can be "meromorphically continued" so that

$$\frac{\alpha_v(F_v,h_v,\phi_v)}{\langle F_v,F_v\rangle \langle h_v,h_v\rangle \langle \phi_v,\phi_v\rangle}\left(\xi_v(2)\xi_v(4)\frac{L_{\psi_v}(\frac{1}{2},\pi_v\times\sigma_v)}{L(1,\pi_v,\mathrm{Ad})L_{\psi_v}(1,\sigma_v,\mathrm{Ad})}\right)^{-1}$$

makes sense and defines a finite number. We assume that this is the case. This in fact holds for almost all places. We will verify this in our particular example in Lemma 4.4. We define

$$\alpha_v^\natural(F_v,h_v,\phi_v) = \frac{\alpha_v(F_v,h_v,\phi_v)}{\langle F_v,F_v\rangle \langle h_v,h_v\rangle \langle \phi_v,\phi_v\rangle}\left(\xi_v(2)\xi_v(4)\frac{L_{\psi_v}(\frac{1}{2},\pi_v\times\sigma_v)}{L(1,\pi_v,\mathrm{Ad})L_{\psi_v}(1,\sigma_v,\mathrm{Ad})}\right)^{-1}. \tag{2.2}$$

Then for almost all places, $\alpha_v^\natural(F_v,h_v,\phi_v) = 1$.

Conjecture 2.1. *Assume that* $\mathrm{Hom}_{R(\mathbb{A}_\mathbb{Q})}(\pi\otimes\sigma\otimes\overline{\omega_\psi},\mathbb{C}) \neq 0$. *Then*

$$L(s) = \frac{L_\psi(s,\pi\times\sigma)}{L(s+\frac{1}{2},\pi,\mathrm{Ad})L_\psi(s+\frac{1}{2},\sigma,\mathrm{Ad})}$$

is holomorphic at $s = \frac{1}{2}$. *Moreover there is an integer* β *so that*

$$\frac{|\mathcal{FJ}_\psi(F,h,\phi)|^2}{\langle F,F\rangle\langle h,h\rangle\langle\phi,\phi\rangle} = 2^{-\beta}\xi(2)^2\xi(4)L\left(\frac{1}{2}\right)\cdot\prod_v \alpha_v^\natural(F_v,h_v,\phi_v).$$

We now assume the conjecture of Arthur as explained in [2, Section 25]. Let $\mathcal{L}$ be the (conjectural) automorphic Langlands group of $\mathbb{Q}$. Then π gives rise to an elliptic Arthur parameter

$$\Psi_\pi : \mathcal{L}\times\mathrm{SL}_2(\mathbb{C}) \to \mathrm{SO}_5(\mathbb{C}),$$

and σ gives rise to an elliptic Arthur parameter

$$\Psi_\sigma : \mathcal{L} \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SL}_2(\mathbb{C}).$$

Let S_π (resp. S_σ) be the centralizer of the image of Ψ_π in $\mathrm{SO}_5(\mathbb{C})$ (resp. $\mathrm{SL}_2(\mathbb{C})$).

Conjecture 2.2. *Let the notation be as above. Suppose that π and σ are tempered. Then $2^\beta = |S_\pi||S_\sigma|$.*

However, if π or σ are not tempered, then in general, $2^\beta \neq |S_\pi||S_\sigma|$. The integer β should depend on both the global data (the global endoscopy) and the local data (we do not know what it should be now). Theorem 4.6 below provides an example of Conjecture 2.1 when π is a Saito–Kurokawa lift, which is nontempered.

3. Automorphic forms

In this section, we review the notion of various classical automorphic forms and their adelization. Some of the materials are not well documented in the literature. So we include some discussions here for completeness.

Let k be a positive integer. By a classical Jacobi form of weight k, level one and index one, we mean a holomorphic function $f : \mathcal{H} \times \mathbb{C} \to \mathbb{C}$, with the following transformation properties:

(1) $f(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}) = (c\tau+d)^k e^{\frac{2\pi\sqrt{-1}cz^2}{c\tau+d}} f(\tau, z), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$;

(2) $f(\tau, z+\lambda\tau+\mu) = e^{-2\pi\sqrt{-1}(\lambda^2\tau+2\lambda z)} f(\tau, z), \quad \lambda, \mu \in \mathbb{Z}$.

The space of all cuspidal Jacobi forms of weight k level one and index one is denoted by $J^0_{k,1}$.

We define an action of $R(\mathbb{R})$ on $\mathcal{H} \times \mathbb{C}$ by

$$g.(\tau, z) = (\frac{a\tau+b}{c\tau+d}, \frac{z+\lambda\tau+\mu}{c\tau+d}), \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} (\lambda, \mu, \kappa).$$

We define a factor of automorphy by

$$j_{k,1}(g, (\tau, z)) = e^{-2\pi\sqrt{-1}(\kappa - \frac{c(z-\lambda\tau+\mu)^2}{c\tau+d} + \lambda^2\tau + 2\lambda z + \lambda\mu)} (c\tau+d)^k.$$

Any element $g \in R(\mathbb{A}_\mathbb{Q})$ can be written as

$$g = \gamma g_\infty k_0, \quad \gamma \in R(\mathbb{Q}),\ g_\infty \in R(\mathbb{R}),\ k_0 \in R(\widehat{\mathbb{Z}}).$$

Let f be a Jacobi form of weight k level one and index one, then we define a function $\mathbf{f}$ on $R(\mathbb{A}_\mathbb{Q})$ via

$$\mathbf{f}(\gamma g_\infty k_0) = j_{k,1}(g_\infty, (\sqrt{-1}, 0))^{-1} f(g_\infty(\sqrt{-1}, 0)).$$

This is called the adelization of f.

Let $F\left(\begin{pmatrix}\tau & z\\ z & \tau'\end{pmatrix}\right) = F(\tau, z, \tau')$ be a Siegel modular form of genus two weight k and level one. As a function of τ', F has the following Fourier expansion

$$F(\tau, z, \tau') = \sum_{m=0}^{\infty} f_m(\tau, z) e^{2m\pi\sqrt{-1}\tau'}.$$

Let $f(\tau, z) = f_1(\tau, z)$ be its first Fourier coefficient. Then f is a Jacobi form of weight k.

Let $\mathbf{F}$ be the adelization of F. It is a function on $\mathrm{Sp}_4(\mathbb{A}_{\mathbb{Q}})$ defined by,

$$\mathbf{F}(g) = \det(C\sqrt{-1} + D)^{-k} F((A\sqrt{-1} + B)(C\sqrt{-1} + D)^{-1}),$$

for $g = \gamma g_\infty k$, $\gamma \in \mathrm{Sp}_4(\mathbb{Q})$, $g_\infty = \begin{pmatrix} A & B\\ C & D\end{pmatrix} \in \mathrm{Sp}_4(\mathbb{R})$, $k \in \mathrm{Sp}_4(\widehat{\mathbb{Z}})$.

Lemma 3.1. *Let the notation be as above, i.e. f is the first Fourier–Jacobi coefficient of F and $\mathbf{F}$ and $\mathbf{f}$ are the adelization of F and f respectively. For any $r \in R(\mathbb{A}_{\mathbb{Q}})$,*

$$\int_{\mathbb{Q}\backslash\mathbb{A}_{\mathbb{Q}}} \mathbf{F}((0,0,\kappa)r)\overline{\psi(\kappa)}\mathrm{d}\kappa = e^{-2\pi}\mathbf{f}(r).$$

Proof. This is a direct computation. $\square$

Let $h \in S_{k-\frac{1}{2}}(\Gamma_0(4))$ be a holomorphic cusp form of weight $k - \frac{1}{2}$. The group $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ acts on $\mathcal{H}$ via projecting to $\mathrm{SL}_2(\mathbb{R})$. Define a factor of automorphy by

$$\widetilde{j}((g,\epsilon),\tau) = \begin{cases} \epsilon\sqrt{d}, & c = 0, d > 0;\\ -\epsilon\sqrt{d}, & c = 0, d < 0;\\ (c\tau + d)^{\frac{1}{2}}, & c \neq 0.\end{cases}, \quad g = \begin{pmatrix} a & b\\ c & d\end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}), \quad \tau \in \mathcal{H}.$$

The adelization $\mathbf{h}$ of h is a function on $\widetilde{\mathrm{SL}}_2(\mathbb{A}_{\mathbb{Q}})$ given by

$$\mathbf{h}(\gamma g_\infty k) = \widetilde{j}(g_\infty, \sqrt{-1})^{-2k+1} h(g_\infty\sqrt{-1}), \gamma \in \mathrm{SL}_2(\mathbb{Q}),\ g_\infty \in \widetilde{\mathrm{SL}}_2(\mathbb{R}),\ k \in K_1(4, \widehat{\mathbb{Z}}).$$

Kohnen's plus space $S^+_{k-\frac{1}{2}}(\Gamma_0(4))$ is a subspace of $S_{k-\frac{1}{2}}(\Gamma_0(4))$, consisting of forms whose Fourier expansion takes the form

$$\sum_{n=1}^{\infty} c(n)q^n, \qquad c(n) = 0 \text{ unless } (-1)^k n \equiv 0, -1 \bmod 4.$$

For $\tau \in \mathcal{H}$ and $h \in S_{k-\frac{1}{2}}(\Gamma_0(4))$, we define

$$h|W_4^\nu(\tau) = (-2\sqrt{-1}\tau)^{-(k-\frac{1}{2})} h\left(\frac{2\nu\tau - 1}{4\tau}\right), \quad \nu = 0, 1.$$

For $g \in \widetilde{\mathrm{SL}}_2(\mathbb{A}_{\mathbb{Q}})$, we also define

$$\mathsf{W}^\nu(\mathbf{h})(g) = \mathbf{h}\left(g w_2 n(-2\nu)_2 m(2)_2\right),\ \nu = 0, 1,$$

where the subscript 2 means that we view those elements as elements in $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)$.

Lemma 3.2. *The adelization of* $h|W_4^\nu$ *is* $\left(\frac{1+\sqrt{-1}}{\sqrt{2}}\right)^{-2k+1}\mathsf{W}^\nu(\mathbf{h})$, $\nu=0,1$.

Proof. This can be proved in the same way as [5, Lemma 3.2]. □

Lemma 3.3. *Let* $\phi=\prod_v \phi_v$ *with*

(1) $\phi_v=\mathbf{1}_{\mathbb{Z}_p}$ *if* $v=p\neq 2,\infty$;
(2) $\phi_v=\mathbf{1}_{\frac{1}{2}\mathbb{Z}_2}$ *if* $v=2$;
(3) $\phi_v=e^{-2\pi x^2}$ *if* $v=\infty$,

then $\Theta(r,\phi)$ *(as defined in Section 2) is the adelization of the classical Jacobi form*

$$\theta(\tau,z)=\sum_{n\in\mathbb{Z}} q^{\frac{n^2}{4}}\zeta^n,\quad q=e^{2\pi\sqrt{-1}\tau},\ \zeta=e^{2\pi\sqrt{-1}z}.$$

Let ϕ^ν $(\nu=0,1)$ *be such that* $\phi_v^\nu=\phi_v$ *if* $v\neq 2$ *and* $\phi_2^\nu=\mathbf{1}_{\frac{\nu}{2}+\mathbb{Z}_2}$, *then* $\Theta(g,\phi^\nu)$ *is the adelization of*

$$\theta_\nu(\tau,z)=\sum_{n\equiv\nu \bmod 2} q^{\frac{n^2}{4}}\zeta^n.$$

Proof. This is a direct computation. □

We assume from now on that k is an even integer. Classically, there is a one-to-one correspondence between $J_{k,1}^0$ and $S_{k-\frac{1}{2}}^+(\Gamma_0(4))$ given as follows, c.f. [15]. Let $f\in J_{k,1}^0$ with the Fourier expansion

$$f(\tau,z)=\sum_{\substack{n,m\in\mathbb{Z}\\ 4n\geq m^2}} A(n,m)q^n\zeta^m,\quad q=e^{2\pi\sqrt{-1}\tau},\ \zeta=e^{2\pi\sqrt{-1}z}.$$

By the transformation properties of the Jacobi form, if m is even, then $A(n,m)=A(n,0)$ and if m is odd, then $A(n,m)=A(n-\frac{m^2-1}{4},1)$. Put $c_0(n)=A(n,0)$ and $c_1(n)=A(n,1)$. Then we may write

$$f(\tau,z)=h_0(\tau)\theta_0(\tau,z)+h_1(\tau)\theta_1(\tau,z), \tag{3.1}$$

where

$$h_\nu(\tau)=\sum_{n=0}^{\infty} c_\nu(n)q^{n-\frac{\nu}{4}},\quad \nu=0,1.$$

Let $h(\tau)=h_0(4\tau)+h_1(4\tau)$. Then $h\in S_{k-\frac{1}{2}}^+(\Gamma_0(4))$. A more concise form of this correspondence is given by

$$\sum_{\substack{n\geq 1\\ n\equiv 0,-1 \bmod 4}} c(n)q^n \longleftrightarrow \sum_{n\equiv -m^2 \bmod 4} c(n)q^{\frac{n+m^2}{4}}\zeta^m.$$

Lemma 3.4. *Let the notation be as above, i.e.* $\mathbf{f}$ *(resp.* $\mathbf{h}$*) is the adelization of* f *(resp.* h*) where* $h \in S^+_{k-\frac{1}{2}}(\Gamma_0(4))$ *and* f *is the Jacobi form which corresponds to* h *via* (3.1)*. Then*

$$2^{-k+1}\frac{1-\sqrt{-1}}{\sqrt{2}}\mathbf{f}(ng) = \Theta(ng,\phi^0)\mathsf{W}^0(\mathbf{h})(g) + \Theta(ng,\phi^1)\mathsf{W}^1(\mathbf{h})(g),$$

where the theta functions are as in Lemma 3.3.

Proof. The transformation properties of h_0 and h_1 are the following, c.f. [15]:

$$h_0(\tau+1) = h_0(\tau), \qquad h_0(-\frac{1}{\tau}) = \frac{1+\sqrt{-1}}{2}\tau^{k-\frac{1}{2}}(h_0(\tau)+h_1(\tau));$$
$$h_1(\tau+1) = -\sqrt{-1}h_1(\tau), \qquad h_1(-\frac{1}{\tau}) = \frac{1+\sqrt{-1}}{2}\tau^{k-\frac{1}{2}}(h_0(\tau)-h_1(\tau)).$$

It follows that

$$h|W_4^\nu = 2^{-k+1}(-1)^{\frac{k}{2}}h_\nu, \quad \nu = 0,1.$$

The lemma then follows from Lemma 3.3 and Identity (3.1). □

4. The main theorem

Let k be an odd integer. Let $f \in S_{2k}(\Gamma_0(1))$ be a normalized Hecke eigenform of weight $2k$ and level one. Let $h \in S^+_{k+\frac{1}{2}}(\Gamma_0(4))$ be the Hecke eigenform of weight $k+\frac{1}{2}$ associated to f by the Shimura correspondences. The Fourier expansion of h takes the form

$$h(\tau) = \sum_{n\equiv 0,-1 \bmod 4} c(n)q^n, \quad \tau\in\mathcal{H}, q = e^{2\pi\sqrt{-1}\tau}.$$

Let $F \in S_{k+1}(\mathrm{Sp}_4(\mathbb{Z}))$ be the Saito–Kurokawa lifting of h whose Fourier expansion is given by

$$F(\tau,z,\tau') = \sum_{\substack{n,r,m\in\mathbb{Z}\\ n,m\geq 0,\ 4nm-r^2>0}} A(n,r,m)e^{2\pi\sqrt{-1}(n\tau+rz+m\tau')}, \tag{4.1}$$

where

$$A(n,r,m) = \sum_{d|(n,r,m)} d^k c\left(\frac{4nm-r^2}{d^2}\right).$$

The Saito–Kurokawa lifting has the following Fourier–Jacobi expansion

$$F(\tau,z,\tau') = \sum_{m=0}^{\infty} g_m(\tau,z)e^{2\pi\sqrt{-1}m\tau'}.$$

We let $g(\tau,z) = g_1(\tau,z)$. Then $g \in J^0_{k+1,1}$ is the Jacobi form corresponding to h via (3.1). This is one of the main results of [15].

We define

$$\langle h,h\rangle = \frac{1}{6}\int_{\Gamma_0(4)\backslash\mathcal{H}}|h(\tau)|^2y^{k+\frac{1}{2}}\frac{\mathrm{d}x\mathrm{d}y}{y^2}, \quad \langle F,F\rangle = \int_{\mathrm{Sp}_4(\mathbb{Z})\backslash\mathcal{H}_2}|F(Z)|^2(\det\Im Z)^{k-2}\mathrm{d}Z.$$

Then by Kohnen–Skoruppa [8] (See also [6, p. 1049] for some comments)

$$\frac{\langle F,F\rangle}{\langle h,h\rangle}=2^{k-2}\pi^{-1}\xi(2)L(\frac{3}{2},\tau), \tag{4.2}$$

where τ is the automorphic representation of $\mathrm{PGL}_2(\mathbb{A}_F)$ generated by f.

Let $\mathbf{f}$, $\mathbf{h}$, $\mathbf{g}$, $\mathbf{F}$ be the adelization of f, h, g and F respectively. Define

$$\mathbf{h}^{(2)}(g)=\mathbf{h}\left(gm(2)_2\right),\quad g\in\widetilde{\mathrm{SL}}_2(\mathbb{A}_\mathbb{Q}),\ m(2)_2\in\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2). \tag{4.3}$$

Let $\phi=\prod_v\phi_v\in\mathcal{S}(\mathbb{A}_\mathbb{Q})$ be that

- $\phi_v=\mathbf{1}_{\mathbb{Z}_p}$ if $v=p\neq 2,\infty$;
- $\phi_v=\mathbf{1}_{\frac{1}{2}\mathbb{Z}_2}$ if $v=2$;
- $\phi_v=e^{-2\pi x^2}$ if $v=\infty$,

Proposition 4.1. *We have*

$$\frac{|\mathcal{FJ}_\psi(\mathbf{F},\overline{\mathbf{h}^{(2)}},\phi)|^2}{\langle\mathbf{F},\mathbf{F}\rangle\langle\mathbf{h}^{(2)},\mathbf{h}^{(2)}\rangle\langle\phi,\phi\rangle}=2^{k+1}\pi e^{-4\pi}\xi(4)\xi(2)^{-1}L\left(\frac{3}{2},\tau\right)^{-1}.$$

Proof. Recall that $Z=\{(0,0,\kappa)\}\subset R$. By Lemma 3.1 and 3.4, we have

$$\begin{aligned}&\int_{Z(\mathbb{Q})\backslash Z(\mathbb{A}_\mathbb{Q})}\mathbf{F}(zng)e^{-2\pi\sqrt{-1}z}\mathrm{d}z\\ =&2^k\frac{1+\sqrt{-1}}{\sqrt{2}}e^{-2\pi}(\Theta(ng,\phi^0)\mathsf{W}^0(\mathbf{h})(g)+\Theta(ng,\phi^1)\mathsf{W}^1(\mathbf{h})(g)).\end{aligned}$$

Note that

$$\mathsf{W}^0(\mathbf{h})(g)+\mathsf{W}^1(\mathbf{h})(g)=(1+\sqrt{-1})\mathbf{h}^{(2)}(g).$$

Therefore

$$\mathcal{FJ}_\psi(\mathbf{F},\overline{\mathbf{h}^{(2)}},\phi)=-2^{k-\frac{1}{2}}e^{-2\pi}\int_{\mathrm{SL}_2(\mathbb{Q})\backslash\mathrm{SL}_2(\mathbb{A}_\mathbb{Q})}|\mathbf{h}^{(2)}(g)|^2\mathrm{d}g.$$

It follows that

$$\frac{|\mathcal{FJ}_\psi(\mathbf{F},\overline{\mathbf{h}^{(2)}},\phi)|^2}{\langle\mathbf{F},\mathbf{F}\rangle\langle\mathbf{h}^{(2)},\mathbf{h}^{(2)}\rangle\langle\phi,\phi\rangle}=2^{2k-1}e^{-4\pi}\frac{\langle\mathbf{h}^{(2)},\mathbf{h}^{(2)}\rangle}{\langle\mathbf{F},\mathbf{F}\rangle}.$$

Note that $\langle\mathbf{h}^{(2)},\mathbf{h}^{(2)}\rangle=\langle\mathbf{h},\mathbf{h}\rangle$. By [6, Section 9], we have $\langle h,h\rangle=2\xi(2)\langle\mathbf{h},\mathbf{h}\rangle$ and $\langle F,F\rangle=2\xi(2)\xi(4)\langle\mathbf{F},\mathbf{F}\rangle$ (the extra factor 2 comes from the fact that we are integrating over Sp_4 instead of SO_5 as in [6]). The desired identity then follows from (4.2). □

Proposition 4.2. *Let π_∞ be the holomorphic discrete series representation of $\mathrm{Sp}_4(\mathbb{R})$ of scalar weight $(k+1,k+1)$ and σ_∞ be a holomorphic discrete series representation of $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ of weight $k+\frac{1}{2}$. Let F_∞ (resp. h_∞) be the lowest weight vector in π_∞ (resp. σ_∞). Let $\phi_\infty(x)=e^{-2\pi x^2}\in\mathcal{S}(\mathbb{R})$. Then*

$$\frac{\alpha_\infty(F_\infty,\overline{h_\infty},\phi_\infty)}{\langle F_\infty,F_\infty\rangle\langle h_\infty,h_\infty\rangle\langle\phi_\infty,\phi_\infty\rangle}=\frac{2^{2k+5}e^{-4\pi}\pi^{k+2}}{(2k-1)\Gamma(k+1)}.$$

This will be proved in Section 5.

Now let $v \neq \infty$ be a finite place of $\mathbb{Q}$. Let $\lambda \in \sqrt{-1}\mathbb{R}$. Let $\eta_v : \mathbb{Q}_v^\times \to \{\pm 1\}$ be the quadratic character of $\mathbb{Q}_v^\times$ associated to the quadratic extension $\mathbb{Q}_v(\sqrt{-1})/\mathbb{Q}_v$ by the local class field theory. Let $\sigma_v = \mathrm{Ind}_{\widetilde{B}(\mathbb{Q}_v)}^{\widetilde{\mathrm{SL}}_2(\mathbb{Q}_v)} \eta_v|\cdot|^\lambda$. If $v \neq 2$, then σ_v contains a nonzero $\mathrm{SL}_2(\mathbb{Z}_v)$-fixed vector h_v. We may normalize so that $h_v(1) = 1$. Let $\phi_v = \mathbf{1}_{\mathbb{Z}_v} \in \mathcal{S}(\mathbb{Q}_v)$ be a Schwartz function. We now assume that $v = 2$. For any $h_2^+ \in \sigma_2$, define

$$\mathsf{W}_2^\nu(h_2^+)(g) = h_2^+\left(gwn(2\nu)m(2)\right),\ \nu = 0, 1.$$

Let $\epsilon_2 : K_0(4, \mathbb{Z}_2) \to \mathbb{C}$ be the character

$$\epsilon_2(k) = \begin{cases} \gamma_{\mathbb{Q}_2}(d, \psi_2)(c, d)_{\mathbb{Q}_2}, & c \neq 0; \\ \gamma_{\mathbb{Q}_2}(d, \psi_2)^{-1}, & c = 0. \end{cases}$$

Let

$$V_0 = \{f \in \sigma_2 \mid \sigma_2(k)f = \epsilon_2(k)^{-1}f,\ k \in K_0(4, \mathbb{Z}_2)\}$$

By [5, p. 571], $\dim V_0 = 2$ and there are two elements $f_1, f_w \in V_0$ with the following property

$$f_1(1) = 1 \quad f_1(w) = 0 \quad f_1(k_1) = 0$$
$$f_w(1) = 0 \quad f_w(w) = 1 \quad f_w(k_1) = 0,$$

where $k_1 = \begin{pmatrix} 1 & \\ 2 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Q}_2)$. Put

$$h_2 = 2^{2\lambda} f_1 + \frac{1 + \sqrt{-1}}{4} f_w. \tag{4.4}$$

We let $\phi_2 = \mathbf{1}_{\frac{1}{2}\mathbb{Z}_2} \in \mathcal{S}(\mathbb{Q}_2)$ be a Schwartz function. Put also $\phi_2^\nu = \mathbf{1}_{\frac{\nu}{2} + \mathbb{Z}_2}$.

We note that for any finite place v, the tensor product $\sigma_v^{\psi_v} = \sigma_v \otimes \omega_{\psi_v}$ is in fact an irreducible representation of $R(\mathbb{Q}_v)$. In fact, it is isomorphic to the principal series representation

$$\mathrm{Ind}_{B_R(\mathbb{Q}_v)}^{R(\mathbb{Q}_v)} |\cdot|^\lambda = \{f : R(\mathbb{Q}_v) \to \mathbb{C} \mid f((0, b, c)n(x)m(y)r) = \psi(\kappa)|y|^{\lambda + \frac{3}{2}} f(r)\},$$

where the isomorphism is given by

$$h^+ \otimes \phi \mapsto (ug \mapsto h^+(g)\omega_\psi(ug)\phi(0)), \quad h^+ \in \sigma,\ \phi \in \mathcal{S}(\mathbb{Q}_v),\ u \in H(\mathbb{Q}_v),\ g \in \mathrm{SL}_2(\mathbb{Q}_v).$$

There is a natural pairing between σ^ψ and $\overline{\sigma^\psi}$, given by

$$\langle \Phi_1, \Phi_2 \rangle = \int_{\mathbb{Q}_2} \int_{\mathrm{SL}_2(\mathbb{Z}_2)} \Phi_1((a, 0, 0)k)\overline{\Phi_2((a, 0, 0)k)} \mathrm{d}k\mathrm{d}a.$$

Moreover, $\sigma_v^{\psi_v}$ contains an $R(\mathbb{Z}_v)$ fixed vector. In fact, if $v \neq 2$, then it is clear that $\Phi_v = h_v \otimes \phi_v$ is such an element. If $v = 2$, we show in Lemma 6.1 that $\Phi_2 = \mathsf{W}^0(h_2) \otimes \phi_2^0 + \mathsf{W}^1(h_2) \otimes \phi_2^1$ is $R(\mathbb{Z}_2)$-fixed. We will use $\mathrm{Ind}_{B_R(\mathbb{Q}_v)}^{R(\mathbb{Q}_v)} |\cdot|^\lambda$ as a model for $\sigma_v^{\psi_v}$ in the following.

For any finite place v ($v = 2$ or not), let $I(\lambda)$ be the degenerate principal series representation

$$\pi_v = \{F : \mathrm{Sp}_4(\mathbb{Q}_v) \to \mathbb{C} \mid F(ang) = |\det a|^{\lambda+\frac{3}{2}} F(g)\}$$

of $\mathrm{Sp}_4(\mathbb{Q}_v)$. We let F_v be an $\mathrm{Sp}_4(\mathbb{Z}_v)$ fixed vector in π_v. Let $I(\mu_1, \mu_2)$ be the unramified principal series representation of $\mathrm{Sp}_4(\mathbb{Q}_v)$ which is induced from the character $\mathrm{diag}[t_1, t_2, t_1^{-1}, t_2^{-1}] \mapsto |t_1|^{\mu_1}|t_2|^{\mu_2}$. Then π_v is a subrepresentation of $I(\lambda - \frac{1}{2}, \lambda + \frac{1}{2})$ and F_v is an $\mathrm{Sp}_4(\mathbb{Z}_v)$ fixed vector of $I(\lambda - \frac{1}{2}, \lambda + \frac{1}{2})$. For general μ_1, μ_2, we again use F_v to denote an $\mathrm{Sp}_4(\mathbb{Z}_v)$ fixed vector in $I(\mu_1, \mu_2)$. They are all unique up to a scalar. We normalize so that $F_v(1) = 1$.

Proposition 4.3. *Let the notation be as above. Suppose that $\lambda, \mu_1, \mu_2 \in \sqrt{-1}\mathbb{R}$. Then*

$$\alpha_v^\natural(F_v, \overline{h_v'}, \phi_v) = \begin{cases} 1, & v \neq 2; \\ \dfrac{1}{2}, & v = 2, \end{cases} \tag{4.5}$$

where $h_v' = h_v$ if $v \neq 2$ and $h_2' = h_2^{(2)}$.

If $v \neq 2$, then this is proved in [14, Section 4]. If $v = 2$, this will be proved in Section 6.

We now explain how to define $\alpha_v^\natural(\xi_v, \varphi_v)$ for all $\xi_v \in I(\mu_1, \mu_2)$ and $\varphi_v \in \sigma_v^{\psi_v}$ and all $\lambda, \mu_1, \mu_2 \in \mathbb{C}$. By a flat section $\xi_v \in I(\mu_1, \mu_2)$ (resp. $\varphi_v \in \sigma_v^{\psi_v}$), we mean that $\xi_v|_{\mathrm{Sp}_4(\mathbb{Z}_v)}$ (resp. $\varphi_v|_{R(\mathbb{Z}_v)}$) is a function on $\mathrm{Sp}_4(\mathbb{Z}_v)$ (resp. $R(\mathbb{Z}_v)$) that is independent of μ_1, μ_2 (resp. λ). Recall that we have the $\mathrm{Sp}_4(\mathbb{Z}_v)$ fixed vector F_v of $I(\mu_1, \mu_2)$ and the $R(\mathbb{Z}_v)$ fixed vector Φ_v of $\sigma_v^{\psi_v}$. Then (for varying λ, μ_1, μ_2), F_v is a flat section of $I(\mu_1, \mu_2)$. If $v \neq 2$, then Φ_v is a flat section of $\sigma_v^{\psi_v}$. It follows from the computation in Lemma 6.1 that $2^{-\lambda}\Phi_2$ and $2^{-\lambda}h_2^{(2)} \otimes \phi_2$ are flat sections of $\sigma_2^{\psi_2}$. We also note that every section ξ_v of $I(\mu_1, \mu_2)$ and φ_v of $\sigma_v^{\psi_v}$ (for a fixed $\lambda, \mu_1, \mu_2 \in \mathbb{C}$) can be embeded into a unique flat section of $I(\mu_1, \mu_2)$ and $\sigma_v^{\psi_v}$ (for varying λ, μ_1, μ_2) respectively.

It follows from [10] that for all $\lambda, \mu_1, \mu_2 \in \mathbb{C}$, we have that the space

$$\mathrm{Hom}_{R(\mathbb{Q}_v)}(I(\mu_1, \mu_2) \otimes \overline{\sigma_v^{\psi_v}}, \mathbb{C})$$

is at most one dimensional. Let $\ell_v \in \mathrm{Hom}_{R(\mathbb{Q}_v)}(I(\mu_1, \mu_2) \otimes \overline{\sigma_v^{\psi_v}}, \mathbb{C})$ be a basis. Results of [10] also show that $\ell_v(F_v, \overline{\Phi_v}) \neq 0$ if $\ell_v \neq 0$. We normalize ℓ_v so that $\ell_v(F_v, \overline{\Phi_v}) = 1$ if $\ell_v \neq 0$.

It follows from Proposition 4.3 that

$$\alpha_v^\natural = c_v \cdot \ell_v \otimes \overline{\ell_v},$$

where $c_v = 1$ (resp. $\frac{1}{2}$) if $v \neq 2$ (resp. $v = 2$).

Lemma 4.4. *Let ξ_v (resp. φ_v) be a flat section of $I(\mu_1, \mu_2)$ (resp. $\sigma_v^{\psi_v}$). Suppose that $\lambda, \mu_1, \mu_2 \in \sqrt{-1}\mathbb{R}$. Then $\alpha_v^\natural(\xi_v, \overline{\varphi_v})$ is a polynomial functions in $q_v^{\pm\lambda}, q_v^{\pm\mu_1}, q_v^{\pm\mu_2}$ where q_v is the number of elements in the residue field of v.*

Proof. It is enough to prove the same statement for $\ell_v(\xi_v, \varphi_v)$. The proof is identical to the argument in [13, Section 5], using a theorem of Bernstein and its corollary due to Banks [1]. We omit the details which need lots of additional notation. The argument in [13] actually proves that $\alpha_v^\natural(\xi_v, \overline{\varphi_v})$ is a rational function. However, the result of Banks [1] and the fact that the dimension of $\mathrm{Hom}_{R(\mathbb{Q}_v)}(I(\mu_1,\mu_2)\otimes\overline{\sigma_v^{\psi_v}}, \mathbb{C})$ is at most one guarantee that it is in fact a polynomial function. Note that for our choices of the flat sections $\xi_v = F_v$ and $\varphi_v = h_v \otimes \phi_v$ (if $v \neq 2$) or $2^{-\lambda} h_2^{(2)} \otimes \phi_2$ (if $v = 2$), the lemma is obvious as it is a constant function. $\square$

Now fix $\lambda, \mu_1, \mu_2 \in \mathbb{C}$ and $\xi_v \in I(\mu_1, \mu_2)$ and $\varphi_v \in \sigma_v^{\psi_v}$. We embed ξ_v and φ_v into the (unique) flat sections of $I(\mu_1,\mu_2)$ and $\sigma_v^{\psi_v}$ respectively. We now define $\alpha_v^\natural(\xi_v, \overline{\varphi_v})$ to be the value of the polynomial function in Lemma 4.4 at (the fixed) λ, μ_1, μ_2.

Corollary 4.5. *Let the notation be as above. For any $\lambda, \mu_1, \mu_2 \in \mathbb{C}$, we have*

$$\alpha_v^\natural(F_v, \overline{h_v'}, \phi_v) = \begin{cases} 1, & v \neq 2, \\ \dfrac{1}{2}, & v = 2. \end{cases}$$

where $h_v' = h_v$ if $v \neq 2$ and $h_2' = h_2^{(2)}$.

Proof. It follows directly from Proposition 4.3 and the definition for $\alpha_v^\natural(F_v, \overline{h_v'}, \phi_v)$. $\square$

With these preparations, we now state and prove the main theorem of this note.

Theorem 4.6. *Let $h \in \mathcal{S}_{k+\frac{1}{2}}(\Gamma_0(4))$ be a holomorphic cusp form of weight $k + \frac{1}{2}$ and F be its Saito–Kurokawa lift as defined by (4.1). Let $\mathbf{h}$ and $\mathbf{F}$ be the adelization of h and F respectively. Define $\mathbf{h}^{(2)}$ as in (4.3). Let π be the automorphic representation of $\mathrm{Sp}_4(\mathbb{A}_\mathbb{Q})$ generated by $\mathbf{F}$. Let σ be the automorphic representation of $\widetilde{\mathrm{SL}}_2(\mathbb{A}_\mathbb{Q})$ generated by $\mathbf{h}$. Then*

(4.6)
$$\frac{|\mathcal{F}\mathcal{J}_\psi(\mathbf{F}, \overline{\mathbf{h}^{(2)}}, \phi)|^2}{\langle \mathbf{F}, \mathbf{F}\rangle \langle \mathbf{h}^{(2)}, \mathbf{h}^{(2)}\rangle\langle\phi,\phi\rangle} = \frac{1}{4}\xi(2)^2\xi(4)\frac{L_\psi(s, \pi\times\sigma)}{L(s+\frac{1}{2}, \pi, \mathrm{Ad})L_\psi(s+\frac{1}{2}, \sigma, \mathrm{Ad})}\Big|_{s=\frac{1}{2}} \cdot \prod_v \alpha_v^\natural(\mathbf{F}_v, \overline{\mathbf{h}_v}, \phi_v),$$

where

- $\mathbf{F}_\infty = F_\infty, \mathbf{h}_\infty = h_\infty$ *and* $F_\infty, h_\infty, \phi_\infty$ *are as described in Proposition 4.2;*
- $\mathbf{F}_v \in \pi_v$, $\mathbf{h}_v \in \sigma_v$ *and* $\phi_v \in \mathcal{S}(\mathbb{Q}_v)$ *are as in Corolloary 4.5.*

In particular, Conjecture 2.1 holds with $\beta = 2$.

Proof. Let τ be the automorphic representation of $\mathrm{PGL}_2(\mathbb{A}_\mathbb{Q})$ generated by f. We have

$$\begin{aligned} \frac{L_\psi(s,\pi\times\sigma)}{L(s+\frac{1}{2},\pi,\mathrm{Ad})L_\psi(s+\frac{1}{2},\sigma,\mathrm{Ad})} &= \frac{L(s,\pi\times\tau)}{L(s+\frac{1}{2},\pi,\mathrm{Ad})L(s+\frac{1}{2},\tau,\mathrm{Ad})} \\ &= \frac{L(s-\frac{1}{2},\tau,\mathrm{Ad})}{L(s+1,\tau)\xi(s+\frac{3}{2})L(s+\frac{1}{2},\tau,\mathrm{Ad})}. \end{aligned}$$

Therefore at $s=\frac{1}{2}$, it is holomorphic and equals $\xi(2)^{-1}L(\frac{3}{2},\tau)^{-1}$. Its local component at the infinity place is

$$\frac{\Gamma_{\mathbb{R}}(1)\Gamma_{\mathbb{C}}(2k-1)}{\Gamma_{\mathbb{C}}(k+1)\cdot\Gamma_{\mathbb{R}}(2)\cdot\Gamma_{\mathbb{R}}(2)\Gamma_{\mathbb{C}}(2k)}=\frac{2^{k+1}\pi^{k+4}}{\Gamma(k+1)(2k-1)}.$$

The local components of π are as follows, c.f. [12].

- π_∞ is a discrete series representation of $\mathrm{Sp}_4(\mathbb{R})$ of lowest K-type $(k+1,k+1)$.
- If $v\neq\infty$, π_v is the degenerate principal series representation $I(\lambda_v)$ of $\mathrm{Sp}_4(\mathbb{Q}_v)$ where $\lambda_v\in\mathbb{C}$ is the Satake parameter of τ_v.

The local components of σ are as follows.

- σ_∞ is a discrete series representation of $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ of lowest K-type $k+\frac{1}{2}$.
- If $v\neq\infty$, $\sigma_v\simeq\mathrm{Ind}_{\widetilde{B}(\mathbb{Q}_v)}^{\widetilde{\mathrm{SL}}_2(\mathbb{Q}_v)}\eta_v|\cdot|_v^{\lambda_v}$, where $\lambda_v\in\mathbb{C}$ is the Satake parameter of τ_v.

By Proposition 4.2, we have

$$\alpha_\infty^\natural(\mathbf{F}_\infty,\overline{\mathbf{h}_\infty},\phi_\infty)=\frac{2^{2k+5}e^{-4\pi}\pi^{k+2}}{(2k-1)\Gamma(k+1)}\left(\pi^{-3}\frac{2^{k+1}\pi^{k+4}}{\Gamma(k+1)(2k-1)}\right)^{-1}=2^{k+4}\pi e^{-4\pi}.$$

By Corollary 4.5, we have

$$\alpha_v^\natural(\mathbf{F}_v,\overline{\mathbf{h}_v},\phi_v)=\begin{cases}1, & v\neq 2\\ \frac{1}{2}, & v=2\end{cases}.$$

The desired identity (4.6) then follows from Proposition 4.1. $\square$

5. Computation at the archimedean place

In this section, we prove Proposition 4.2. We remark that some of the integrals in this section are computed by Mathematica 10. Whenever an integral is computed by Mathematica, we mark the equality by $\equiv$ instead of $=$.

To simplify notation, we suppress all subscripts ∞. We recall the following notation.

- k is a positive odd integer.
- π is the holomorphic discrete series representation of $\mathrm{Sp}_4(\mathbb{R})$ of lowest K-type $(k+1,k+1)$. $F\in\pi$ is the lowest weight vector. We may assume that $\langle F,F\rangle=1$.
- σ is the holomorphic discrete series representation of $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ of lowest K-type $k+\frac{1}{2}$. $h\in\sigma$ is the lowest weight vector. We may assume that $\langle h,h\rangle=1$.
- $\phi(x)=e^{-2\pi x^2}\in\mathcal{S}(\mathbb{R})$. Then $\langle\phi,\phi\rangle=\frac{1}{2}$.

The goal is to prove Proposition 4.2, that is

$$\alpha(F,\overline{h},\phi)=\int_{\mathrm{SL}_2(\mathbb{R})}\int_{N(\mathbb{R})}\langle\pi(ng)F,F\rangle\overline{\langle\omega(ng)\phi,\phi\rangle\langle\sigma(g)h,h\rangle}\mathrm{d}n\mathrm{d}g=\frac{2^{2k+4}e^{-4\pi}\pi^{k+2}}{(2k-1)\Gamma(k+1)}.$$

An element $g\in\mathrm{Sp}_4(\mathbb{R})$ has the following Cartan decomposition $g=k_1\Lambda(t_1,t_2)k_2$ where $k_1,k_2\in\mathrm{U}_2(\mathbb{R})$, $\Lambda(t_1,t_2)$ is a diagonal matrix with entries $e^{t_1},e^{t_2},e^{-t_1},e^{-t_2}$, $t_1\geq t_2\geq 0$. We have $\pi(g)F = (\det g)^{k+1}F$ if $g \in \mathrm{U}_2(\mathbb{R})$, where det is the determinant of g as an element in $\mathrm{U}_2(\mathbb{R})$. Then

$$\langle \pi(g)F, F\rangle = (\cosh t_1 \cosh t_2 \det k_1^{-1} \det k_2^{-1})^{-k-1},$$

if $g = k_1\Lambda(t_1, t_2)k_2$ is the Cartan decomposition.

Lemma 5.1. *Suppose that*

$$g = \begin{pmatrix} 1 & & & b \\ a & 1 & b & c \\ & & 1 & -a \\ & & & 1 \end{pmatrix} \begin{pmatrix} e^t & & & \\ & 1 & & \\ & & e^{-t} & \\ & & & 1 \end{pmatrix} \in \mathrm{Sp}_4(\mathbb{R}).$$

Then

$$\langle \pi(g)F, F\rangle = \left(\frac{4e^t}{a^2e^{2t} + b^2 + 2 + 2e^{2t} + \sqrt{-1}(ab(-1+e^{2t}) - c(1+e^{2t}))}\right)^{k+1}.$$

Proof. Suppose that $g = k_1\Lambda(t_1, t_2)k_2$ is the Cartan decomposition and

$$k_1 = \begin{pmatrix} A_1 & B_1 \\ -B_1 & A_1 \end{pmatrix},\ k_2 = \begin{pmatrix} A_2 & B_2 \\ -B_2 & A_2 \end{pmatrix}, \qquad A_1 + \sqrt{-1}B_1,\ A_2 + \sqrt{-1}B_2 \in \mathrm{U}(2, \mathbb{R}).$$

Let $C = \begin{pmatrix} 1_2 & \sqrt{-1}1_2 \\ \sqrt{-1}1_2 & 1_2 \end{pmatrix}$ and $T = \mathrm{diag}[e^{t_1}, e^{t_2}]$. Then

$$\begin{aligned} CgC^{-1} &= \begin{pmatrix} A_1 - \sqrt{-1}B_1 & \\ & * \end{pmatrix} \begin{pmatrix} \frac{1}{2}(T + T^{-1}) & * \\ * & * \end{pmatrix} \begin{pmatrix} A_2 - \sqrt{-1}B_2 & \\ & * \end{pmatrix} \\ &= \begin{pmatrix} \frac{1+e^{2t}}{2e^t} & -\frac{a+\sqrt{-1}b}{2} & * & * \\ \frac{ae^{2t}-\sqrt{-1}b}{2e^t} & \frac{2-\sqrt{-1}c}{2} & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}, \end{aligned}$$

where $*$'s stand for entries of matrix which are not relevant to our computation. Therefore

$$\cosh t_1 \cosh t_2 \det k_1^{-1} \det k_2^{-1} = \frac{a^2e^{2t} + b^2 + 2 + 2e^{2t} + \sqrt{-1}(ab(-1+e^{2t}) - c(1+e^{2t}))}{4e^t}.$$

The lemma then follows. □

Lemma 5.2.

$$\langle \sigma(m(e^t), \epsilon)h, h\rangle = \epsilon(\cosh t)^{-(k+\frac{1}{2})}.$$

Proof. This lemma is well-known to the experts. We include a proof here as we are not able to find a satisfactory reference.

The holomorphic discrete series σ can be realized on

$$L^2\left(\mathcal{H}, k+\frac{1}{2}\right) = \left\{ f \text{ holomorphic on } \mathcal{H} \;\middle|\; \iint_{\mathcal{H}} |f(z)|^2 y^{k-\frac{3}{2}} \mathrm{d}x\mathrm{d}y < \infty \right\},$$

and the action of $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ is given by

$$\widetilde{\sigma}(g,\epsilon)f(z) = \widetilde{j}((g,\epsilon),z)^{-(2k+1)} f\left(\frac{az+b}{cz+d}\right), \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}).$$

Note that $(z+\sqrt{-1})^{-(k+\frac{1}{2})}$ is in $L^2(\mathcal{H}, k+\frac{1}{2})$ and $\sigma(k_\theta,\epsilon)(z+\sqrt{-1})^{-(k+\frac{1}{2})} = \epsilon e^{\sqrt{-1}\theta(k+\frac{1}{2})}(z+\sqrt{-1})^{-(k+\frac{1}{2})}$. We have

$$\iint_{\mathcal{H}} (z+\sqrt{-1})^{-(k+\frac{1}{2})}(\overline{z}-\sqrt{-1})^{-(k+\frac{1}{2})} y^{k-\frac{3}{2}} \mathrm{d}x\mathrm{d}y \equiv \frac{\pi}{4^{k-1}(2k-1)}.$$

We also have

$$\begin{aligned} &\langle \sigma(m(e^t),\epsilon)(z+\sqrt{-1})^{-(k+\frac{1}{2})}, (z+\sqrt{-1})^{-(k+\frac{1}{2})} \rangle \\ =&\epsilon \int_0^\infty \int_{-\infty}^\infty e^{t(k+\frac{1}{2})}(e^{2t}x + \sqrt{-1}(e^{2t}y+1))^{-(k+\frac{1}{2})}(x-\sqrt{-1}(y+1))^{-(k+\frac{1}{2})} y^{k-\frac{3}{2}} \mathrm{d}x\mathrm{d}y. \end{aligned}$$

The inner integral equals

$$\frac{2^{2k}\Gamma(k)\sqrt{\pi}}{\Gamma(k+\frac{1}{2})} e^{-(k+\frac{1}{2})t}(2y+1+e^{-2t})^{-2k}.$$

This can be proved by induction on k. The integral over y equals

$$\epsilon(\cosh t)^{-k-\frac{1}{2}} \frac{\pi}{4^{k-1}(2k-1)}.$$

Since $\langle h, h\rangle = 1$, we have

$$\langle m(e^t)h, h\rangle = \epsilon(\cosh t)^{-k-\frac{1}{2}} \frac{\pi}{4^{k-1}(2k-1)} \cdot \left(\frac{\pi}{4^{k-1}(2k-1)}\right)^{-1} = \epsilon(\cosh t)^{-k-\frac{1}{2}}.$$

□

Lemma 5.3. *Suppose that*

$$g = \begin{pmatrix} 1 & & & b \\ a & 1 & b & c \\ & & 1 & -a \\ & & & 1 \end{pmatrix} \begin{pmatrix} e^t & & & \\ & 1 & & \\ & & e^{-t} & \\ & & & 1 \end{pmatrix} \in \mathrm{Sp}_4(\mathbb{R}).$$

Then

$$\langle \omega(g,\epsilon)\phi, \phi\rangle = \epsilon e^{\frac{t}{2} - \frac{2\pi(a^2e^{2t}+b^2)}{1+e^{2t}} - 2\pi\sqrt{-1}(ab\frac{-1+e^{2t}}{1+e^{2t}} - c)}.$$

Proof. By definition,

$$\langle \omega(g,\epsilon)\phi,\phi\rangle = \epsilon e^{\frac{t}{2}+2\pi\sqrt{-1}c}\int_{\mathbb{R}} e^{-2\pi(x+a)^2e^{2t}+2\pi\sqrt{-1}(2x+a)b}\mathrm{d}x$$

$$\equiv \epsilon\frac{e^{\frac{t}{2}-\frac{2\pi(a^2e^{2t}+b^2)}{1+e^{2t}}-2\pi\sqrt{-1}(ab\frac{-1+e^{2t}}{1+e^{2t}}-c)}}{\sqrt{2}(1+e^{2t})^{\frac{1}{2}}}.$$

$\square$

Proof of Proposition 4.2. Let $A^+ = \{m(e^t) \mid t \geq 0\} \subset \mathrm{SL}_2(\mathbb{R})$. Then

$$(K\times A^+ \times K)/\{\pm 1\} \to \mathrm{SL}_2(\mathbb{R})$$

is a bijection outside the boundary of A^+, where $-1 = (-1,1,-1) \in K\times A^+\times K$. It is well-known that there is a constant C, such that

$$\mathrm{d}g|_{K\times A^+\times K} = C\cdot \sinh 2t\mathrm{d}t\mathrm{d}k_1\mathrm{d}k_2.$$

Let $A(T)$ be the area of a disc of radius T around $\sqrt{-1}$ on the upper half plane. Then

$$A(T)\sim 4\pi\int_0^{\frac{T}{2}}\sinh 2t\mathrm{d}t = \pi T^2, \quad T\to 0.$$

It follows that $C = 8\pi$.

Therefore

$$\begin{aligned}&\int_{\mathrm{SL}_2(\mathbb{R})}\int_{N(\mathbb{R})}\langle\pi(ng)F,F\rangle\overline{\langle\omega(ng)\phi,\phi\rangle\langle\sigma(g)h,h\rangle}\mathrm{d}n\mathrm{d}g\\ =&4\pi\int_0^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty\langle\pi(ng)F,F\rangle\overline{\langle\omega(ng)\phi,\phi\rangle\langle\sigma(g)h,h\rangle}\sinh 2t\mathrm{d}c\mathrm{d}a\mathrm{d}b\mathrm{d}t,\end{aligned}$$

where

$$n = \begin{pmatrix}1 & & & b\\ a & 1 & b & c\\ & & 1 & -a\\ & & & 1\end{pmatrix}, \quad g = \begin{pmatrix}e^t & & & \\ & 1 & & \\ & & e^{-t} & \\ & & & 1\end{pmatrix}.$$

By Lemma 5.1, we have

$$\begin{aligned}&\int_{\mathbb{R}}\langle\pi(ng)\varphi,\varphi\rangle e^{-2\pi\sqrt{-1}c}\mathrm{d}c\\ =&\int_{\mathbb{R}}\left(\frac{4e^t}{a^2e^{2t}+b^2+2+2e^{2t}+\sqrt{-1}(ab(-1+e^{2t})-c(1+e^{2t}))}\right)^{k+1}e^{-2\pi\sqrt{-1}c}\mathrm{d}c\\ \equiv&\frac{(8\pi e^t)^{k+1}e^{-\frac{2\pi(a^2e^{2t}+b^2+2+2e^{2t})}{1+e^{2t}}-2\pi\sqrt{-1}ab\frac{-1+e^{2t}}{1+e^{2t}}}}{\Gamma(k+1)(1+e^{2t})^{k+1}}.\end{aligned}$$

Thus the inner integrals over a and b equals

$$\frac{(8\pi e^{t})^{k+1}}{(1+e^{2t})^{k+1}\Gamma(k+1)}\frac{e^{\frac{t}{2}}}{\sqrt{2}(1+e^{2t})^{\frac{1}{2}}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-\frac{4\pi(a^2e^{2t}+b^2+1+e^{2t})}{1+e^{2t}}}\mathrm{d}a\mathrm{d}b$$
$$=\frac{4^{k+\frac{1}{2}}e^{-4\pi}\pi^{k+1}}{\Gamma(k+1)}(\cosh t)^{-(k+\frac{1}{2})}.$$

Therefore

$$\alpha(F,\overline{h},\phi)=\int_0^\infty\frac{4^{k+\frac{3}{2}}e^{-4\pi}\pi^{k+2}}{\Gamma(k+1)}(\cosh t)^{-2k-1}\sinh 2t\mathrm{d}t=\frac{2^{2k+4}e^{-4\pi}\pi^{k+2}}{(2k-1)\Gamma(k+1)}.$$

Note that $\langle\phi,\phi\rangle=\frac{1}{2}$. Proposition 4.2 then follows. □

6. Computation at the 2-adic place

We prove Proposition 4.3 in this section. The computation is mostly the same as [14, Sectoin 4]. Some additional work is needed to take care of the fact that the residue characteristic is two.

To simplify notation, we suppress all subscripts 2. Recall the following notation.

- $\lambda,\mu_1,\mu_2\in\sqrt{-1}\mathbb{R}$.
- $\pi=I(\mu_1,\mu_2)$ is the principal series representation of $\mathrm{Sp}_4(\mathbb{Q}_2)$ and contains an $\mathrm{Sp}_4(\mathbb{Z}_2)$-fixed element F.
- $\sigma=\mathrm{Ind}_{\widetilde{B}(\mathbb{Q}_2)}^{\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)}\eta|\cdot|^{\lambda}$ and contains the distinguished element h.
- For any element $h^+\in\sigma$, put $h^{+,(2)}=\sigma(m(2))h^+$ and
$$\mathsf{W}^{\nu}(h^+)(g)=h^+\left(gwn(-2\nu)m(2)\right),\nu=0,1.$$
- $\phi^{\nu}=\mathbf{1}_{\frac{\nu}{2}+\mathbb{Z}_2}$, $\nu=0,1$ and $\phi=\phi^0+\phi^1=\mathbf{1}_{\frac{1}{2}\mathbb{Z}_2}$.
- For simplicity, we write $K_0(4)$ for $K_0(4,\mathbb{Z}_2)$.
- $\sigma^{\psi}=\sigma\otimes\omega_{\psi}$ is an unramified principal series representation of $R(\mathbb{Q}_2)$. Let $\Phi=\mathsf{W}^0(h)\otimes\phi^0+\mathsf{W}^1(h)\otimes\phi^1$, $\Phi^-=\mathsf{W}^0(h)\otimes\phi^1+\mathsf{W}^1(h)\otimes\phi^0\in\sigma^{\psi}$.

Lemma 6.1. (1) $2h^{(2)}=(1-\sqrt{-1})(\mathsf{W}^0(h)+\mathsf{W}^1(h))$.

(2) Φ *is $R(\mathbb{Z}_2)$-fixed.*

(3) $\int_{\mathrm{SL}_2(\mathbb{Z}_2)}\sigma^{\psi}(k)\Phi^-\mathrm{d}k=0$.

Proof. Let

$$K(4)=\left\{k\in\mathrm{SL}_2(\mathbb{Z}_2)\mid k,{}^tk\in K_0(4)\right\}.$$

We first investigate the action of $K(4)$ on $h^{(2)},\mathsf{W}^0(h),\mathsf{W}^1(h)$. By [5, Lemma 3.5], $\mathsf{W}^0(h)\in V_0$. Let

$$k=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in K(4).$$

Then $(wm(2))^{-1}kwm(2) = (k', \epsilon')$, and $n(-\frac{1}{2})(k', \epsilon')n(\frac{1}{2}) = (k'', \epsilon'')$, where

$$k' = \begin{pmatrix} d & -\frac{1}{4}c \\ -4b & a \end{pmatrix}, \quad k'' = \begin{pmatrix} d+2b & b - \frac{1}{4}c - \frac{1}{2}(a-d) \\ -4b & a-2b \end{pmatrix} \in K_0(4, \mathbb{Z}_2).$$

It follows from (the proof of) [5, Lemma 3.6] that $\epsilon'\epsilon_2(k') = \epsilon_2(k)$. Moreover it is straight forward to check that $\epsilon' = \epsilon''$ and $\epsilon_2(k') = \epsilon_2(k'')$. It follows that

$$\mathsf{W}^1(h)(gk) = \epsilon_2(k)^{-1}\mathsf{W}^1(h)(g), \quad k \in K(4).$$

It is also checked that

$$\begin{pmatrix} \frac{1}{2} & \\ & 2 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 2 & \\ & \frac{1}{2} \end{pmatrix} = \left(\begin{pmatrix} a & \frac{b}{4} \\ 4c & d \end{pmatrix}, 1 \right).$$

It follows that $h^{(2)}(gk) = \epsilon_2(k)^{-1}h^{(2)}(g)$ if $\in K(4)$. Therefore $h^{(2)}$, $\mathsf{W}^\nu(h)$ are fixed by right multiplication of $K(4)$.

The representatives of the double coset $\widetilde{B}(\mathbb{Q}_2)\backslash\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)/K(4)$ are

$$\left\{ 1, w, k_1; w\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}, k_1\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}, x = 1, 2, 3 \right\}.$$

So to prove

$$2h^{(2)} = (1 - \sqrt{-1})(\mathsf{W}^0(h) + \mathsf{W}^1(h)),$$

it is enough to check this identity at these elements. This can be done via a tedious computation using the explicit description of the element h. For instance, when $g = 1$, we have

$$h^{(2)}(1) = |2|^{\lambda+1}h(1) = 2^{-\lambda-1}h(1),$$

and

$$\mathsf{W}^0(h)(1) = |2|^{-\lambda-1}h(w) = 2^{\lambda+1}h(w), \quad \mathsf{W}^1(h)(1) = |2|^{-\lambda-1}h(wn(-\frac{1}{2})) = 2^{\lambda+1}h(wn(-\frac{1}{2})).$$

By (the proof of) [5, Lemma 3.6], we have

$$h(1) = 2^{2\lambda}, \quad h(w) = \frac{1+\sqrt{-1}}{4}, \quad h(wn(-\frac{1}{2})) = 0.$$

Therefore

$$2h^{(2)}(1) = (1 - \sqrt{-1})(\mathsf{W}^0(h)(1) + \mathsf{W}^1(h)(1)).$$

We omit the details of the rest of the computation. This proves (1).

To prove (2), it is enough to prove the invariance under $\mathrm{SL}_2(\mathbb{Z}_2)$ as the invariance under $H(\mathbb{Z}_2)$ is straight forward to check. Since $\mathrm{SL}_2(\mathbb{Z}_2)$ is generated by $n(1)$ and w, we only need to consider the action of these two elements.

First we have $\sigma(n(1))\mathsf{W}^0(h) = \mathsf{W}^0(h)$ since $\mathsf{W}^0(h) \in V_0$. We have

$$h(gn(1)wn(-2)m(2)) = h\left(gwn(-2)m(2)\begin{pmatrix} -1 & 1 \\ -4 & 3 \end{pmatrix}\right) = \sqrt{-1}h(gwn(-2)m(2)).$$

By the formulae of the Weil representations, we have

$$\omega_\psi(n(1))\phi^0 = \phi^0, \quad \omega_\psi(n(1))\phi^1 = -\sqrt{-1}\phi^1.$$

It follows that $\mathsf{W}^0(h) \otimes \phi^0 + \mathsf{W}^1(h) \otimes \phi^1$ is invariant under the action of $n(1)$.

We then consider the action of w. Note that $\sigma(w)h^{(2)} = \mathsf{W}^0(h)$ and $\sigma(-1)h = \sqrt{-1}h$, thus by (1)

$$2\sigma(w)\mathsf{W}^0(h) = (1+\sqrt{-1})(\mathsf{W}^0(h) + \mathsf{W}^1(h)).$$

By (1) again, we have

$$2\mathsf{W}^0(h) = (1-\sqrt{-1})(\sqrt{-1}h^{(2)} + \sigma(w)\mathsf{W}^1(h)).$$

It follows that, using (1) one more time, we get

$$2\sigma(w)\mathsf{W}^1(h) = (1+\sqrt{-1})(\mathsf{W}^0(h) - \mathsf{W}^1(h)).$$

By the formulae of the Weil representation, we have

$$\omega_\psi(w)\phi^0 = \frac{1-\sqrt{-1}}{2}(\phi^0 + \phi^1), \quad \omega_\psi(w)\phi^1 = \frac{1-\sqrt{-1}}{2}(\phi^0 - \phi^1).$$

Then (2) follows.

It is not hard to check that $h^{(2)} \otimes \phi$ is invariant under the group

$$\{k \in \mathrm{SL}_2(\mathbb{Z}_2) \mid {}^t k \in K_0(4)\}.$$

Therefore Φ^- is invariant under this group since $\Phi + \Phi^- = h^{(2)} \otimes \phi$. So (3) is equivalent to

$$\sum_{x=0}^{3} \sigma^\psi(n(x))\Phi^- = 0.$$

This can be checked directly by the transformation properties of $\mathsf{W}^\nu(h)$ and ϕ^ν under the action of $n(1)$. Then (3) follows. $\square$

Remark 6.2. This lemma is in fact the local counterpart of Lemma 3.4. The validity of (1) can be seen from the transformation properties in the proof of Lemma 3.4. The validity of (2) can be seen from the fact that Φ is the local component of a Jacobi form of level one.

Lemma 6.3. *We have*

$$\alpha(F, \overline{h^{(2)}}, \phi) = \frac{1}{2}\int_{R(\mathbb{Q}_2)} \langle \pi(r)F, F\rangle \overline{\langle \sigma^\psi(r)\Phi, \Phi\rangle} \mathrm{d}r.$$

Proof. By definition,

$$\alpha(F,\overline{h^{(2)}},\phi)=\int_{R(\mathbb{Q}_2)}\langle\pi(r)F,F\rangle\overline{\langle\sigma^\psi(r)h^{(2)}\otimes\phi,h^{(2)}\otimes\phi\rangle}\mathrm{d}r.$$

By Lemma 6.1(1), we have

$$2\langle\sigma^\psi(r)h^{(2)}\otimes\phi,h^{(2)}\otimes\phi\rangle=\langle\sigma^\psi(r)\Phi,\Phi\rangle+\langle\sigma^\psi(r)\Phi,\Phi^-\rangle+\langle\sigma^\psi(r)\Phi^-,\Phi\rangle+\langle\sigma^\psi(r)\Phi^-,\Phi^-\rangle.$$

Note that F is $\mathrm{SL}_2(\mathbb{Z}_2)$-invariant. The lemma then follows from Lemma 6.1(3). □

Before we proceed, let us make some remarks on the computation in [14]. We are in the case Sp of [14]. The 2-adic case is explicitly excluded in the computation. However, some computations there are valid in the 2-adic case. In fact, any computation that does not refer to the metaplectic group is valid. All the "reduction steps" before the proof of [14, Proposition 2.2.3] are valid as they consider Jacobi groups and their representations, not the metaplectic groups. In our present situation, we use [10] instead of [3] which is valid in the 2-adic case. However, the proof of [14, Proposition 2.2.3] in the case $r=0$ (in the notation of [14]) makes use of theta correspondences and the unramified computation of the doubling zeta integrals on the metaplectic group. Thus it does not work in the present 2-adic situation.

Let $\pi_0=\mathrm{Ind}_{B(\mathbb{Q}_2)}^{\mathrm{SL}_2(\mathbb{Q}_2)}|\cdot|^{\mu_2}$ and $F_0\in\pi_0$ be the $\mathrm{SL}_2(\mathbb{Z}_2)$ fixed vector that satisfies $F_0(1)=1$. By the same argument as in [14, Section 4], we have

$$\alpha(F,\overline{h^{(2)}},\phi)=\frac{1}{2}\frac{\xi(4)}{\xi(1)}\frac{L_\psi(\frac{1}{2},\sigma\times\mu_1)L_\psi(\frac{1}{2},\sigma\times\mu_1^{-1})}{L(1,\pi_0\times\mu_1)L(1,\pi_0\times\mu_1^{-1})}\int_{\mathrm{SL}_2(\mathbb{Q}_2)}\langle\pi_0(g)F_0,F_0\rangle\overline{\langle\sigma^\psi(g)\Phi,\Phi\rangle}\mathrm{d}g. \tag{6.1}$$

We need to compute the integral

$$\int_{\mathrm{SL}_2(\mathbb{Q}_2)}\langle\pi_0(g)F_0,F_0\rangle\overline{\langle\sigma^\psi(g)\Phi,\Phi\rangle}\mathrm{d}g. \tag{6.2}$$

Let

$$\eta=\begin{pmatrix}1&&&\\1&1&&\\&&1&-1\\&&&1\end{pmatrix}\begin{pmatrix}&&1&\\&1&&\\1&&&\\&&&1\end{pmatrix}.$$

Let

$$T_{\mu_2,\lambda,\psi}=\int_{\mathrm{SL}_2(\mathbb{Q}_2)}F_0(g)\overline{\Phi(\eta g)}\mathrm{d}g.$$

Via the same computation as [14, Section 4], we have

$$(6.2)=\frac{\xi(2)}{\xi(1)^2}T_{\mu_2,\lambda,\psi}T_{\mu_2^{-1},\lambda^{-1},\psi^{-1}}. \tag{6.3}$$

Therefore it is enough to calculate $T_{\mu_2,\lambda,\psi}$.

Lemma 6.4. *The support of Φ is contained in $B_R(\mathbb{Q}_2)R(\mathbb{Z}_2)$.*

Proof. By Iwasawa decomposition, we have $R = B(\mathbb{Q}_2)H(\mathbb{Q}_2)R(\mathbb{Z}_2)$. Suppose that $(a,0,0) \in \operatorname{supp}\Phi$. Then for any $b \in \mathbb{Z}_2$, we have

$$\Phi((a,0,0)(0,b,0)) = \Phi((a,0,0)) \neq 0.$$

On the other hand, we have

$$\Phi((a,0,0)(0,b,0)) = \psi(2ab)\Phi((a,0,0)).$$

It follows that $a \in \frac{1}{2}\mathbb{Z}_2$. It is straightforward to check that $\Phi(\frac{1}{2}) = 0$. Therefore $a \in \mathbb{Z}_2$. The lemma then follows. $\square$

It follows from Lemma 6.4 and routine calculation that $F_0(m(t)n(x)k) = |t|^{\mu_2+1}$ and

$$\Phi(\eta m(t)n(x)k) = \Phi(1) \times \begin{cases} |t|^{-\lambda-\frac{3}{2}}\,\mathbf{1}_{\mathbb{Z}_2}(t^{-1}), & |x| \leq 1 \\ |tx|^{-\lambda-\frac{3}{2}}\,\mathbf{1}_{\mathbb{Z}_2}(t^{-1}x^{-1})\psi(t^{-2}x^{-1}), & |x| > 1, \end{cases}$$

where $t \in \mathbb{Q}_2^\times$, $x \in \mathbb{Q}_2$, $k \in \mathrm{SL}_2(\mathbb{Z}_2)$. Then

$$T_{\mu_2,\lambda,\psi} = \overline{\Phi(1)}\left(\int_{|x|\leq 1}\int_{|t|\geq 1}|t|^{-\frac{1}{2}-\lambda+\mu_2}\mathrm{d}t\mathrm{d}x + \int_{|x|>1}\int_{|tx|\geq 1}|t|^{-\frac{1}{2}-\lambda+\mu_2}|x|^{-\frac{3}{2}-\lambda}\psi(t^{-2}x^{-1})\mathrm{d}x\mathrm{d}t\right),$$

where $\mathrm{d}t$ is the measure on $\mathbb{Q}_2^\times$ and $\mathrm{d}x$ is the additive measure on $\mathbb{Q}_2$. We have

$$\int_{|x|\leq 1}\int_{|t|\geq 1}|t|^{-\frac{1}{2}-\lambda+\mu_2}\mathrm{d}t\mathrm{d}x = \sum_{n=0}^{\infty}2^{(-\frac{1}{2}-\lambda+\mu_2)n} = \frac{1}{1-2^{-\frac{1}{2}-\lambda+\mu_2}}.$$

Note that

$$\int_{|x|=2^m}\int_{|t|=2^n}\psi(t^{-2}x^{-1})\mathrm{d}t\mathrm{d}x = \begin{cases} 2^{m-1}, & 2n+m \geq 0; \\ -2^{m-1}, & 2n+m = -1; \\ 0, & 2n+m \leq -2. \end{cases}$$

Therefore

$$\begin{aligned}
&\int_{|x|>1}\int_{|tx|\geq 1}|t|^{-\frac{1}{2}-\lambda+\mu_2}|x|^{-\frac{3}{2}-\lambda}\psi(t^{-2}x^{-1})\mathrm{d}x\mathrm{d}t \\
=&\frac{1}{2}\sum_{m\geq 1}\sum_{n\geq -[\frac{m}{2}]}2^{(-\frac{1}{2}-\lambda+\mu_2)n+(-\frac{1}{2}-\lambda)m} - \frac{1}{2}\sum_{\substack{m\geq 1\\ m \text{ odd}}}2^{\frac{m+1}{2}(\frac{1}{2}+\lambda-\mu_2)-m(\frac{1}{2}+\lambda)} \\
=&\frac{2^{-\frac{1}{2}-\lambda}+2^{-\frac{3}{2}-\lambda-\mu_2}-2^{-1-\mu_2}}{(1-2^{-\frac{1}{2}-\lambda-\mu_2})(1-2^{-\frac{1}{2}-\lambda+\mu_2})}.
\end{aligned}$$

In summary

$$T_{\mu_2,\lambda,\psi} = \overline{\Phi(1)}\frac{(1-2^{-1-\mu_2})(1+2^{-\frac{1}{2}-\lambda})}{(1-2^{-\frac{1}{2}-\lambda-\mu_2})(1-2^{-\frac{1}{2}-\lambda+\mu_2})}. \tag{6.4}$$

Thus

$$(6.2) = \frac{\xi(2)}{\xi(1)^2}|\Phi(1)|^2 \frac{(1-2^{-1-\mu_2})(1+2^{-\frac{1}{2}-\lambda})(1-2^{-1+\mu_2})(1+2^{-\frac{1}{2}+\lambda})}{(1-2^{-\frac{1}{2}-\lambda-\mu_2})(1-2^{-\frac{1}{2}-\lambda+\mu_2})(1-2^{-\frac{1}{2}+\lambda+\mu_2})(1-2^{-\frac{1}{2}+\lambda-\mu_2})}$$
$$= \xi(2)|\Phi(1)|^2 \frac{L_\psi(\frac{1}{2}, \pi_0 \times \sigma)}{L(1, \pi_0, \mathrm{Ad}) L_\psi(1, \sigma, \mathrm{Ad})}.$$

Proof of Proposition 4.3. It follows from (6.1) and the computation above that

$$\frac{\alpha(F, h^{(2)}, \phi)}{\langle F, F\rangle \langle \Phi, \Phi\rangle} = \frac{1}{2}\xi(2)\xi(4) \frac{L_\psi(\frac{1}{2}, \pi \times \sigma)}{L(1, \pi, \mathrm{Ad}) L_\psi(1, \sigma, \mathrm{Ad})}.$$

It is not hard to check that $\langle \Phi, \Phi\rangle = \frac{1}{2}$, $\langle h^{(2)}, h^{(2)}\rangle = \frac{1}{4}$ and $\langle \phi, \phi\rangle = 2$. Proposition 4.3 then follows. □

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