ARITHMETIC THETA LIFTS AND THE ARITHMETIC GAN–GROSS–PRASAD CONJECTURE FOR UNITARY GROUPS

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Abstract. We propose a precise formula relating the height of certain diagonal cycles on the product of unitary Shimura varieties and the central derivative of some tensor product L-functions. This can be viewed as a refinement of the arithmetic Gan–Gross–Prasad conjecture. We use the theory of arithmetic theta lifts to prove some endoscopic cases of it for $\mathrm{U}(2) \times \mathrm{U}(3)$.

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1. Introduction

In 1980s, Gross–Zagier [GZ86] established a formula that relates the Neron–Tate height of Heegner points on modular curves to the central derivative of certain L-functions associated to modular forms. Around the same time, Waldspurger proved a formula, relating toric periods of modular forms to the central value of certain L-functions. Gross put both of these formula in the framework of representation theory in his MSRI lecture in 2001 [Gro04]. In this framework, the formula of Waldspurger concerns the toric periods of automorphic forms on quaternion algebras, while the formula of Gross–Zagier maybe viewed as a formula for the "periods" of "automorphic forms" on the incoherent quaternion algebras. The proof of the most general form of the Gross–Zagier formula given in [YZZ13] has been largely inspired by the proof of Waldspurger's formula.

Gross–Prasad [GP92] formulated a conjecture which generalizes the work of Waldspurger to relate the nonvanishing of $\mathrm{SO}(n)$-periods of automorphic forms on $\mathrm{SO}(n) \times \mathrm{SO}(n+1)$ and the nonvanishing of the central value of certain Rankin–Selberg L-functions, with Waldspurger's formula being the case $n = 2$. Gan, Gross and Prasad [GGP12] further generalized this framework to include all classical groups. These conjectures are usually referred to as the Gan–Gross–Prasad (GGP) conjectures. Parallel to the periods of automorphic forms, a conjectural generalization of the Gross–Zagier formula to higher-dimensional Shimura varieties has been proposed, for instance, in [GGP12, Zha12]. These are generally referred to as the arithmetic Gan–Gross–Prasad conjectures, or arithmetic GGP conjectures for short.

The goal of this paper is to prove some endoscopic cases of the arithmetic GGP conjecture for $\mathrm{U}(2)\times\mathrm{U}(3)$ using theta correspondences and arithmetic theta lifts of Kudla [Kud04] and Liu [Liu11a, Liu11b]. In principle, our argument generalizes to the case of all n, yielding a relation among the GGP conjecture for $\mathrm{U}(n) \times \mathrm{U}(n)$, Liu's conjecture on the arithmetic inner product formula and some endocopic cases of the arithmetic GGP conjecture for $\mathrm{U}(n)\times\mathrm{U}(n+1)$. We stick to the case of $\mathrm{U}(2)\times\mathrm{U}(3)$, as if $n > 2$, Liu's conjectural arithmetic inner product formula is not available currently, and the GGP conjecture for $\mathrm{U}(n) \times \mathrm{U}(n)$ is known only in some cases which is not sufficient for the application of the method of this paper. Moreover, in this situation, we can formulate our main result unconditionally, without appealing to the standard conjectures of Beilinson and Bloch.

We hope that the results of this paper provide some further motivation, in addition to the already amply demonstrated ones, for the study of the GGP conjecture for $\mathrm{U}(n) \times \mathrm{U}(n)$ and the arithmetic inner product formulae.

A byproduct of our investigation is that it enables us to predict a precise conjectural formula for the height and the central derivative of L-functions for $\mathrm{U}(n) \times \mathrm{U}(n+1)$, in the style of the Ichino–Ikeda's conjecture [II10, Har14], as a refinement to the original Gross–Prasad conjecture. Our main result is to verify this formula for $\mathrm{U}(2) \times \mathrm{U}(3)$ in some endoscopic cases. In the appendix, we check that the case of $\mathrm{U}(1) \times \mathrm{U}(2)$ is compatible with the main result of [YZZ13]. It turns out that even the case $\mathrm{U}(1) \times \mathrm{U}(2)$ is not merely a triviality since the formulation of the results in [YZZ13] is different from ours. This also provides strong evidence that the complicated constant involving the measures and the power of two in the precise formula is correct.

1.1. **The conjecture and the main result.** Let us briefly recall the arithmetic GGP conjecture for $\mathrm{U}(n) \times \mathrm{U}(n+1)$. The details will be given in Section 5. Let F be a totally real field and E/F a CM extension. Let $\mathbb{W} \subset \mathbb{V}$ be a pair of *incoherent* Hermitian spaces over $\mathbb{A}_E$ of rank n and $n+1$ respectively. Assume that $\mathbb{W}$ and $\mathbb{V}$ are both positive definite at all infinite places of E. Put $H = \mathrm{U}(\mathbb{W})$ and $\mathbb{G} = \mathrm{U}(\mathbb{W}) \times \mathrm{U}(\mathbb{V})$. These are the reductive groups over $\mathbb{A}_F$. There is a (projective system of) Shimura variety Y of dimension $n-1$ (resp. X of dimension n) attached to $\mathrm{U}(\mathbb{W})$ (resp. $\mathrm{U}(\mathbb{V})$). Put $M = Y \times X$. We have an embedding $Y \to X$ induced by the inclusion $\mathbb{W} \subset \mathbb{V}$. Thus we have a diagonal embedding $Y \to M$ and in this way Y defines a cycle of codimension n in M, which we denote by y. Let $\mathrm{cl} : \mathrm{Ch}^n(M) \to \mathrm{H}^{2n}(M)$ be the cycle class map and $\mathrm{Ch}^n(M)_0$ be the kernel of cl which consists of cohomologically trivial cycles. It is expected that there is a Hecke equivariant projection $\mathrm{Ch}^n(M) \to \mathrm{Ch}^n(M)_0$. We assume such a projection exists and denote by y_0 the image of y under this projection. We are going to construct this y_0 when $n = 1$ or 2.

Let $\mathcal{A}$ be the set of irreducible admissible representations of $\mathbb{G}(\mathbb{A}_F)$ which appear in $\mathrm{H}^{2n-1}(M)$. Note that by definition $\mathbb{G}(F_\infty)$ acts trivially on $\mathrm{H}^{2n-1}(M)$. Then we have a surjective map

$$C_c^\infty(\mathbb{G}(\mathbb{A}_{F,f})) \to \bigoplus_{\pi \in \mathcal{A}} \pi \otimes \widetilde{\pi}.$$

Let us fix an inner product on π so as to identify $\widetilde{\pi}$ with $\overline{\pi}$. Let $\pi \in \mathcal{A}$ and $\varphi \in \pi$. We choose a function $t \in C_c^\infty(\mathbb{G}(\mathbb{A}_{F,f}))$ which maps to $\varphi \otimes \overline{\varphi}$. Let $T(t)$ be the Hecke correspondence on M given by t.

We need to invoke the Beilinson–Bloch height pairing. This is a highly conjectural pairing for the cohomologically trivial cycles. To proceed, we propose the following hypothesis.

Hypothesis 1.1. *We have the following hypothesis of Beilinson and Bloch.*

(1) *The height pairing* $\langle -, - \rangle_{\mathrm{BB}}$ *is well-defined, cf. the Hypothesis (BB1) and (BB2) in Section 2.3.*
(2) *Suppose that* S *is a smooth projective variety defined over a number field and* δ *be a correspondence on* S*. If* δ_* *acts trivially on* $\mathrm{H}^{2r-1}(S)$*, then it acts trivially on* $\mathrm{Ch}^r(S)_0$*.*

The arithmetic GGP conjecture then predicts the following identity

$$\langle T(t)_* y_0, y_0 \rangle_{\mathrm{BB}} = (*) L'(\frac{1}{2}, \pi),$$

where

– $(*)$ is some explicit nonzero constant which we will specify in Conjecture 5.1;
– $L(s, \pi)$ is a certain tensor product L-function attached to π.

By Hypothesis 1.1, the left hand side does not depend on the choice of t, but only on φ. Moreover the height pairing is well-defined.

Theorem 1.2 (provisional form)**.** *Assume* $n = 2$ *and Hypothesis 1.1. Assume that*

– *the Shimura varieties in question are all projective, e.g.* $F \neq \mathbb{Q}$*;*
– *both automorphic representations on* $\mathrm{U}(3)$ *and* $\mathrm{U}(2)$ *are theta lifts from the quasi-split* $\mathrm{U}(2)$*.*

Then the arithmetic GGP conjecture for $\mathrm{U}(2) \times \mathrm{U}(3)$ *holds.*

We refer the readers to Theorem 5.2 for the precise statement of the theorem.

One drawback of this formulation of the theorem is that it is conditional on Hypothesis 1.1 for the 3-folds, especially (2), which is impossible to check even for some very simple varieties, e.g. triple product of smooth projective curves. Therefore we formulate the main result of this paper in a different way, cf. Theorem 5.3. Under Hypothesis 1.1, these two formulations are equivalent. In the formulation of Theorem 5.3, we do not assume Hypothesis 1.1, but only the existence of regular models of X and $X \times Y$. This is of course expected for all surfaces by the conjectural resolution of singularities. In practice, this assumption can be verified when the level of the Picard modular surface is simple.

The main result of this paper should be considered as some "degenerate" case of the arithmetic Gan–Gross-Prasad conjecture. In fact, let us write $\pi = \pi_2 \boxtimes \pi_3$ where π_2 (resp. π_3) is an admissible representation of $\mathrm{U}(2)$ (resp. $\mathrm{U}(3)$). Under the assumption of theorem, there are two irreducible cuspidal automorphic representations σ_1 and σ_2 such that π_2 (resp. π_3) is a theta lift of σ_1 (resp. σ_2), as abstract representations. Therefore the L-function factorizes as

$$L(s, \pi) = L(s, \pi_3 \times \pi_2) = L(s, \sigma_1 \times \sigma_2) L(s, \sigma_1).$$

Here $L(s, \sigma_1 \times \sigma_2)$ is some tensor product L-function of σ_1 and σ_2 and $L(s, \sigma_1)$ is the standard L-function of σ_1 defined by the doubling zeta integrals. There are some twists in these L-functions, but just to fix ideas, let us ignore this issue here. Under our assumption, $L(\frac{1}{2}, \sigma_1) = 0$ as the sign of the functional equation is -1. Therefore

$$L'(\frac{1}{2},\pi) = L(\frac{1}{2},\sigma_1 \times \sigma_2)L'(\frac{1}{2},\sigma_1).$$

This means that the picture on the L-function side is essentially known. More precisely, $L(\frac{1}{2},\sigma_1 \times \sigma_2)$ is the one appearing in the GGP conjecture for $\mathrm{U}(2) \times \mathrm{U}(2)$ and $L'(\frac{1}{2},\sigma_1)$ is the one appearing in Liu's arithmetic inner product formula. So it is not too surprising that the height pairing on M should be reduced to some known height pairings, i.e. a height pairing of arithmetic theta lifts on Y. Indeed, this was the very first observation which led to this paper.

One should also compare Theorem 1.2 to the "degenerate" case of the formula for the central derivative of the triple product L-function [YZZ]. Namely, using the same technique as in this paper, plus the arithmetic inner product formula of Kudla–Rapoport–Yang [KRY06], we should be able to deduce certain degenerate cases of central derivative formula of the triple product L-function, in particular [YZZ, Corollary 1.3.2].

1.2. **The method.** Jacquet–Rallis proposed a relative trace formula approach to the GGP conjecture for $\mathrm{U}(n) \times \mathrm{U}(n+1)$. This is by far the most successful approach. It proves the (nonvanishing part) of the GGP conjecture under the assumption that the representation in question is supercuspidal at some split place [Zha14b, Xue]. Inspired by this approach, W. Zhang proposed a relative trace formula to attack the arithmetic GGP conjecture for $\mathrm{U}(n) \times \mathrm{U}(n+1)$ [Zha12]. As a first step, an arithmetic fundamental lemma was conjectured and proved in the case of $\mathrm{U}(2) \times \mathrm{U}(3)$. A smooth transfer conjecture has been formulated in [RSZ] and verified for $\mathrm{U}(2) \times \mathrm{U}(3)$ in some special cases. These results strongly support the solidity of the relative trace formula approach.

There is a different approach to the GGP conjecture via theta correspondences. This approach proves the GGP conjecture for $\mathrm{SO}(2) \times \mathrm{SO}(3)$ and $\mathrm{SO}(3) \times \mathrm{SO}(4)$ in full generality and is capable of obtaining some endoscopic cases of $\mathrm{U}(n) \times \mathrm{U}(n+1)$. Due to technical limitations, mainly the lack of a fine spectral expansion, the relative trace formula only handles the case where the automorphic representations of $\mathrm{U}(n) \times \mathrm{U}(n+1)$ are stable, i.e. their base change remain cuspidal. Contrary to this, the theta correspondence approach has the limitation that, besides some low rank situations, it only handles certain endoscopic cases. So at present, these two methods seem to be complementary to each other. Of course, the relative trace formula approach is much more powerful and has the potential of proving the conjectures in full generality. Nevertheless, the argument via theta correspondence is still useful, for its clarity and simplicity. Moreover, the method of theta correspondences yields directly precise identities between central L-values and periods.

In this paper, we use arithmetic theta lifts to attack the arithmetic GGP conjecture. We in fact state a more precise version of the conjecture. It is clear that such a formulation is directly borrowed from the conjecture of Ichino–Ikeda [II10]. Our method is again largely inspired by the theta correspondence approach to the GGP conjecture.

We now describe our method. First we introduce some notation. Let H be the quasi-split unitary group in two variables. Then we have a Weil representation ω of $H(\mathbb{A}_F) \times \mathrm{U}(\mathbb{V})(\mathbb{A}_F)$, realized on $\mathcal{S}(\mathbb{V})$. It depends on a nontrivial additive character ψ of $F\backslash\mathbb{A}_F$ and a multiplicative character χ of $E^\times\backslash\mathbb{A}_E^\times$. Write $\pi = \pi_2 \boxtimes \pi_3$ where π_2 (resp. π_3) is an irreducible admissible representation of $\mathrm{U}(\mathbb{W})(\mathbb{A}_F)$ (resp. $\mathrm{U}(\mathbb{V})(\mathbb{A}_F)$). We may assume that $\varphi = \varphi_2 \otimes \varphi_3$ where $\varphi_i \in \pi_i$, $i = 2, 3$. Then $T(t) = T(t_2) \times T(t_3)$ where $T(t_2)$ (resp. $T(t_3)$) is a Hecke operator on Y (resp. X). By assumption, there is an irreducible cuspidal automorphic representation σ_2 of $H(\mathbb{A}_F)$ such that π_3 is a theta lift of σ_2 (as an abstract representation). This means that there is a nonzero $H(\mathbb{A}_F) \times \mathrm{U}(\mathbb{V})(\mathbb{A}_F)$-equivariant map $\overline{\sigma_2} \otimes \omega \to \pi_3$. Let us fix such a map and choose $f_2 \in \sigma_2$ and $\phi_3 \in \mathcal{S}(\mathbb{V})$ so that (f_2, ϕ_3) maps to φ_3. We may further assume that φ_3 has the property that ϕ_3 can be chosen to be of the form $\phi_2 \otimes \phi_1$ where $\phi_2 \in \mathcal{S}(\mathbb{W})$ and $\phi_1 \in \mathcal{S}(\mathbb{A}_E)$. The proof of Theorem 1.2 now proceeds in the following steps. For brevity, we do not pay much attention on the constants, expect for the central values and derivatives of the L-functions.

(1) Interpreting Hecke correspondences in terms of arithmetic theta liftings, cf. Subsection 4.6. Let $\Theta = \Theta_{f_2}^{\phi_3}$ be the arithmetic theta lift from $H(\mathbb{A}_F)$ to X in the sense of Liu [Liu11a]. This is a (formal sum of) divisor(s) on X. We have that $T(t_3)$ and $\Theta \times \Theta$ define the same cohomology class in $\mathrm{H}^4(X \times X)$, cf. Proposition 4.10. Therefore by Hypothesis 1.1, we have

$$\langle (T(t_2) \times T(t_3))_* y_0, y_0 \rangle_{\mathrm{BB}} = \langle (T(t_2) \times \Theta \times \overline{\Theta})_* y_0, y_0 \rangle_{\mathrm{BB}}.$$

(2) Reducing the height pairing on $Y \times X$ to a height pairing on Y, cf. Subsection 2.4. A little computation shows that we have

$$\langle (T(t_2) \times \Theta \times \overline{\Theta})_* y_0, y_0 \rangle_{\mathrm{BB}} = \langle T(t_2)_* (\Theta|_Y)_0, (\overline{\Theta}|_Y)_0 \rangle_{\mathrm{NT}},$$

where $\langle -, - \rangle_{\mathrm{NT}}$ stands for the Neron–Tate height pairing on Y and $(\Theta|_Y)_0$ is the projection of $\Theta|_Y$ to the cohomologically trivial part of $\mathrm{Ch}^1(Y)$. We will prove that the height pairing on the left hand side is well-defined, without assuming Hypothesis 1.1.

(3) A pullback formula for Θ, cf. Subsection 4.4. Let $Z(h, \phi_2)$ be the generating series on $H(\mathbb{A}_F)$ valued in $\mathrm{Ch}^1(Y)$ (c.f. [Liu11a]) and $Z(h, \phi_2)_0$ be its projection the cohomologically trivial part of $\mathrm{Ch}^1(Y)$. Let $\theta(h, \phi_1)$ be the theta function on $H(\mathbb{A}_F)$. We have

$$(\Theta|_Y)_0 = \int_{H(\mathbb{A}_F)} \overline{f_2(h)} Z(h, \phi_2)_0 \theta(h, \phi_1) \mathrm{d}h.$$

An analogous result for the generating series on the symplectic groups and valued in the Chow group of orthogonal Shimura varieties was proved in [YZZ09].

(4) An arithmetic seesaw, cf. Subsection 5.4. Unravelling the definitions, we have

$$\begin{aligned} &\langle T(t_2)_* (\Theta|_Y)_0, (\Theta|_Y)_0 \rangle_{\mathrm{NT}} \\ &= \iint_{H(\mathbb{A})^2} \overline{f_2(h)} f_2(h') \langle T(t_{\varphi_2,\varphi_2})_* Z(h, \phi_2)_0, \overline{Z(h', \phi_2)_0} \rangle_{\mathrm{NT}} \theta(h, \phi_1) \overline{\theta(h', \phi_1)} \mathrm{d}h \mathrm{d}h'. \end{aligned} \tag{1.1}$$

This is the arithmetic analogue of the computation in [Xue16], where we deduce some endoscopic cases of the refined GGP conjecture for $\mathrm{U}(n)\times\mathrm{U}(n+1)$ from the refined GGP conjecture for $\mathrm{U}(n)\times\mathrm{U}(n)$. The seesaw diagram we use is

$$\begin{array}{ccc} \mathrm{U}(2)\times\mathrm{U}(2) & & \mathrm{U}(3) \\ | & \times & | \\ \mathrm{U}(2) & & \mathrm{U}(2)\times\mathrm{U}(1). \end{array}$$

In the case of period integrals, a seesaw argument amounts to changing the order of integration. In our current situation, it is changing order of integration and height pairing.

(5) An arithmetic inner product formula for $\mathrm{U}(2)$, cf. Subsection 4.5. Note that the Neron–Tate height $\langle T(t_2)_* Z(h,\phi_2)_0, \overline{Z(h',\phi_2)_0}\rangle_{\mathrm{NT}}$ defines a cusp form on $H(\mathbb{A}_F)\times H(\mathbb{A}_F)$ which is of the form $f_1(h)\overline{f_1(h')}$. We have the following variant of Liu's inner product formula [Liu11b]:

$$\int_{H(F)\backslash H(\mathbb{A}_F)} \langle T(t_2)_* Z(h,\phi_2)_0, \overline{Z(h,\phi_2)_0}\rangle_{\mathrm{NT}} \mathrm{d}h = L'(\frac{1}{2},\sigma_1)\prod_v Z_v^\natural(\varphi_{2,v},\varphi_{2,v},\phi_{2,v},\phi_{2,v}),$$

where $Z_v^\natural$ stands for the normalized doubling zeta integral.

(6) Making use of the refined GGP conjecture for $\mathrm{U}(2)\times\mathrm{U}(2)$ and the inner product formula to compute (1.1), cf. Subsection 5.4. As noted above, $\langle T(t_2)_* Z(h,\phi_2)_0, Z(h',\phi_2)_0\rangle_{\mathrm{NT}} = f_1(h)\overline{f_1(h')}$ where f_1 is a cusp form on $H(\mathbb{A}_F)$. The integral is computed by the refined GGP conjecture for $\mathrm{U}(2)\times\mathrm{U}(2)$, which is known and can be in fact deduced from the triple product formula. The inner product of f_1 is computed using (the above variant of) Liu's arithmetic inner product formula for $\mathrm{U}(2)$.

Our unconditional formulation of the main theorem, Theorem 5.3, can be extracted from the above steps. Note that Hypothesis 1.1 is used only in the first step. Instead of using the Hecke operators $T(t_3)$ as projectors on the Chow groups, we use arithmetic theta lifts as projectors. Under Hypothesis 1.1 these two projectors are the same. This eliminates the dependence of the main theorem on Hypothesis 1.1.

Remark 1.3. A technical point in our argument is that in the second step, we need to show that $(\Delta_{X,1,*}\Theta)|_Y = 0$, cf. Lemma 5.6. Here $\Delta_{X,1}$ is the first Künneth–Chow component of the surface X, and the map $\Theta\mapsto\Delta_{X,1,*}\Theta$ is a Hecke equivariant projection $\mathrm{Ch}^1(X)\to\mathrm{Ch}^1(X)_0$ where $\mathrm{Ch}^1(X)_0$ is the subgroup of cohomologically trivial divisors on X. Indeed, we even have $\Delta_{1,X,*}\Theta = 0$. This means that arithmetic theta lift from $\mathrm{U}(2)$ in this case does not provide us with nontrivial elements in $\mathrm{Ch}^1(X)_0$. In other words, one does not have a nontrivial Neron–Tate height pairing between any arithmetic theta lift from $\mathrm{U}(2)$ and 0-cycles on the surface.

Remark 1.4. Another technical but important point here is that in the variant of Liu's inner product formula, the local doubling zeta integral is on the group $\mathrm{U}(\mathbb{W})$ whereas in the original formula it is on the group H. So we need to relate the doubling integrals on these two groups. This relation itself and its proof may be of independent interest. It turns out that such a relation is a generalization of the fact that the equal rank local theta correspondence preserves the formal degree in the case of discrete series representations. We refer the readers to Subsection 3.4 for a more detailed discussion.

1.3. **Organization of the paper.** This paper is organized as follows. In Section 2, we review how to construct cohomologically trivial cycle classes and the theory of height pairing. As the theory of height pairing is still highly conjectural, to work with it, we need some working hypothesis. We state these hypothesis in this section. We also study the height pairing on the product of a curve and a surface. The main result is Proposition 2.2. It proves that in some special cases, the height pairing of 1-cycles on the product of a surface and a curve is well defined and can be reduced to the Neron–Tate height pairing on a curve. In Section 3, we review the theory of theta lifts and doubling zeta integrals. The new result is Proposition 3.4, which handles the second technical point mentioned in the previous subsection. In Section 4, we review the theory of arithmetic theta lifts following [Liu11a, Liu11b]. We prove two results. The first is an identity between the Hecke correspondences and the arithmetic theta lifts. The second is a variant of Liu's arithmetic inner product formula. The key input in this variant is Proposition 3.4. Section 5 contains the main results of this paper. We first state the precise form of the arithmetic GGP conjecture. Then combining all results from the previous sections, we prove this conjecture for $\mathrm{U}(2) \times \mathrm{U}(3)$ in the endoscopic case. We state two versions of our main theorem. The version depending on Hypothesis 1.1 is Theorem 5.2. The unconditional version is Theorem 5.3. In the appendix, we check that the arithmetic GGP conjecture, in its precise form, is compatible with the Gross–Zagier formula proved in [YZZ13].

1.4. **Notation.** Throughout this paper, we fix the following notation and convention.

- Let F be a number field and E/F a quadratic extension. We write $\mathbb{A}_{F,f}$ for the group of finite adeles and $F_\infty = \prod_{v|\infty} F_v$. We fix a nontrivial additive character $\psi : F\backslash\mathbb{A}_F \to \mathbb{C}^\times$, such that for each archimedean place v of F, $\psi_v(x) = e^{2\pi i x}$. Put $\psi_E(x) = \psi(\frac{1}{2}\mathrm{Tr}_{E/F}\, x)$. Let $\eta : F^\times\backslash\mathbb{A}_F^\times \to \{\pm 1\}$ be the quadratic character associated to the extension E/F.
- By a Hermitian space $\mathbb{V}$ over $\mathbb{A}_E$, we mean a restricted tensor product $\mathbb{V} = \otimes \mathbb{V}_v$ where $\mathbb{V}_v$ is a Hermitian space over E_v. It is said to be coherent if there is a Hermitian space V over E so that $\mathbb{V} = V \otimes \mathbb{A}_E$. It is said to be incoherent if such a V does not exist.
- By the Hermitian space $\mathbb{A}_E$ (over $\mathbb{A}_E$), we mean the one dimensional hermitian space over $\mathbb{A}_E$, with the Hermitian inner product given by $(x, y) \mapsto x\overline{y}$.
- For any algebraic group G over F, we put $[G] = G(F)\backslash G(\mathbb{A}_F)$.
- For any algebraic variety X of F, we let $\mathrm{Ch}^*(X)$, $\mathrm{Pic}(X)$, $\mathrm{H}^*(X)$ be the Chow group, the Picard group of X and the (Betti) cohomology group of $X(\mathbb{C})$ (for some embedding $F \to \mathbb{C}$ which is clear from the context). Without saying explicitly to the contrary, they all have $\mathbb{C}$ coefficients. Thus we may take complex conjugation of elements in these groups.

1.5. **Measures.** Let us fix some measures. Recall that we have fixed a nontrivial additive character $\psi : F\backslash\mathbb{A}_F \to \mathbb{C}^\times$.

For any place v of F, let V be a Hermitian space over E_v of dimension n. Let $\mathfrak{u}(V)$ be the Lie algebra of $\mathrm{U}(V)$. Let $\mathfrak{c}_v : \mathfrak{u}(V) \to \mathrm{U}(V)$ be the Cayley transform, namely,

$$\mathfrak{c}_v(X) = (1+X)(1-X)^{-1}, \quad X \in \mathfrak{u}(V).$$

We have a self-dual measure on $\mathfrak{u}(V)$ and we let $\mathrm{d}'h_v$ be the unique measure on $\mathrm{U}(V)$ so that the Caylay transform is measure preserving. Suppose that $n = 2r$. Then this measure satisfies the property that

$$\int_{\mathsf{Herm}_{2r}(E_v)} \left(\int_{V^{2r}} \phi(x)\psi(\mathrm{Tr}\, nQ(x))\mathrm{d}x \right) \psi(-\mathrm{Tr}\, nQ)\mathrm{d}n = \gamma_V \int_{\mathrm{U}(V)(F_v)} \phi(h_v^{-1}x_Q)\mathrm{d}'h_v,$$

where Herm_{2r} stands for the space of $2r \times 2r$ Hermitian matrices, $\mathrm{d}T$ is the self-dual measure on Herm_{2r}, and x_Q is any fixed element in V^{2r} with $Q(x) = Q$ where $Q(x)$ stands for the moment matrix of x. Put

$$\mathrm{d}h_v = L(1, \eta_v)\zeta_{F_v}(2) \cdots L(n, \eta_v^n)\mathrm{d}'h_v.$$

We shall call $\mathrm{d}'h_v$ the unnormalized local measure and $\mathrm{d}h_v$ the normalized local measure. Thus the normalized local measure coincides with the measure $\mathrm{d}'h_{\mathfrak{p}}$ in [Liu12, Definition 4.3.3] (in the notation there).

Let V be a Hermitian space over E of dimension n. Then the Tamagawa measure on $[\mathrm{U}(V)]$ equals

$$(L(1, \eta) \cdots L(n, \eta^n))^{-1} \prod_v \mathrm{d}h_v.$$

Let $\mathbb{V}$ be an incoherent Hermitian space over $\mathbb{A}_E$ of dimension n. By abuse of terminology, we call the measure

$$(L(1, \eta)\zeta(2) \cdots L(n, \eta^n))^{-1} \prod_v \mathrm{d}h_v$$

the Tamagawa measure on $\mathrm{U}(\mathbb{V})(\mathbb{A}_F)$.

Let $K \subset \mathrm{U}(V)(\mathbb{A}_{F,f})$ an open compact subgroup. By $\mathrm{vol}\, K$, we mean the volume of K with respect to the measure

$$(2L(1, \eta)\zeta(2) \cdots L(n, \eta^n))^{-1} \prod_{v \text{ finite}} \mathrm{d}h_v,$$

where $\mathrm{d}h_v$ is the normalized local measure at v. This coincides with the measure given in [Liu11a, Definition 4.3.3]. The volume of K computed using other measures will be denoted by $\mathrm{vol}'\, K$.

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2. The height pairing

The goal of this section is to review the (conjectural) height pairing of Beilinson–Bloch. We also study the height pairing on a product of a curve and a surface. In this case, the height pairing can be defined under some mild conditions for a large class of cohomologically trivial cycles. We suggest the readers look at only Subsection 2.1 and the statement of Proposition 2.2 for the first reading.

2.1. **Trivializing cohomology classes.** In this subsection, we review how to construct cohomologically trivial cycle classes in some low dimensional cases.

Let X be a smooth projective variety over F of dimension n. Let $\mathrm{Ch}^i(X)$ (resp. $\mathrm{Ch}_i(X)$) be $\mathbb{C}$-vector space of codimension i (resp. dimension i) cycles on X. There is an intersection paring between $\mathrm{Ch}^*(X)$ and $\mathrm{Ch}_*(X)$, which we denote by $\alpha \cdot \beta$, $\alpha \in \mathrm{Ch}^*(X)$, $\beta \in \mathrm{Ch}_*(X)$.

Let $\mathrm{cl} : \mathrm{Ch}^i(X) \to \mathrm{H}^{2i}(X)$ be the cycle class map and $\mathrm{Ch}^i(X)_0$ be the kernel of cl. It is expected that there is a splitting

$$\mathrm{Ch}^i(X) \simeq \mathrm{Ch}^i(X)_0 \oplus \mathrm{Im}\,\mathrm{cl}\,.$$

In the case X being a Shimura variety, it is expected that this splitting is Hecke equivariant. If X is the Shimura variety attached to a unitary group, W. Zhang recently constructs a candidate of it using Hecke operators. His construction indeed gives a splitting if we assume Hypothesis 1.1. In certain low dimensional cases, we may also use Künneth–Chow decomposition to construct such a splitting. Even though it is very hard to show the existence of such a decomposition in higher dimensions, in the low dimensional cases, it has the advantage of being concrete and geometric. The idea of using Künneth–Chow decomposition to trivialize cohomology classes in the low dimensional cases is also due to W. Zhang [Zha].

Let Δ be the diagonal cycle in $X \times X$. By a Künneth–Chow decomposition, we mean a sum

$$\Delta = \Delta_{X,0} + \Delta_{X,1} + \cdots \Delta_{X,2n} \in \mathrm{Ch}^n(X \times X),$$

such that the natural map $\Delta_{X,i,*} : \mathrm{H}^*(X) \to \mathrm{H}^*(X)$ is the projection to the i-th component. We call $\Delta_{X,i}$ the i-th Künneth–Chow component of X. When there is no confusion, we write Δ_i instead of $\Delta_{X,i}$. The existence of the Künneth–Chow decomposition is one of the standard conjectures on algebraic cycles. The essentially known cases are curves and surfaces. Let $z \in \mathrm{Ch}^r(X)$ be a codimension r cycle on X. Then it follows from the definition of the Künneth–Chow decomposition that $\Delta_{X,2r-1,*}z \in \mathrm{Ch}^r(X)_0$. This defines a map

$$\mathrm{Ch}^r(X) \to \mathrm{Ch}^r(X)_0, \quad z \mapsto \Delta_{X,2r-1,*}z,$$

and for some good choice of the Künneth–Chow decomposition, it is expected to be the splitting that we are looking for.

Let us now recall the construction of Künneth–Chow decomposition in the low dimensional cases.

Suppose that $n = 1$, i.e. X is a smooth projective curve. Choose an ample class $\xi \in \mathrm{Pic}(X)$ of degree one and put

$$\Delta_0 = X \times \xi, \quad \Delta_2 = \xi \times X, \quad \Delta_1 = \Delta - \Delta_0 - \Delta_2.$$

It is well-known (and easy to check) that $\Delta = \Delta_0 + \Delta_1 + \Delta_2$ is a Künneth–Chow decomposition.

Suppose that $n = 2$, i.e. X is a smooth projective surface. Choose an ample class $\xi \in \mathrm{Pic}(X)$ and let $e = (\deg \xi \cdot \xi)^{-1}(\xi \cdot \xi) \in \mathrm{Ch}^2(X)$. Let $\mathrm{Alb}(X)$ be the Albanese variety of X and $\underline{\mathrm{Pic}}^0(X)$ be the neutral component of the Picard variety of X. Let $\alpha : \underline{\mathrm{Pic}}^0(X) \to \mathrm{Alb}(X)$ be the isogeny given by $L \mapsto L \cdot \xi$ and $\alpha^\vee : \mathrm{Alb}(X) \to \underline{\mathrm{Pic}}^0(X)$ the dual isogeny. Assume that the degree of α is d. Let $j : X \to \underline{\mathrm{Pic}}^0(X)$ be the composition of the natural map $X \to \mathrm{Alb}(X)$ given by $x \mapsto x - e$ and the isogeny $\alpha^\vee$. Then we have a morphism

$$j \times 1 : X \times X \to \underline{\mathrm{Pic}}^0(X) \times X,$$

where 1 stands for the identity morphism. Let $\mathcal{P}$ be the Poincare bundle on $\underline{\mathrm{Pic}}^0(X) \times X$ and $\beta = (j \times 1)^*\mathcal{P}$. Put

$$\Delta_0 = X \times e,\ \Delta_1 = \frac{1}{d} p_1^*\xi \cdot \beta,\ \Delta_3 = {}^t\Delta_1,\ \Delta_4 = e \times X,\ \Delta_2 = \Delta - \Delta_0 - \Delta_1 - \Delta_3 - \Delta_4,$$

where $p_1 : X \times X \to X$ is the projection to the first factor. Then $\Delta = \Delta_0 + \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4$ is a Künneth–Chow decomposition for X.

Suppose that $X = C \times S$ where C is a smooth projective curve and S is a smooth projective surface. Choose an ample class $\xi \in \mathrm{Pic}(X)$. Then we have a Künneth–Chow decomposition for S (with respect to ξ) and C (with respect to $\xi|_C$) respectively. Let

$$\Delta_{X,i} = \sum_{j=0}^{i} \Delta_{C,j} \times \Delta_{S,i-j}, \quad i = 0, 1, \cdots, 6$$

Then $\Delta_X = \sum_{i=0}^6 \Delta_{X,i}$ is a Künneth–Chow decomposition.

2.2. **Arithmetic intersection theory.** By an arithmetic variety of dimension $n+1$ over $\mathfrak{o}_F$, we mean an integral scheme $\mathcal{X}$, projective and flat over $\mathfrak{o}_F$ such that the generic fiber $\mathcal{X}_F = \mathcal{X} \times \mathrm{Spec}\, F$ is smooth. We mainly follow the exposition in [Zha10, Section 2.1].

By a (homological) cycle of dimension p, we mean a pair $\widehat{Z} = (Z, g)$ where Z is a finite linear combination of integral closed subschemes of $\mathcal{X}$ of dimension p and $g = \{g_\iota\}$, where ι ranges over all archimedean places of F, is a collection of Green currents of Z. Recall that this means that for each archimedean place ι of F, we have

$$\mathrm{curv}(g_\iota) = \frac{\partial\overline{\partial}}{\pi i} g_\iota + \delta_{Z_\iota(\mathbb{C})}$$

is a smooth form on $X_\iota(\mathbb{C})$, where Z_ι and X_ι are base change of Z and X to $\mathbb{C}$ along the embedding $\iota : F \to \mathbb{C}$ respectively. An cycle is called vertical if Z is contained in the closed fibers of $\mathcal{X}$. We define the (homological) arithmetic cycles to be the $\mathbb{C}$-linear combination of cycles, modulo the relations that $(\mathrm{div}(f), -\log|f|) = 0$ for all f being rational function on some closed subschemes $\mathcal{Y}$

of $\mathcal{X}$ and that $(0, \partial\alpha + \overline{\partial}\beta) = 0$, $a(0,g) - (0, ag) = 0$. For a morphism $\phi : \mathcal{X} \to \mathcal{Y}$, we may define the pushforward $\phi_* : \mathrm{Ch}_*(\mathcal{X}) \to \mathrm{Ch}_*(\mathcal{Y})$ if ϕ is proper and generically smooth and the pullback $\phi^* : \mathrm{Ch}_*(\mathcal{Y}) \to \mathrm{Ch}_*(\mathcal{X})$ if ϕ is flat.

Let $\widehat{K}(\mathcal{X})$ be the arithmetic K-group of Hermitian vector bundles and smooth forms (with $\mathbb{C}$ coefficients) modulo the usual secondary Chern class relation for exact sequences. Then we have an arithmetic Chern character

$$\widehat{\mathrm{ch}} : \widehat{K}(\mathcal{X}) \to \mathrm{End}(\widehat{\mathrm{Ch}}_*(\mathcal{X})).$$

This is the usual Chern character for Hermitian vector bundles and is given by the following formula for smooth forms α on $\mathcal{X}(\mathbb{C})$:

$$\widehat{\mathrm{ch}}(\alpha)(Z, g) = (0, \alpha \wedge \mathrm{curv}(g)),$$

We define the (cohomological) arithmetic Chow group $\widehat{\mathrm{Ch}}^*(\mathcal{X})$ to be the quotient of $\widehat{K}(\mathcal{X})$ by the subgroup of elements t with the property that $\widehat{\mathrm{ch}}(\phi^* t) = 0$ for any morphism $\phi : \mathcal{Y} \to \mathcal{X}$. If $\mathcal{X}$ is regular, then $\widehat{\mathrm{ch}}$ is an isomorphism and we have $\widehat{K}(\mathcal{X}) \simeq \widehat{\mathrm{Ch}}^*(\mathcal{X}) \simeq \widehat{\mathrm{Ch}}_*(\mathcal{X})$. There is a natural intersection pairing

$$\widehat{\mathrm{Ch}}^p(\mathcal{X}) \times \widehat{\mathrm{Ch}}_q(\mathcal{X}) \to \widehat{\mathrm{Ch}}_{q-p}(\mathcal{X}).$$

If $\mathcal{X}$ is regular, the intersection pairing makes $\widehat{\mathrm{Ch}}_*(\mathcal{X})$ a commutative graded ring.

There is a degree map given by

$$\widehat{\deg} : \widehat{\mathrm{Ch}}_0(\mathcal{X}) \to \widehat{\mathrm{Ch}}_0(\mathrm{Spec}\, \mathfrak{o}_F) \to \mathbb{C}.$$

where the first map is given by pushing forward via the structure morphism and the second one is the usual degree map. Without saying to the contrary, we implicitly compose the intersection pairing $\widehat{\mathrm{Ch}}^p(\mathcal{X}) \times \widehat{\mathrm{Ch}}_p(\mathcal{X}) \to \widehat{\mathrm{Ch}}_0(\mathcal{X})$ with the degree map so that we end up with a complex number.

Let $\widehat{\mathcal{L}}$ be a Hermitian line bundle. Then we have $\widehat{c}_1(\widehat{\mathcal{L}}) \in \mathrm{End}(\widehat{\mathrm{Ch}}_*(\mathcal{X}))$ such that $\widehat{\mathrm{ch}}(\widehat{\mathcal{L}}) = \exp(\widehat{c}_1(\widehat{\mathcal{L}}))$. Then for Hermitian line bundles $\widehat{\mathcal{L}_1}, \cdots, \widehat{\mathcal{L}_k}$ and $\alpha \in \widehat{\mathrm{Ch}}_k(\mathcal{X})$, we have the intersection number

$$\widehat{c}_1(\widehat{\mathcal{L}_1}) \cdots \widehat{c}_1(\widehat{\mathcal{L}_k}) \cdot \alpha.$$

To simplify notation, we usually write $\widehat{\mathcal{L}_1} \cdot \cdots \cdot \widehat{\mathcal{L}_k}$ for $\widehat{c}_1(\widehat{\mathcal{L}_1}) \cdots \widehat{c}_1(\widehat{\mathcal{L}_k})$ and write $\widehat{\mathcal{L}_1} \cdot \cdots \cdot \widehat{\mathcal{L}_k} \cdot \alpha$ for $\widehat{c}_1(\widehat{\mathcal{L}_1}) \cdots \widehat{c}_1(\widehat{\mathcal{L}_k}) \cdot \alpha$.

Assume that $\mathcal{X}$ is regular. By an arithmetic correspondence, we mean an element in $\widehat{\mathrm{Ch}}_*(\mathcal{X} \times \mathcal{X})$, say $\widehat{\Gamma} = (\Gamma, g)$, with the property that for any smooth form ω on $X(\mathbb{C})$, the forms $p_{2,*}(\mathrm{curv}(g) \wedge p_1^* \omega)$ and $p_{1,*}(\mathrm{curv}(g) \wedge p_2^* \omega)$ are smooth. It defines endomorphisms $\widehat{\Gamma}_*$ and $\widehat{\Gamma}^*$ via the usual formulae. Moreover if $\alpha, \beta \in \widehat{\mathrm{Ch}}_*(\mathcal{X})$, we have $\alpha \cdot \widehat{\Gamma}_* \beta = \widehat{\Gamma}^* \alpha \cdot \beta$.

2.3. **Height pairing: Generalities.** Again we follow the exposition of [Zha10, Section 2.1]. Let X be a smooth projective variety over a number field F of dimension $2d+1$. Let $\alpha \in \mathrm{Ch}^i(X)_0$ and $\beta \in \mathrm{Ch}_{i-1}(X)_0$. Beilinson and Bloch conditionally define a height pairing $\langle \alpha, \beta \rangle_{\mathrm{BB}}$. The definition is made under the following two hypothesis.

(BB1) Any model $\mathcal{X}'$ of X over $\mathfrak{o}_F$ can be dominated by a regular model $\mathcal{X}$, possibly after a finite extension of F.

(BB2) At least one of α and β can be extended to an arithmetically flat cycle on a regular model $\mathcal{X}$ of X.

Here by an arithmetically flat cycle, we mean an element $(Z, g) \in \widehat{\mathrm{Ch}}_*(\mathcal{X})$ such that restriction to any vertical component in $\mathcal{X}$ is numberically trivial and $\mathrm{curv}\, g_\iota = 0$ for all archimedean places ι of F.

While both hypothesis can be verified for $d = 0$ and are expected to hold in general, they are wide open when $d > 0$.

With these two hypothesis, the height pairing can be defined as follows. Let $\mathcal{X}$ be a regular model and $\widehat{\alpha} \in \widehat{\mathrm{Ch}}_{2d-i}(\mathcal{X})$, $\widehat{\beta} \in \widehat{\mathrm{Ch}}_i(\mathcal{X})$ be extensions of α and β respectively. Assume that $\widehat{\alpha}$ is arithmetically flat. Then define

$$\langle \widehat{\alpha}, \widehat{\beta} \rangle_{\mathrm{BB}} = \widehat{\alpha} \cdot \widehat{\beta},$$

where the right hand side is the arithmetic intersection taken on $\mathcal{X}$. Note that the height pairing is symmetric bilinear, not Hermitian.

Lemma 2.1. *Let $\alpha \in \mathrm{Ch}_*(X)_0, \beta \in \mathrm{Ch}^*(X)_0$, and $\Gamma \in \mathrm{Ch}_*(X \times X)$ be a correspondence. Let $\mathcal{X}$ be a regular model of X. Suppose that $\widehat{\beta}$ is an arithmetically flat extension of β to $\mathcal{X}$ and $\widehat{\Gamma}$ be any arithmetic correspondence on $\mathcal{X} \times \mathcal{X}$ extending Γ. Then*

(1) *$\widehat{\Gamma}_* \widehat{\beta}$ is arithmetically flat;*

(2) *$\langle \alpha, \Gamma_* \beta \rangle_{\mathrm{BB}} = \langle \Gamma^* \alpha, \beta \rangle_{\mathrm{BB}}$.*

Proof. Let V be a vertical cycle in $\mathcal{X}$. Then $\widehat{\Gamma}_* \widehat{\beta} \cdot V = \widehat{\beta} \cdot \widehat{\Gamma}^* V$. Again $\widehat{\Gamma}^* V$ is vertical. So statement (1) follows. The second statement follows easily from the first one. Let $\widehat{\alpha}$ be any extension of α to $\mathcal{X}$. Then

$$\langle \alpha, \Gamma_* \beta \rangle_{\mathrm{BB}} = \widehat{\alpha} \cdot \widehat{\Gamma}_* \widehat{\beta} = \widehat{\Gamma}^* \widehat{\alpha} \cdot \widehat{\beta} = \langle \Gamma^* \alpha, \beta \rangle_{\mathrm{BB}}.$$

This proves the second statement. ☐

2.4. **Height pairing: the case of a product of a curve and a surface.** In this subsection, let Y be a smooth curve and X be a smooth surface with an embedding $Y \to X$. Let us fix an ample class on X and define Künneth–Chow decomposition for X, Y and $X \times Y$ using it. Let $\Delta_{?,i}$ be the i-th Künneth–Chow component for the variety ?. The curve Y defines a cycle in $\mathrm{Ch}^2(X \times Y)$, which we denote by y. Then $\Delta_{X \times Y, 3, *} y \in \mathrm{Ch}^2(X \times Y)_0$.

Proposition 2.2. *Let L_1 be a line bundle on X and L_2 be a degree zero line bundle on Y. Assume that X and Y have regular models $\mathcal{X}$ and $\mathcal{Y}$ respectively over $\mathfrak{o}_F$ and that $X \times Y$ has a regular model $\mathcal{Z}$ with a dominant map $\pi : \mathcal{Z} \to \mathcal{X} \times \mathcal{Y}$. We (by abuse of notation) denote by p_1 both the projections $X \times Y \to X$ and $\mathcal{X} \times \mathcal{Y} \to \mathcal{X}$, and by p_2 both the projections $X \times Y \to Y$ and*

$\mathcal{X}\times\mathcal{Y}\to\mathcal{Y}$. Let $\widehat{\mathcal{L}_2}$ be an arithmetically flat Hermitian line bundle on $\mathcal{Y}$ extending L_2. Let $\widehat{\mathcal{L}_1}$ be any Hermitian line bundle on $\mathcal{X}$ that extends L_1. Then

$$\pi^*(p_1^*\widehat{\mathcal{L}_1}\cdot p_2^*\widehat{\mathcal{L}_2})\in\widehat{\mathrm{Ch}}^2(\mathcal{Z})$$

is arithmetically flat and it extends $p_1^*L_1\cdot p_2^*L_2$. In particular, the height pairing

$$\langle p_1^*L_1\cdot p_2^*L_2,\Delta_{X\times Y,3,*}y\rangle_{\mathrm{BB}}$$

is well-defined. Moreover, we have

$$\langle p_1^*L_1\cdot p_2^*L_2,\Delta_{X\times Y,3,*}y\rangle_{\mathrm{BB}}=\langle\Delta_{Y,1,*}(\Delta_{X,2,*}L_1)|_Y,L_2\rangle_{\mathrm{NT}},$$

where the right hand side is the Neron–Tate height on Y.

Proof. Any irreducible vertical 2-cycle V in $\mathcal{X}\times\mathcal{Y}$ is contained in $X_1\times Y_1$ where X_1 is an irreducible component of a closed fiber of $\mathcal{X}$ and Y_1 is an irreducible component of a closed fiber of $\mathcal{Y}$. The restriction $(p_1^*\widehat{\mathcal{L}_1}\cdot p_2^*\widehat{\mathcal{L}_2})|_{X_1\times Y_1}$ is numerically trivial since $\widehat{\mathcal{L}_2}|_{Y_1}$ is. This shows that the restriction of $\pi^*(p_1^*\widehat{\mathcal{L}_1}\cdot p_2^*\widehat{\mathcal{L}_2})$ to any vertical component of $\mathcal{Z}$ is numerically trivial. The curvature of $\pi^*(p_1^*\widehat{\mathcal{L}_1}\cdot p_2^*\widehat{\mathcal{L}_2})$ is zero since it is the product of those of $\widehat{\mathcal{L}_1}$ and $\widehat{\mathcal{L}_2}$ and the curvature of $\widehat{\mathcal{L}_2}$ is zero. This proves that $\pi^*(p_1^*\widehat{\mathcal{L}_1}\cdot p_2^*\widehat{\mathcal{L}_2})$ is arithmetically flat and the height pairing $\langle p_1^*L_1\cdot p_2^*L_2,\Delta_{X\times Y,3,*}y\rangle_{\mathrm{BB}}$ is well-defined.

It remains to calculate the height pairing. First by definition we have

$$\Delta_{X\times Y,3}=\Delta_{X,3}\times\Delta_{Y,0}+\Delta_{X,2}\times\Delta_{Y,1}+\Delta_{X,1}\times\Delta_{Y,2}.$$

The terms on the right hand side are idempotents. By Lemma 2.1, we have

$$\langle p_1^*L_1\cdot p_2^*L_2,(\Delta_{X,3}\times\Delta_{Y,0})_*y\rangle_{\mathrm{BB}}=\langle(\Delta_{X,3}\times\Delta_{Y,0})^*(p_1^*L_1\cdot p_2^*L_2),(\Delta_{X,3}\times\Delta_{Y,0})_*y\rangle_{\mathrm{BB}}.$$

Since L_2 is cohomologically trivial on Y, $\Delta_{Y,0}^*L_2=0$. Thus the above expression equals zero. Similarly we have

$$\langle p_1^*L_1\cdot p_2^*L_2,(\Delta_{X,1}\times\Delta_{Y,2})_*y\rangle_{\mathrm{BB}}=0.$$

Therefore

$$\langle p_1^*L_1\cdot p_2^*L_2,\Delta_{X\times Y,3,*}y\rangle_{\mathrm{BB}}=\langle p_1^*L_1\cdot p_2^*L_2,(\Delta_{X,2}\times\Delta_{Y,1})_*y\rangle_{\mathrm{BB}}.$$

Put $\widetilde{p_i}=p_i\circ\pi$, $i=1,2$. Let $\widetilde{\mathcal{Y}}$ be the normalization of Y in $\mathcal{Z}$ which defines a 2-cycle which we denote by $[\widetilde{\mathcal{Y}}]$. Let $q:\widetilde{\mathcal{Y}}'\to\widetilde{\mathcal{Y}}$ be a regular model of Y which dominates $\widetilde{\mathcal{Y}}$. Let $\widehat{\Delta_{X,2}}$ (resp. $\widehat{\Delta_{Y,1}}$) be any arithmetic correspondence on $\mathcal{X}$ (resp. $\mathcal{Y}$) which extends $\Delta_{X,2}$ (resp. $\Delta_{Y,1}$). Choose an arithmetic correspondence $\widehat{\Delta_{21}}$ on $\mathcal{Z}\times\mathcal{Z}$ which extends $\Delta_{X,2}\times\Delta_{Y,1}$. By definition,

$$\langle p_1^*L_1\cdot p_2^*L_2,(\Delta_{X,2}\times\Delta_{Y,1})_*y\rangle_{\mathrm{BB}}=\widetilde{p_1}^*\widehat{\mathcal{L}_1}\cdot\widetilde{p_2}^*\widehat{\mathcal{L}_2}\cdot\widehat{\Delta_{21}}_*[\widetilde{\mathcal{Y}}]=\widehat{\Delta_{21}}^*(\widetilde{p_1}^*\widehat{\mathcal{L}_1}\cdot\widetilde{p_2}^*\widehat{\mathcal{L}_2})\cdot[\widetilde{\mathcal{Y}}].$$

We claim that the last term equals $\widetilde{p_1}^*(\widehat{\Delta_{X,2}}^*\widehat{\mathcal{L}_1})\cdot\widetilde{p_2}^*(\widehat{\Delta_{Y,1}}^*\widehat{\mathcal{L}_2})\cdot[\widetilde{\mathcal{Y}}]$. Indeed, both

$$\widetilde{p_1}^*(\widehat{\Delta_{X,2}}^*\widehat{\mathcal{L}_1})\cdot\widetilde{p_2}^*(\widehat{\Delta_{Y,1}}^*\widehat{\mathcal{L}_2})\cdot[\widetilde{\mathcal{Y}}],\quad\widehat{\Delta_{21}}^*(\widetilde{p_1}^*\widehat{\mathcal{L}_1}\cdot\widetilde{p_2}^*\widehat{\mathcal{L}_2})\cdot[\widetilde{\mathcal{Y}}]$$

are arithmetically flat and their difference is a vertical 2-cycle which is again arithmetically flat. Denote by V this difference. Then $V \cdot [\widetilde{\mathcal{Y}}] = \widehat{\deg}(V|_{\widetilde{\mathcal{Y}}'}) = 0$. This proves our claim.

Thus

$$\langle p_1^* L_1 \cdot p_2^* L_2, (\Delta_{X,2} \times \Delta_{Y,1})_* Y\rangle_{\mathrm{BB}} = \widetilde{p_1}^* (\widehat{\Delta_{X,2}}^* \widehat{\mathcal{L}_1}) \cdot \widetilde{p_2}^* (\widehat{\Delta_{Y,1}}^* \widehat{\mathcal{L}_2}) \cdot [\widetilde{\mathcal{Y}}].$$

This equals

$$\pi^* (p_1^* (\widehat{\Delta_{X,2}}^* \widehat{\mathcal{L}_1})) \cdot p_2^* (\widehat{\Delta_{Y,1}}^* \widehat{\mathcal{L}_2}) \cdot [\widetilde{\mathcal{Y}}] = p_1^* (\widehat{\Delta_{X,2}}^* \widehat{\mathcal{L}_1}) \cdot p_2^* (\widehat{\Delta_{Y,1}}^* \widehat{\mathcal{L}_2}) \cdot \pi_* [\widetilde{\mathcal{Y}}].$$

By definition, the right hand side equals

$$(\widehat{\Delta_{X,2}}^* \widehat{\mathcal{L}_1})|_{\widetilde{\mathcal{Y}}'} \cdot (\widehat{\Delta_{Y,1}}^* \widehat{\mathcal{L}_2})|_{\widetilde{\mathcal{Y}}'}.$$

Note that both $\mathcal{Y}$ and $\widetilde{\mathcal{Y}}'$ are regular models of Y. We then conclude that

$$\langle p_1^* L_1 \cdot p_2^* L_2, (\Delta_{X,2} \times \Delta_{Y,1})_* y\rangle_{\mathrm{BB}} = q_*((\widehat{\Delta_{X,2}}^* \widehat{\mathcal{L}_1})|_{\widetilde{\mathcal{Y}}'}) \cdot \widehat{\Delta_{Y,1}}^* \widehat{\mathcal{L}_2} = \widehat{\Delta_{Y,1}}_* (q_*((\widehat{\Delta_{X,2}}^* \widehat{\mathcal{L}_1})|_{\widetilde{\mathcal{Y}}'})) \cdot \widehat{\mathcal{L}_2}.$$

Note that $\widehat{\Delta_{Y,1}}_* (q_*((\widehat{\Delta_{X,2}}^* \widehat{\mathcal{L}_1})|_{\widetilde{\mathcal{Y}}'}))$ is an extension of $\Delta_{Y,1,*}(\Delta_{X,2,*} L_1)|_Y$ and $\widehat{\mathcal{L}_2}$ is an arithmetic flat extension of L_2. Thus by the definition of the Neron–Tate height pairing on Y, we have

$$\langle p_1^* L_1 \cdot p_2^* L_2, (\Delta_{X,2} \times \Delta_{Y,1})_* y\rangle_{\mathrm{BB}} = \langle \Delta_{Y,1,*}(\Delta_{X,2,*} L_1)|_Y, L_2\rangle_{\mathrm{NT}}.$$

This proves the proposition. □

3. Theta lifts

We review the theory of theta lifts and doubling zeta integrals in this section. Most of the results are not new. One exception is Proposition 3.4, which relates the doubling zeta integrals on different unitary groups of the same rank.

3.1. **Weil representations.**

Let H_r be the quasi-split unitary group of rank $2r$ over F, i.e. the unitary group attached to the skew-Hermitian form given by the matrix

$$\begin{pmatrix} & 1_r \\ -1_r & \end{pmatrix}.$$

Let v be a place of F. Let $(V_v, (-,-))$ be a Hermitian space over E_v of dimension m and $\mathrm{U}(V_v)$ be the corresponding unitary group. Fix a character $\chi_v : E_v^\times \to \mathbb{C}^\times$ so that $\chi_v|_{F_v^\times} = \eta_v^m$. Then we have the Weil representation of $H_r(F_v) \times \mathrm{U}(V_v)$, realized on $\mathcal{S}(V_v^r)$, the Schwartz space of V_v^r. We denote it by ω_{χ_v} or ω_v for short when there is no confusion with the characters. It is given by the following formulae.

- $\omega_{\chi_v}(n(b))\phi(x) = \psi_v(\mathrm{Tr}\, bQ(x))\phi(x)$;
- $\omega_{\chi_v}(m(a))\phi(x) = |\det a|_{E_v}^{\frac{m}{2}} \chi_v(\det a)\phi(xa)$;
- $\omega_{\chi_v}(w)\phi(x) = \gamma_{V_v} \widehat{\phi}(x)$;
- $\omega_{\chi_v}(h)\phi(x) = \phi(h^{-1}x)$;

where

$$n(b) = \begin{pmatrix} 1_r & b \\ & 1_r \end{pmatrix},\ m(a) = \begin{pmatrix} a & \\ & {}^t\overline{a}^{-1} \end{pmatrix},\ w = \begin{pmatrix} & 1_r \\ -1_r & \end{pmatrix} \in H_r(F_v); \quad h \in \mathrm{U}(V_v)(F_v),$$

and $x = (x_1, \cdots, x_r) \in V^r$, $Q(x) = \frac{1}{2}((x_i, x_j))_{1 \le i,j \le r}$ is the moment matrix of x, $\widehat{\phi}$ is the Fourier transform defined by

$$\widehat{\phi}(x) = \int_{V_v^r} \phi(y)\psi_E((x,y))\mathrm{d}y,$$

using the self-dual measure on V_v^r, γ_{V_v} is the Weil index associated to V_v.

Let σ_v be an irreducible admissible representation of $H_r(F_v)$. By the theta lift of σ_v to $\mathrm{U}(V_v)(F_v)$, we mean an irreducible admissible representation $\theta_{\chi_v}(\sigma_v)$ of $\mathrm{U}(V_v)(F_v)$ such that

$$\sigma_v \boxtimes \theta_{\chi_v}(\sigma_v)$$

is the maximal semisimple quotient of the σ_v-isotypic part of ω_{χ_v}. Similarly we speak of the theta lift of an irreducible admissible representation of $\mathrm{U}(V_v)(F_v)$. The existence of the theta lift of σ_v has been proved by Howe [How89] and Waldspurger [Wal90] in the archimedean case and the p-adic case ($p \neq 2$) respectively, and recently by Gan–Takeda [GT16] in general. Moreover by [How89, Wal90, LST11] we have the multiplicity one result

$$\dim \mathrm{Hom}_{H_r(F_v)\times \mathrm{U}(V_v)}(\omega_{\chi_v}, \sigma_v \boxtimes \theta_{\chi_v}(\sigma_v)) \le 1. \tag{3.1}$$

Let $\mathbb{V} = \otimes V_v$ be a Hermitian module over $\mathbb{A}_E$. Let $\chi = \otimes \chi_v : E^\times \backslash \mathbb{A}_E^\times \to \mathbb{C}^\times$ be a multiplicative character. Taking restricted tensor product of all local Weil representations, we have a global Weil representation ω_χ of $H_r(\mathbb{A}_F) \times \mathrm{U}(\mathbb{V})(\mathbb{A}_F)$ which is realized on $\mathcal{S}(\mathbb{V}^r)$. Let $\sigma = \otimes \sigma_v$ be an irreducible admissible representation of $H_r(\mathbb{A}_F)$. By the abstract theta lift of σ to $\mathrm{U}(\mathbb{V})(\mathbb{A}_F)$, we mean an irreducible admissible representation $\pi = \otimes \pi_v$ of $\mathrm{U}(V)(\mathbb{A}_F)$ so that for all places v of F, $\pi_v \simeq \theta_{\chi_v}(\sigma_v)$. Suppose that $\mathbb{V}$ is coherent, i.e. there is a Hermitian space V over E such that $\mathbb{V} = V \otimes \mathbb{A}_E$. Then for $\phi \in \mathcal{S}(\mathbb{V}^r)$, we have the theta series

$$\theta_\chi(g, h, \phi) = \sum_{x \in V^r(F)} \omega_\chi(g,h)\phi(x), \quad g \in H_r(\mathbb{A}_F),\ h \in \mathrm{U}(V)(\mathbb{A}_F).$$

3.2. **The archimedean case.** We have a more concrete description of theta lifts of discrete series representations of unitary groups. Let $\mathrm{U}(p,q)$ be the real unitary group of signature (p,q).

The Harish-Chandra parameter of a discrete series representation of $\mathrm{U}(p,q)$ is a sequence of integers or half-integers, $(a_1, \cdots, a_p; b_1, \cdots, b_q)$, $a_i, b_i \in \mathbb{Z}$ if $p+q$ is odd and in $\frac{1}{2} + \mathbb{Z}$ if $p+q$ is even, $a_1 > \cdots > a_p$, $b_1 > \cdots > b_q$. The Harish-Chandra parameter of the trivial representation of $\mathrm{U}(n,0)$ is $(\frac{n-1}{2}, \frac{n-3}{2}, \cdots, -\frac{n-1}{2})$.

Assume that the character χ in the definition of the Weil representation of $\mathrm{U}(r,r) \times \mathrm{U}(p,q)$ is of the form $\chi(z) = (z/\sqrt{z\overline{z}})^\alpha$, where α has the same parity as $p+q$.

Lemma 3.1. *Under the above choice of the characters, we have the following statements.*

(1) *The theta lift of the trivial representation of* $\mathrm{U}(2r,0)$ *to* $\mathrm{U}(r,r)$ *is a discrete series representation with Harish-Chandra parameter*

$$\left(\frac{2r-1+\alpha}{2}, \frac{2r-3+\alpha}{2}, \dots, \frac{-(2r-1)+\alpha}{2}\right).$$

The lowest K*-type is* $(k_1,k_2) \mapsto (\det k_1)^{r+\frac{\alpha}{2}}(\det k_2)^{-r+\frac{\alpha}{2}}$.

(2) *The theta lift of the trivial representation of* $\mathrm{U}(2r+1,0)$ *to* $\mathrm{U}(r,r)$ *is a discrete series representation with Harish-Chandra parameter*

$$\left(\frac{2r+\alpha}{2}, \frac{2r-2+\alpha}{2}, \dots, \frac{-(2r-2)+\alpha}{2}\right).$$

The lowest K*-type is* $(k_1,k_2) \mapsto (\det k_1)^{\frac{2r+1+\alpha}{2}}(\det k_2)^{-\frac{2r+1-\alpha}{2}}$. *This representation is not a theta lift of any representation of* $\mathrm{U}(p,q)$ *with* $p+q = 2r-1$.

Proof. This follows from the result of Paul [Pau98]. Her results are stated in the form of metaplectic representations. Harris [Har07, Subsection 2.3] reinterprets these results in term of the "usual" representations. □

3.3. **Doubling zeta integrals.** Let $\mu : E^\times\backslash\mathbb{A}_E^\times \to \mathbb{C}^\times$ be a character. Let $I_{2r}(s,\mu)$ be the degenerate principal series representation of $H_{2r}(\mathbb{A}_F)$. It is an induced representation from the usual Siegel parabolic subgroup $P_{2r} = M_{2r}N_{2r}$ of H_{2r}. It consists of functions Φ_s on $H_{2r}(\mathbb{A}_F)$ with the property that

$$\Phi_s(m(a)n(b)g) = \mu(\det a)|\det a|^{s+r}\Phi_s(g).$$

Let $\Phi_s \in I_{2r}(s,\mu)$. We may then form the Siegel Eisenstein series

$$E(g,\Phi_s) = \sum_{\gamma\in P_{2r}(F)\backslash H_{2r}(F)} \Phi_s(\gamma g).$$

We define the embeddings $i_0, i : H_r \times H_r \to H_{2r}$ as follows. Let $h_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in H_r$, $i = 1,2$. Put

$$i_0(h_1,h_2) = \begin{pmatrix} a_1 & & b_1 & \\ & a_2 & & b_2 \\ c_1 & & d_1 & \\ & c_2 & & d_2 \end{pmatrix} \in H_{2r}, \quad i(h_1,h_2) = i_0(h_1,h_2^\vee), \tag{3.2}$$

where

$$h^\vee = \begin{pmatrix} 1_r & \\ & -1_r \end{pmatrix} h \begin{pmatrix} 1_r & \\ & -1_r \end{pmatrix}^{-1}.$$

Let $\sigma = \otimes\sigma_v$ be an irreducible cuspidal automorphic representation of $H_r(\mathbb{A}_F)$ and $f_1 \in \sigma$, $f_1' \in \widetilde{\sigma}$. Let $\Phi_s \in I_{2r}(s,\mu)$. We have the global doubling zeta integral

$$Z(f_1,f_1',\Phi_s) = \int_{[H_r]}\int_{[H_r]} \overline{f_1(h)} f_1'(h') E(i(h,h'),\Phi_s)\mathrm{d}h\mathrm{d}h'.$$

This integral is convergent for all s where the Eisenstein series does not have pole.

We denote by $L(s, \sigma \times \mu)$ the standard L-function of σ twisted by μ. We also have the local L-function which we denote by $L(s, \sigma_v \times \mu_v)$. It is known that the doubling zeta integral represents this standard L-function. Moreover if σ_v is unramified, then $L(s, \sigma_v \times \mu_v) = L(s, \mathrm{BC}(\sigma_v) \otimes \mu_v)$ where BC stands for the (unramified) local base change of σ_v to $\mathrm{GL}_r(E_v)$. It is expected that this equality holds for all σ_v's. We are not going to use this stronger (conjectural) equality. For a detailed discussion, cf. [LR05].

Put

$$\gamma_0 = \begin{pmatrix} & & 1_r & \\ & 1_r & & \\ -1_r & 1_r & & \\ & & 1_r & 1_r \end{pmatrix} \in H_{2r}.$$

For each place v of F, we have the local doubling zeta integral

$$Z_v(f_{1,v}, f'_{1,v}, \Phi_{s,v}) = \int_{H_r(F_v)} \overline{\langle \sigma_v(h) f_{1,v}, f'_{1,v} \rangle} \Phi_{s,v}(\gamma_0 i(h,1)) \mathrm{d}h.$$

Define

$$b_m(s) = \prod_{i=1}^{m} L(2s+i, \mu|_{\mathbb{A}_F^\times} \eta^{n-i}),$$

and

$$Z_v^\natural(f_{1,v}, f'_{1,v}, \Phi_{s,v}) = \left(\frac{L(s+\frac{1}{2}, \sigma_v \times \mu_v)}{b_{2r,v}(s)} \right)^{-1} Z_v(f_{1,v}, f'_{1,v}, \Phi_{s,v}).$$

Suppose that we choose the measure at each place v so that $\mathrm{d}h = \prod \mathrm{d}h_v$. Then we have the decomposition of doubling zeta integrals [LR05, (23)]

$$Z(f_1, f'_1, \Phi_s) = \frac{L(s+\frac{1}{2}, \sigma \times \mu)}{b_{2r}(s)} \prod_v Z_v^\natural(f_{1,v}, f'_{1,v}, \Phi_{s,v}). \tag{3.3}$$

We now fix a place v of F. Let us consider the Weil representation $\omega_\chi^\square$ of $H_{2r}(F_v) \times \mathrm{U}(\mathbb{V})(F_v)$, realized on $\mathcal{S}(\mathbb{V}_v^{2r})$. By [HKS96, Proposition 2.2], we have an isomorphism $\omega_{\chi_v} \otimes \overline{\omega_{\chi_v}} \simeq \omega_{\chi_v}^\square \circ i$, as representations of $H_r(F_v) \times H_r(F_v) \times \mathrm{U}(\mathbb{V})(F_v)$. This isomorphism is realized as a partial Fourier transform

$$\mathcal{S}(\mathbb{V}_v^r) \otimes \overline{\mathcal{S}(\mathbb{V}_v^r)} \mapsto \mathcal{S}(\mathbb{V}_v^{2r}), \quad \phi_1 \otimes \overline{\phi_2} \mapsto \mathcal{F}^{\phi_1 \otimes \overline{\phi_2}},$$

with the property that $\mathcal{F}^{\phi_1 \otimes \overline{\phi_2}}(0) = \langle \phi_1, \phi_2 \rangle$, cf. [Li92, Section 2]. The map

$$h' \mapsto \omega_{\chi_v}^\square(h') \mathcal{F}^{\phi_1 \otimes \overline{\phi_2}}(0)$$

defines a section of $I_{2r}(0, \chi_v)$ which extends to a section $f_s^{\phi_1 \otimes \overline{\phi_2}} \in I_{2r}(s, \chi_v)$.

Assume that σ_v is tempered. For any $f_1, f_2 \in \sigma_v$, we have the doubling zeta integral $Z_v(f_1, f_2, f_s^{\phi_1 \otimes \overline{\phi_2}})$ whose value at $s=0$ equals

$$Z_v(f_1, f_2, \phi_1, \phi_2) = \int_{H_r(F_v)} \overline{\langle \sigma_v(h) f_1, f_2 \rangle} \langle \omega_v(h) \phi_1, \phi_2 \rangle \mathrm{d}h,$$

which is absolutely convergent. Similarly if π_v is tempered, we have a doubling zeta integral for $\varphi_1, \varphi_2 \in \pi_v$ whose value at $s = 0$ equals

$$Z_v(\varphi_1, \varphi_2, \phi_1, \phi_2) = \int_{\mathrm{U}(\mathbb{V})(F_v)} \overline{\langle \pi_v(g)\varphi_1, \varphi_2\rangle} \langle \omega_v(g)\phi_1, \phi_2\rangle \mathrm{d}g.$$

We also have the normalized version $Z_v^\natural$ of these integrals, namely

$$Z_v^\natural(f_1, f_2, \phi_1, \phi_2) = \left(\frac{L(\frac{1}{2}, \sigma_v \times \mu_v)}{b_{2r}(0)} \right)^{-1} Z_v(f_1, f_2, \phi_1, \phi_2),$$

and

$$Z_v^\natural(\varphi_1, \varphi_2, \phi_1, \phi_2) = \left(\frac{L(\frac{1}{2}, \pi_v \times \mu_v)}{b_{2r}(0)} \right)^{-1} Z_v(\varphi_1, \varphi_2, \phi_1, \phi_2).$$

3.4. **A duality property of doubling zeta integrals.** In this subsection, F is $\mathbb{R}$ or a nonarchimedean local field of characteristic zero, E is a quadratic étale algebra over F. We consider a slightly more general situation than the previous subsections. Let W (resp. V) be a skew-Hermitian (resp. Hermitian) space of dimension n over E. Put $H = \mathrm{U}(W)$ and $G = \mathrm{U}(V)$. We have a Weil representation ω realized on certain Schwartz space $\mathcal{S}$. For simplicity, we write H for $H(F)$ and G for $G(F)$. Note that to define the Weil representation and theta lifts, we need to fix some characters. However, the following discussion is independent from the choice of these characters. So we tacitly fix some choices, and suppress them from all the discussions below.

Let $\mathrm{Temp}(H)$ be the set of isomorphism classes of irreducible tempered representations of H. Let $\mathcal{X}_{\mathrm{temp}}(H)$ be the set of isomorphism classes of representations of H of the form $i_P^H \sigma_0$ where $P = MN$ is a parabolic subgroup of H whose Levi component is M and σ_0 is square-integrable representation of M. The set $\mathcal{X}_{\mathrm{temp}}(H)$ has a natural structure of a smooth manifold. Let $\lambda \in i\mathfrak{a}_{M,\mathbb{R}}^*$ and $\sigma_{0,\lambda} = \sigma_0 \otimes \lambda$. Then the connected component of $\mathcal{X}_{\mathrm{temp}}(H)$ containing $i_P^H \sigma_0$ consists of representations of the form $i_P^H \sigma_{0,\lambda}$.

Recall that we fix a measure $\mathrm{d}h$ on H. There is a natural measure $\mathrm{d}\sigma$ on $\mathcal{X}_{\mathrm{temp}}(H)$, called the Plancherel measure, cf. [BP, Section 2.6]. Let $\mathcal{C}(H)$ be the Harish-Chandra Schwartz space on H [BP, Section 1.5]. Then we have the following Plancherel formula. For any $\alpha \in \mathcal{C}(H)$, we have

$$\alpha(h) = \int_{\mathcal{X}_{\mathrm{temp}}(H)} \mathrm{Tr}(\sigma(h)^{-1}\sigma(\alpha))\mathrm{d}\sigma. \tag{3.4}$$

The representation $i_P^H \sigma_0$ might not be irreducible. When it is reducible, it decomposes as a direct sum of finitely many irreducible tempered representations. All tempered representations arise in this way. The decomposition of $i_P^H \sigma_0$ is governed by a certain elementary abelian 2-group R, called the R-group of $i_P^H \sigma_0$. Moreover, the subset of $\mathcal{X}_{\mathrm{temp}}(H)$ consisting of irreducible representations is open and dense.

The above discussion also applies to G.

Lemma 3.2. *Suppose that the Levi subgroup of P is isomorphic to $H_0 \times L$ where H_0 is a unitary group of smaller size and L is a general linear group. Suppose that $\sigma \subset i_P^H(\sigma_0 \boxtimes \tau)$ is an irreducible subrepresentation, where σ_0 is an irreducible square-integrable representation of H_0 and τ is an irreducible square integrable representation of L. Assume that $\theta(\sigma) \neq 0$. Then there is a parabolic subgroup $Q = M'N'$ of G so that $M' \simeq G_0 \times L$ where G_0 is a unitary group of the same type as G and the same rank as H_0, such that $\theta(\sigma) \subset i_Q^G(\theta(\sigma_0) \boxtimes \tau)$.*

This is an easy consequence of [GI16, Lemma 8.3]. Indeed, an $G \times H$ equivariant map $T : \omega \otimes (i_P^H(\sigma_0 \boxtimes \tau))^\vee \to i_Q^G(\theta(\sigma_0) \boxtimes \tau)$ is constructed in [GI16, Subsection 8.1]. It is shown in [GI16, Lemma 8.3] that given any $f \in (i_P^H(\sigma_0 \boxtimes \tau))^\vee$, there is a $\varphi \in \mathcal{S}$ so that $T(\varphi, f) \neq 0$. One may choose $f \in \sigma^\vee \subset (i_P^H(\sigma_0 \boxtimes \tau))^\vee$. The restriction of T to $\omega \otimes \sigma^\vee$ is thus nonzero and it factors through $\theta(\sigma)$ because $i_Q^G(\theta(\sigma_0) \boxtimes \tau)$ is semisimple. Therefore $\theta(\sigma)$ is a subrepresentation of $i_Q^G(\theta(\sigma_0) \boxtimes \tau)$. Even though [GI16] deals with the case that F is nonarchimedean, the argument goes without change in the archimedean case.

By definition, the R-groups of $i_P^H(\sigma_0 \boxtimes \tau)$ and $i_Q^G(\theta(\sigma_0) \boxtimes \tau)$ are canonically identified. Moreover, by [GI16, Proposition 8.4], theta correspondence gives a bijections between the irreducible subrepresentations of $i_P^H(\sigma_0 \boxtimes \tau)$ and those of $i_Q^G(\theta(\sigma_0) \boxtimes \tau)$. Moreover, $\mathfrak{a}_{M,\mathbb{R}}^*$ and $\mathfrak{a}_{M',\mathbb{R}}^*$ are identified. Thus the components of $\mathcal{X}_{\mathrm{temp}}(H)$ and $\mathcal{X}_{\mathrm{temp}}(G)$ containing $i_P^H(\sigma_0 \boxtimes \tau)$ and $i_Q^G(\theta(\sigma_0) \boxtimes \tau)$ respectively can be identified and under such an identification the Plancherel measures on them are the same. This follows from the definition of the Plancherel measure and the fact that the theta correspondence preserves the formal degree [GI14, Theorem 15.1] and the Plancherel measure [GI14, Theorem 12.1].

The doubling zeta integral still makes sense for $\sigma \in \mathcal{X}_{\mathrm{temp}}(H)$, even if it is reducible, as σ is of finite length. In the following, we write $\sum_{f \in \sigma}$ or just $\sum_f$ when the representation σ is clear to mean that f runs over an orthonormal basis of σ. Similar notation also applies to the group G and the representation π.

Lemma 3.3. *Fix two Schwartz functions ϕ_1, ϕ_2. The map*

$$\mathcal{X}_{\mathrm{temp}}(H) \to \mathbb{C}, \quad \sigma \mapsto \sum_f Z(f, f, \phi_1, \phi_2)$$

is continuous.

Proof. Let us choose an $\alpha \in C_c^\infty(H)$ so that $\alpha^* * \phi_1 = \phi_1$, where $\alpha^*(g) = \overline{\alpha(g^{-1})}$. This is always possible. Then

$$\sum_f Z(f, f, \phi_1, \phi_2) = \int_H \overline{\mathrm{Tr}(\sigma(h)\sigma(\alpha))} \langle \omega(h)\phi_1, \phi_2 \rangle \mathrm{d}h.$$

Let $\mathcal{C}^w(H)$ be the weak Harish-Chandra Schwartz space [BP, Section 1.5]. Then we have the following assertions.

(1) Fix some $T \in \mathrm{End}(\sigma)^\infty$ (the space of smooth endomorphisms). Then the map

$$\mathcal{X}_{\mathrm{temp}}(H) \to \mathcal{C}^w(H), \quad \sigma \mapsto (h \mapsto \mathrm{Tr}(\sigma(h)T))$$

is continuous.

(2) For any fixed ϕ_1, ϕ_2, the linear form on $\mathcal{C}^w(H)$

$$\alpha \mapsto \int_H \alpha(h)\langle\omega(h)\phi_1, \phi_2\rangle \mathrm{d}h, \quad \alpha \in \mathcal{C}^w(H)$$

is continuous.

The first assertion follows from [BP, Lemma 2.3.1(ii)]. The second assertion can be verified easily. Lemma 3.3 then follows from these two assertions. □

Proposition 3.4. *Suppose that $\sigma \in \mathrm{Temp}(H)$ or $\mathcal{X}_{\mathrm{temp}}(H)$. Put $\pi = \theta(\sigma)$ if $\sigma \in \mathrm{Temp}(H)$ and $\pi = i_Q^G(\theta(\sigma_0) \boxtimes \tau)$ if $\sigma = i_P^H(\sigma_0 \boxtimes \tau) \in \mathcal{X}_{\mathrm{temp}}(H)$ (notation as in the lemma above). Then for all $\phi_1, \phi_2 \in \mathcal{S}$, we have*

$$\sum_{f \in \sigma} Z(f, f, \phi_1, \phi_2) = \sum_{\varphi \in \pi} Z(\varphi, \varphi, \phi_1, \phi_2). \tag{3.5}$$

Proof. We proceed in four steps.

Step 1: Proposition 3.4 holds for $\sigma \in \mathrm{Temp}(H)$ up to a constant.

It is clear that we may (and will) assume that $\theta(\sigma) \neq 0$. If σ is an irreducible tempered representation of H, then there is a positive real number $c(\sigma)$, depending on σ only, but not on ϕ_1, ϕ_2, such that

$$\sum_{\varphi} Z(\varphi, \varphi, \phi_1, \phi_2) = c(\sigma) \sum_{f} Z(f, f, \phi_1, \phi_2).$$

Indeed, the doubling zeta integral on G (resp. H) defines a nonzero $H \times G$ equivariant map

$$\theta : \omega \to \sigma \boxtimes \pi, \quad \text{resp. } \theta' : \omega \to \sigma \boxtimes \pi.$$

The left (resp. right) hand side equals $\langle\theta(\phi_1), \theta(\phi_2)\rangle$ (resp. $\langle\theta'(\phi_1), \theta'(\phi_2)\rangle$) where $\langle -, -\rangle$ stands for the inner product on $\sigma \boxtimes \pi$. By (3.1), θ and θ' are proportional. Suppose that $\theta = \lambda\theta'$. Then $c(\sigma) = |\lambda|^2$.

Step 2: Proposition 3.4 holds all $\sigma \in \mathcal{X}_{\mathrm{temp}}(H) \cap \mathrm{Temp}(H)$.

Let $\alpha \in C_c^\infty(H)$ and $\beta \in C_c^\infty(G)$. Consider the integral

$$J = \int_H \int_G \overline{\alpha(h^{-1})\beta(g^{-1})}\langle\omega(h, g)\phi_1, \phi_2\rangle \mathrm{d}g\mathrm{d}h.$$

It is not hard to check that this double integral is absolutely convergent. We apply the Plancherel formula to α and get

$$J = \int_H \int_{\mathcal{X}_{\mathrm{temp}}(H)} \overline{\mathrm{Tr}(\sigma(h)\sigma(\alpha))}\langle\omega(h)\phi_1, \omega(\beta)\phi_2\rangle \mathrm{d}\sigma\mathrm{d}h.$$

We claim that this double integral is absolutely convergent. If F is nonarchimedean, this follows from the facts that $|\mathrm{Tr}(\sigma(h)\sigma(\alpha))| \ll \Xi(h)$, where Ξ is the Harish-Chandra function [BP, Section 1.5], and that the map $\sigma \mapsto \sigma(\alpha)$ is compactly supported. If F is archimedean, we have a more precise estimate

$$|\mathrm{Tr}(\sigma(h)\sigma(\alpha))| \leq C\Xi(h)|||\sigma(\alpha)|||_{\Delta_K^n, \Delta_K^n},$$

where C is some absolute constant which does not depend on σ, and $|||\cdot|||_{\Delta_K^n,\Delta_K^n}$ is a certain norm on $\mathrm{End}(\sigma)^\infty$, cf. [BP, Section 2.2]. Moreover it follows from [BP, Theorem 2.6.1(i)] that the integral of $|||\sigma(\alpha)|||_{\Delta_K^n,\Delta_K^n}$ over all $\mathcal{X}_{\mathrm{temp}}(H)$ (with respect to the Plancherel measure) is convergent. This proves the claim.

We can switch the order of integration and conclude that

$$\begin{aligned} J &= \int_{\mathcal{X}_{\mathrm{temp}}(H)} \int_H \overline{\mathrm{Tr}(\sigma(h)\sigma(\alpha))} \langle \omega(h)\phi_1, \omega(\beta)\phi_2\rangle \mathrm{d}h\mathrm{d}\sigma \\ &= \int_{\mathcal{X}_{\mathrm{temp}}(H)} \sum_f \int_H \overline{\langle \sigma(h)\sigma(\alpha)f, f\rangle} \langle \omega(h)\phi_1, \omega(\beta)\phi_2\rangle \mathrm{d}h\mathrm{d}\sigma \\ &= \int_{\mathcal{X}_{\mathrm{temp}}(H)} \sum_f Z(f,f,\phi_1,\omega(\alpha)\omega(\beta)\phi_2)\mathrm{d}\sigma. \end{aligned}$$

Similarly we have

$$J = \int_{\mathcal{X}_{\mathrm{temp}}(G)} \sum_{\varphi} Z(\varphi,\varphi,\phi_1,\omega(\alpha)\omega(\beta)\phi_2)\mathrm{d}\pi.$$

Therefore we conclude that

$$\int_{\mathcal{X}_{\mathrm{temp}}(H)} \sum_{f\in\sigma} Z(f,f,\phi_1,\omega(\alpha)\omega(\beta)\phi_2)\mathrm{d}\sigma = \int_{\mathcal{X}_{\mathrm{temp}}(G)} \sum_{\varphi\in\pi} Z(\varphi,\varphi,\phi_1,\omega(\alpha)\omega(\beta)\phi_2)\mathrm{d}\pi. \tag{3.6}$$

Let us fix a small neighbourhood Ω of $\sigma = i_P^H(\sigma_0 \boxtimes \tau)$ in $\mathcal{X}_{\mathrm{temp}}(H)$. By the identification of the component containing σ and the component containing π, this determines a small neighbourhood Ω' of π in $\mathcal{X}_{\mathrm{temp}}(G)$. This identification of Ω and Ω' identifies the Plancherel measures of Ω and Ω'. Since σ and hence π are irreducible, we may assume that any representations in Ω and Ω' are irreducible. We choose α and β so that the maps $\sigma \mapsto \sigma(\alpha)$ and $\pi \mapsto \pi(\beta)$ are supported in Ω and Ω' respectively. It then follows from the identity (3.6) that

$$\int_\Omega (c(\sigma)-1)\sum_f Z(f,f,\phi_1,\omega(\alpha)\omega(\beta)\phi_2)\mathrm{d}\sigma = 0.$$

We may write it as

$$\int_\Omega (c(\sigma)-1)\sum_f Z(f,\sigma(\alpha^*)f,\phi_1,\omega(\beta)\phi_2)\mathrm{d}\sigma = 0. \tag{3.7}$$

Suppose that F is nonarchimedean. Consider the tempered Bernstein center of H [SZ07]. Elements in the tempered center can be viewed as functions on $\mathcal{X}_{\mathrm{temp}}(H)$ which separate points on $\mathcal{X}_{\mathrm{temp}}(H)$. It also acts on $\mathcal{C}(H)$ and let us denote this action by $\circ$. Let z be any element in the tempered center, then $\sigma(z\circ\alpha) = z(\sigma)\sigma(\alpha)$. It follows that if $\sigma\mapsto\sigma(\alpha)$ is supported on Ω, then so is $\sigma\mapsto\sigma(z\circ\alpha)$. So we conclude, by replacing α by $z\circ\alpha$ in the identity (3.7) that

$$\int_\Omega (c(\sigma)-1)\overline{z(\sigma)}\sum_f Z(f,\sigma(\alpha^*)f,\phi_1,\omega(\beta)\phi_2)\mathrm{d}\sigma = 0.$$

Since c is continuous on Ω by Lemma 3.3, all such z's separate points on $\mathcal{X}_{\rm temp}(H)$, in particular on Ω, we conclude that

$$(c(\sigma)-1)\sum_f Z(f,\sigma(\alpha^*)f,\phi_1,\omega(\beta)\phi_2)=0$$

at all points $\sigma\in\Omega$ for all α,β satisfying the support condition above and all Schwartz functions ϕ_1 and ϕ_2. It is clear that for any σ we can find some α,β, ϕ_1,ϕ_2 such that

$$\sum_f Z(f,\sigma(\alpha^*)f,\phi_1,\omega(\beta)\phi_2)\neq 0.$$

It then follows that $c(\sigma)=1$ for all $\sigma\in\Omega$, in particular for $\sigma=i_P^H(\sigma_0\boxtimes\tau)$.

If F is archimedean, then we use the center of the enveloping algebra instead of the Bernstein center, and repeat the above argument.

Step 3: Proposition 3.4 holds for all $\sigma\in\mathcal{X}_{\rm temp}(H)$.

By Lemma 3.3, two sides of the identity (3.5) are continuous linear forms on $\mathcal{X}_{\rm temp}(H)$ and $\mathcal{X}_{\rm temp}(G)$ respectively. The subset consisting of irreducible representations of $\mathcal{X}_{\rm temp}(H)$ and $\mathcal{X}_{\rm temp}(G)$ are dense. So we conclude that for all $\sigma\in\mathcal{X}_{\rm temp}(H)$ and $\pi\in\mathcal{X}_{\rm temp}(G)$, the identity (3.5) holds.

Step 4: Proposition 3.4 holds for all $\sigma\in\mathrm{Temp}(H)$.

Suppose that $\sigma\subset i_P^H(\sigma_0\boxtimes\tau)$ is an irreducible subrepresentation and $\pi=\theta(\sigma)\subset i_Q^G(\theta(\sigma_0)\boxtimes\tau)$. We may choose a Schwartz function ϕ such that its image in $\sigma\boxtimes\pi$ is not zero, but for all other irreducible subrepresentations $\sigma'\subset i_P^H(\sigma_0\boxtimes\tau)$, the image in $\sigma'\boxtimes\theta(\sigma')$ is zero. Therefore

$$\sum_{f\in i_P^H(\sigma_0\boxtimes\tau)} Z(f,f,\phi,\phi)=\sum_{f\in\sigma}Z(f,f,\phi,\phi)\neq 0.$$

Similarly

$$\sum_{\varphi\in i_Q^G(\theta(\sigma_0)\boxtimes\tau)} Z(\varphi,\varphi,\phi,\phi)=\sum_{\varphi\in\pi}Z(\varphi,\varphi,\phi,\phi)\neq 0.$$

Then we conclude that $c(\sigma)=1$ for all $\sigma\in\mathrm{Temp}(H)$ from Step 3. □

4. Arithmetic theta lifts

4.1. **Shimura varieties attached to incoherent unitary groups.** From now on, we let F be a totally real field and E/F a CM extension. Following the exposition of [Liu11a, Section 3.1], we introduce Shimura varieties attached to incoherent unitary groups. For a more thorough discussion, see [KM90].

Let $\mathbb{V}$ be an m-dimensional incoherent Hermitian space over $\mathbb{A}_E$ which is totally positive at all archimedean places. Let $\mathbb{G}=\mathrm{U}(\mathbb{V})$. This is a reductive group over $\mathbb{A}_F$. For each embedding $\iota:E\to\mathbb{C}$, there is a unique Hermitian space $V(\iota)$ over E, called the nearby Hermitian space at ι, such that for all $v\neq\iota$, $\mathbb{V}_v\simeq V(\iota)_v$ and $V(\iota)_\iota$ is of signature $(m-1,1)$. Let $\mathrm{U}(V(\iota))$ be the unitary group attached to $V(\iota)$ and $\mathrm{U}(V(\iota))(\mathbb{A}_{F,f})\simeq\mathrm{U}(\mathbb{V})(\mathbb{A}_{F,f})$. Let K be an open compact subgroup of $\mathrm{U}(\mathbb{V})(\mathbb{A}_{F,f})$. There is a variety $X_K = \mathrm{Sh}(\mathbb{G})_K$ over E such that we have the following ι-adic uniformization

$$X_{K,\iota}(\mathbb{C}) = \mathrm{U}(V(\iota)(F))\backslash(D(\iota) \times \mathrm{U}(\mathbb{V})(\mathbb{A}_{F,f})/K).$$

Here the subscript means that E is embedded in $\mathbb{C}$ via ι. The Hermitian domain $D(\iota)$ is the set of all negative $\mathbb{C}$-lines in $V(\iota)_\iota$, $D(\iota) = \{v \in V(\iota)_\iota \mid q_\iota(v,v) < 0\}/\mathbb{C}^\times$ where q_ι is the Hermitian form on $V(\iota)_\iota$. We usually denote an element in it by $[z,g]_K$ where $z \in D(\iota)$ and $g \in \mathrm{U}(\mathbb{V})(\mathbb{A}_{F,f})$.

These varieties are projective if $F \neq \mathbb{Q}$. For the rest of this section, to avoid technical difficulties, we assume that all the Shimura varieties we consider are projective. We also assume that the level K is small so that X_K is a smooth projective variety (instead of a stack).

Remark 4.1. If $F = \mathbb{Q}$ and they are not projective, we may replace them by their smooth toroidal compactifications and develop a similar theory for these compactified Shimura varieties. For the Shimura varieties discussed above, the compactifications are canonical. For a quick discussion of the compactification of the Shimura varieties at hand, we refer the readers to [Liu11a, Section 3C].

We have a natural ample class $\mathscr{L}_K \in \mathrm{Pic}(X_K)$ for each K, called the Hodge bundle on X_K, which is naturally metrized. Indeed, it is easier to describe its dual line bundle. Let $\iota : E \to \mathbb{C}$ be an embedding. There is a homogeneous line bundle over the hermitian domain $D(\iota)$ whose fiber over a point $z \in D(\iota)$ is the $\mathbb{C}$-line in $V(\iota)$ represented by z. This space has a natural metric given by $|q_\iota|^{1/2}$. This line bundle naturally descents to a line bundle on X_K, which is the dual of the Hodge bundle. Let Ω_K be the Chern form of $\mathscr{L}_K$. Then $\Omega_K^{\dim X_K}$ is a volume form on X_K. When we talk about the volume of X_K, we always mean the volume of X_K with respect to this volume form. As K varies, $\mathscr{L}_K$ is compatible with pullbacks.

There is a cycle class map $\mathrm{cl} : \mathrm{Ch}^*(X_K) \to \mathrm{H}^{2*}(X_K)$. Let $\mathrm{Ch}^*(X_K)_0$ be its kernel. As in Section 2, at least conjecturally, we have a projection $\mathrm{Ch}^*(X_K) \to \mathrm{Ch}^*(X_K)_0$ which commutes with the Hecke action. If $m \leq 3$, then this projection is constructed unconditionally via the Künneth–Chow decomposition, using the ample class $\mathscr{L}_K$.

4.2. **Hecke actions.** We have a Hecke algebra $C_c^\infty(\mathbb{G}(\mathbb{A}_{F,f}))$. Let $x \in \mathbb{G}(\mathbb{A}_{F,f})$ and K be a finite level. We have a natural isomorphism $\gamma_x : X_{xKx^{-1}} \to X_K$, which on the level of uniformization is given by "the right multiplication by x". Put $K_x = K \cap xKx^{-1}$. Then we define the correspondence $T(x)_K$ to be the image of the map

$$(\pi_{K_x,K}, \pi_{K_x,K} \circ \gamma_x) : X_{K_x} \to X_K \times X_K,$$

In terms of the complex uniformation described in the previous subsection, the pushforward map $T(x)_{K,*}$ is given by

$$T(x)_{K,*}[z,g]_K = \sum_{i=1}^{s} [z, gx_i]_K,$$

where $x_1, \cdots, x_s$ is a set of representatives of KxK/K. Moreover the transpose of $T(x)_K$ is nothing but $T(x^{-1})_K$

Let $C_c^\infty(K\backslash\mathbb{G}(\mathbb{A}_{F,f})/K)$ be the sub-algebra of $C_c^\infty(\mathbb{G}(\mathbb{A}_{F,f}))$ consisting of functions which are bi-K-invariant. Then we may define a cycle $T(\phi)_K \subset X_K \times X_K$ for each $\phi \in C_c^\infty(K\backslash\mathbb{G}(\mathbb{A}_{F,f})/K)$ by

$$T(\phi)_K = \sum_{x\in K\backslash\mathbb{G}(\mathbb{A}_{F,f})/K} \phi(x)T(x)_K.$$

Let us fix a measure $\mathrm{d}g$ on $C_c^\infty(\mathbb{G}(\mathbb{A}_{F,f}))$. Let $\mathcal{A}(\mathbb{G})$ be the set of irreducible admissible representations of $\mathbb{G}(\mathbb{A}_F)$ which appear in the cohomology $\mathrm{H}^{m-1}(X) = \varinjlim_K \mathrm{H}^{m-1}(X_K)$. Note that $\pi \in \mathcal{A}(\mathbb{G})$ implies that $\pi_\infty = \otimes_{v|\infty}\pi_v$ is the trivial representation of $\mathbb{G}(F_\infty)$ (by assumption $\mathbb{G}(F_\infty)$ is compact). There is a natural surjective map

$$C_c^\infty(\mathbb{G}(\mathbb{A}_{F,f})) \to \bigoplus_{\sigma\in\mathcal{A}(\mathbb{G})} \sigma \otimes \widetilde{\sigma}, \tag{4.1}$$

where $\widetilde{\sigma}$ stands for the contragradient representation of σ. It is bi-$C_c^\infty(\mathbb{G}(\mathbb{A}_F))$-invariant if we require that $C_c^\infty(\mathbb{G}(F_\infty))$ acts on the left hand side trivially. Let us fix an inner product $\langle -,-\rangle$ on π and identify $\widetilde{\pi}$ and $\overline{\pi}$ via this inner product. Let $\pi \in \mathcal{A}(\mathbb{G})$ and $\varphi, \varphi' \in \pi$. We view $\varphi \otimes \overline{\varphi'}$ as an element in $\bigoplus_{\sigma\in\mathcal{A}(\mathbb{G})} \sigma \otimes \overline{\sigma}$ and choose an element $t_{\varphi,\varphi'} \in C_c^\infty(\mathbb{G})$ which maps to $\varphi \otimes \overline{\varphi'}$.

4.3. **Generating series.** We follow [Liu11a] to define the generating series of Kudla special cycles on X_K. It is an automorphic form on $H_r(\mathbb{A}_F)$ valued in $\mathrm{Ch}^*(X_K)$.

Let v be an archimedean place of F. We define a subspace $\mathcal{S}(\mathbb{V}_v^r)^{\mathrm{U}_v} \subset \mathcal{S}(\mathbb{V}_v^r)$ consisting of functions of the form

$$P(Q(x))e^{-2\pi \operatorname{Tr} Q(x)},$$

where P is a polynomial function on the space of Hermitian matrices. Let

$$\mathcal{S}(\mathbb{V}^r)^{\mathrm{U}_\infty} = \left(\bigotimes_{v|\infty} \mathcal{S}(\mathbb{V}_v^r)^{\mathrm{U}_v}\right) \otimes \mathcal{S}(\mathbb{V}_f^r), \quad \mathcal{S}(\mathbb{V}^r)^{\mathrm{U}_\infty K} = \left(\bigotimes_{v|\infty} \mathcal{S}(\mathbb{V}_v^r)^{\mathrm{U}_v}\right) \otimes \mathcal{S}(\mathbb{V}_f^r)^K,$$

for any open compact subgroup K of $\mathbb{G}(\mathbb{A}_{F,f})$.

Let $V_1 \subset \mathbb{V}_f$ be an E-subspace. We say that it is admissible if $(-,-)|_{V_1}$ takes values in E is totally positive. For $x \in \mathbb{V}_f^r$, we let V_x be the E-subspace of $\mathbb{V}_f$ generated by the components of x. We say that $x \in \mathbb{V}_f^r$ is admissible if V_x is. Suppose that V_x is admissible. Then $\mathbb{V}_1 = V_x^\perp \subset \mathbb{V}$ is an incoherent Hermitian space over $\mathbb{A}_E$ which is totally positive definite at all archimedean places. Let K be an open compact subgroup of $\mathbb{G}(\mathbb{A}_{F,f})$ and $K_1 = K \cap \mathrm{U}(\mathbb{V}_1)(\mathbb{A}_{F,f})$. Then we have a natural map

$$\mathrm{Sh}(\mathrm{U}(\mathbb{V}_1))_{K_1} \to X_K = \mathrm{Sh}(\mathrm{U}(\mathbb{V}))_K,$$

whose image defines a cycle $Z(V_x)$ in X_K. In terms of the complex uniformization described before, the cycle $Z(V_x)$ can be described as follows. Suppose that we are given an admissible $x \in \mathbb{V}_f^r$ and an embedding $\iota : E \to \mathbb{C}$. According to [Liu11a, Lemma 3.1], there is an $h \in \mathrm{SU}(\mathbb{V})(\mathbb{A}_{F,f})$ (SU stands for the derived group of the unitary group), such that $hV_x \subset V(\iota)_\iota \subset \mathbb{V}_f$. Then the cycle

$Z(V_x)$ is represented, in the ι-adic uniformization, by $[z, h_1 h]_K$, where $z \perp hV_x$ and h_1 fixes every element in hV_x.

We put

$$Z(x) = \begin{cases} Z(V_x) c_1(\mathscr{L}_K)^{r-\dim V_x}, & x \text{ is admissible;} \\ 0, & \text{otherwise,} \end{cases}.$$

Lemma 4.2. *For any $g \in \mathrm{U}(\mathbb{V})(\mathbb{A}_{F,f})$, and any $x \in \mathbb{V}_f^r$, we have*

$$T(g)_{K,*} Z(x) = \sum_{i=1}^{s} Z(g_i^{-1} x)_K,$$

where $g_1, \cdots, g_s$ is a set of representatives of KgK/K.

Proof. This can be checked directly on the level of uniformization. Fix an embedding $\iota : E \to \mathbb{C}$. By the description above, the cycle $Z(V_x)$ is represented by the points $[z, h_1 h]_K$ where $z \perp hV_x$ and h_1 fixes all elements in hV_x. Note that $V_{g_i^{-1}x} = g_i^{-1} V_x$ and $hg_i g_i^{-1} V_x \subset V(\iota)_\iota$. Therefore $Z(g_i^{-1}x)_K$ is represented $[z, h_1 h g_i]_K$ where $z \perp hV_x$ and h_1 fixes all elements in hV_x. The lemma then follows from the description of the Hecke operator in term of the uniformization. $\square$

Let $\phi \in \mathcal{S}(\mathbb{V}^r)^{\mathrm{U}_\infty K}$ and we define

$$Z(h, \phi)_K = \sum_{x \in K \backslash \mathbb{V}_f^r} \omega_\chi(h) \phi(x, Q(x)) Z(x)_K, \quad h \in H_r(\mathbb{A}_F).$$

This is a formal series valued in $\mathrm{Ch}^r(X_K)$. Note that since the support of the finite component of $\omega_\chi(h)\phi$ is compact, the sum is indeed over some countable set. Here we write a Schwartz function ϕ as $\phi_f \phi_\infty$ and $\phi(x, Q(x)) = \phi_f(x)\phi_\infty(y)$ where $x \in \mathbb{V}_f^r$ and $y \in \mathbb{V}_\infty^r$ is any element with $Q(x) = Q(y)$. Such a y exists because x is admissible. As K varies, the generating series $Z(h, \phi)_K$ is compatible with pullback. It has been proved by Liu [Liu11a, Theorem 3.5] that for any linear form ℓ on $\mathrm{Ch}^r(X_K)$, if $\ell(Z(h, \phi)_K)$ is absolutely convergent, then it is an automorphic form on $H_r(\mathbb{A}_F)$. If $r = 1$, Liu [Liu11a, Theorem 3.5] proved that $\ell(Z(h, \phi)_K)$ is convergent for all ℓ. For general r, Bruinier and Westerholt-Raum [BWR15] shows the absolute convergence for the generating series in the case of orthogonal-symplectic dual pairs. It is reasonable to believe that with the same technique, one can establish the convergence of $\ell(Z(h, \phi)_K)$ for all ℓ in our situation.

If $r = m - 1$, then $Z(h, \phi)_K$ is a formal series valued in the 0-cycle on X_K. So we can speak of its degree. By Kudla–Millson [KM90], $\deg Z(h, \phi)_K$ is absolutely convergent. The following result of Kudla relates this degree to the value of the Siegel Eisenstein series defined in Subsection 3.3. Let $g = m(h)n(h)k(h)$ be the Iwasawa decomoposition of h. If $\phi \in \mathcal{S}(\mathbb{V}(\mathbb{A}_F)^r)$, then

$$f_s^\phi(h) = \omega_\chi(h)\phi(0) |\det m(h)|^{s - \frac{1}{2}} \in I_r(s, \chi).$$

Proposition 4.3 ([Kud97])**.** *Assume $r = m - 1$. Suppose that $\phi \in \mathcal{S}(\mathbb{V}^r)^{\mathrm{U}_\infty K}$. Then $E(g, f_s^\phi)$ is holomorphic at $s = \frac{1}{2}$ and*

$$\deg Z(h, \phi)_K = \mathrm{vol}\, X_K \cdot E(h, f_s^\phi)|_{s = \frac{1}{2}}.$$

4.4. **Pullback of generating series.** Let us recall here two formulae for the pullback of the generating series. The analogous formulae in the setting of orthogonal-symplectic dual pairs were established in [YZZ09, Theorem 1.1, Proposition 3.1]. The same technique there also applies to the current situation. So we only state the results and leave the proof to the interested reader.

Let $\phi_1, \phi_2 \in \mathcal{S}(\mathbb{V}^r)$ be Schwartz functions. Then we have the generating series $Z(h, \phi_1)_K$ and $Z(h, \phi_2)_K$ valued in $\mathrm{Ch}^r(X_K)$ where $h \in H_r(\mathbb{A}_F)$. We also have $\phi_1 \otimes \overline{\phi_2} \in \mathcal{S}(\mathbb{V}^{2r})$ and the generating series $Z(h, \phi_1 \otimes \overline{\phi_2})_K \in \mathrm{Ch}^{2r}(X_K)$ where $h \in H_{2r}(\mathbb{A}_F)$. Note that the Weil representation is the one for $H_{2r}(\mathbb{A}_F) \times \mathrm{U}(\mathbb{V})(\mathbb{A}_F)$. The restriction of this Weil representation to $H_r(\mathbb{A}_F) \times H_r(\mathbb{A}_F) \times \mathrm{U}(\mathbb{V})(\mathbb{A}_F)$, along the embedding $i : H_r \times H_r \to H_{2r}$ given in (3.2), is isomorphic to $\omega_\chi \otimes \overline{\omega_\chi}$. The following is the analogue of [YZZ09, Theorem 1.1].

Proposition 4.4. *We have*

$$Z(h_1, \phi_1)_K \cdot \overline{Z(h_2, \phi_2)_K} = Z(i(h_1, h_2), \phi_1 \otimes \overline{\phi_2})_K,$$

where the left hand side stands for the intersection pairing on X.

Assume that $\mathbb{V}$ has an orthogonal decomposition $\mathbb{V} = \mathbb{W} + \mathbb{A}_E$ where $\mathbb{W}$ is a codimension one Hermitian module over $\mathbb{A}_E$. Let $K' = K \cap \mathrm{U}(\mathbb{W})(\mathbb{A}_{F,f})$. Then we have a Shimura variety $Y_{K'} = \mathrm{Sh}(\mathrm{U}(\mathbb{W}))_K$ with an embedding $j : Y_{K'} \to X_K$. Suppose that $\phi_{m-1} \in \mathcal{S}(\mathbb{W}^r)$ and $\phi_1 \in \mathcal{S}(\mathbb{A}_E^r)$. Then we have the generating series $Z(h, \phi_{m-1} \otimes \phi_1)_K$ valued in $\mathrm{Ch}^r(X_K)$ and $Z(h, \phi_{m-1})_{K'}$ valued in $\mathrm{Ch}^r(Y_{K'})$ where $h \in H_r(\mathbb{A}_F)$. For the next proposition, we need to be more careful with the characters of the Weil representation. We fix a character $\mu : E^\times \backslash \mathbb{A}_E^\times \to \mathbb{C}^\times$ so that $\mu|_{\mathbb{A}_F^\times} = \eta$. We always use the character $\chi_{\mathbb{V}} = \mu^{\dim \mathbb{V}}$ to define the Weil representation of $H_r(\mathbb{A}_F) \times \mathrm{U}(\mathbb{V})(\mathbb{A}_F)$ for any Hermitian module $\mathbb{V}$ over $\mathbb{A}_E$ (incoherent or not). The following is the analogue of [YZZ09, Proposition 3.1].

Proposition 4.5. *With the above choice of the characters, we have*

$$j^* Z(h, \phi_{m-1} \otimes \phi_1)_K = Z(h, \phi_{m-1})_{K'} \cdot \theta(h, \phi_1),$$

where $\theta(h, \phi_1)$ is the theta function on $H_r(\mathbb{A}_F)$ attached to ϕ_1.

4.5. **Arithmetic theta lifts.** Let σ be an irreducible cuspidal automorphic representation of $H_r(\mathbb{A}_F)$. Let $f \in \sigma$ and $\phi \in \mathcal{S}(\mathbb{V}^r)$. Put

$$\Theta_f^\phi = \int_{[H_r]} \overline{f(h)} Z(h, \phi)_K \mathrm{d}h.$$

This is a (formal sum of) element(s) in $\mathrm{Ch}^r(X_K)$. For any linear form $\ell : \mathrm{Ch}^r(X_K) \to \mathbb{C}$, if $\ell(Z(h, \phi)_K)$ is absolutely convergent, then

$$\ell(\Theta_f^\phi) = \int_{[H_r]} \overline{f(h)} \ell(Z(h, \phi)_K) \mathrm{d}h.$$

If $\ell(Z(h, \phi)_K)$ is absolutely convergent for all ℓ (e.g. $r = 1$ and according to [BWR15] it is reasonable to believe this is true for all r), then Θ_f^ϕ is a well-defined element in $\mathrm{Ch}^r(X_K)$. By the result of

Kudla–Milson [KM90], $\ell(Z(h,\phi)_K)$ is convergent for all ℓ factoring through $\mathrm{H}^{2r}(X_K)$, thus the cohomology class of $\Theta_f^\phi \in \mathrm{H}^{2r}(X_K)$ is always well-defined. When K varies, Θ_f^ϕ is compatible with the pullbacks when varying K. We write it as $\Theta_{f,K}^\phi$ if we need to specify K.

Before we proceed, let us note that the arithmetic theta lift, at least formally, gives an $H_r(\mathbb{A}_F)$ invariant and $C_c^\infty(K\backslash\mathbb{G}(\mathbb{A}_{F,f})/K)$ invariant map

$$\overline{\sigma}\otimes\mathcal{S}(\mathbb{V}^r)^{\mathrm{U}_\infty K}\to \mathrm{Ch}^r(X_K),\quad f\otimes\phi\mapsto\Theta_f^\phi.$$

Here the group $H_r(\mathbb{A}_F)$ acts on σ and $\mathcal{S}(\mathbb{V}^r)^{\mathrm{U}_\infty K}$ and the Hecke algebra $C_c^\infty(K\backslash\mathbb{G}(\mathbb{A}_{F,f})/K)$ acts on $\mathcal{S}(\mathbb{V}^r)^{\mathrm{U}_\infty K}$ via the Weil representation and on $\mathrm{Ch}^r(X_K)$ via pullback. The invariance by $H_r(\mathbb{A}_F)$ is clear by a simple change of variable in the integration. The Hecke invariance means that for any $g\in\mathbb{G}(\mathbb{A}_{F,f})$, we have

$$Z(h,\mathbf{1}_{KgK}*\phi)_K = T(g)_K^* Z(h,\phi)_K. \tag{4.2}$$

Here $\mathbf{1}_{KgK}*\phi$ stands for the "convolution"

$$\mathbf{1}_{KgK}*\phi(x)=\int_{KgK}\phi(h^{-1}x)\mathrm{d}h,$$

and the measure on the right hand side is the one with $\mathrm{vol}\,K=1$. Suppose that $g_1,\cdots,g_s$ is a set of representatives of $Kg^{-1}K/K$. Then

$$Z(h,\mathbf{1}_{KgK}*\phi)_K=\sum_{x\in K\backslash\mathbb{V}_f^r}\sum_{i=1}^s\omega(h)\phi(g_ix,Q(x))Z(x)_K=\sum_{x\in K\backslash\mathbb{V}_f^r}\sum_{i=1}^s\omega(h)\phi(x,Q(x))Z(g_i^{-1}x)_K.$$

The desire invariance (4.2) then follows from Lemma 4.2.

Assume $m=2r$. Then by [Liu11a, Proposition 3.9], Θ_f^ϕ is cohomologially trivial. Let $\langle -,-\rangle_{\mathrm{BB}}$ be the (conjectural) Beilinson–Bloch height pairing.

Conjecture 4.6 ([Liu11a, Conjecture 3.7]). *Assume that K is a sufficiently small open compact subgroup of $\mathbb{G}(\mathbb{A}_{F,f})$. Assume that the abstract theta lifting of σ to $\mathbb{G}(\mathbb{A}_F)$ is nonzero. We choose measures $\mathrm{d}h$ on $H_r(\mathbb{A}_F)$ and $\mathrm{d}h_v$ on $H_r(F_v)$ for all v so that $\mathrm{d}h=\prod_v\mathrm{d}h_v$. We choose an inner product $\langle -,-\rangle:\sigma_v\otimes\overline{\sigma_v}\to\mathbb{C}$ for each v so that $\prod_v\langle f_{1,v},f_{2,v}\rangle=\int_{[H_r]}f_1(h)\overline{f_2(h)}\mathrm{d}h$ if $f_1=\otimes f_{1,v}, f_2=\otimes f_{2,v}\in\sigma$. Then we have*

$$\mathrm{vol}\,K\cdot\langle\Theta_{f_1}^{\phi_1},\overline{\Theta_{f_2}^{\phi_2}}\rangle_{X_K,\mathrm{BB}}=\frac{L'(\frac12,\sigma\times\chi)}{b_m(0)}\prod_v Z_v^\natural(f_{1,v},f_{2,v},\phi_{1,v},\phi_{2,v}).$$

Recall that $\mathrm{vol}\,K$ *is calculated in terms of the measure specified in Subsection 1.5, and the normalized local doubling zeta integral is defined at the end of Subsection 3.3.*

Note that by Lemma 3.1, for any archimedean place v of F, the theta lift of σ_v to $\mathbb{G}(F_v)$ is the trivial representation. The abstract theta lift of σ to $\mathbb{G}(\mathbb{A}_F)$ being nonzero also implies that $\epsilon(\sigma,\chi)=-1$ and hence $L(\frac12,\sigma\times\chi)=0$.

The main theorem of [Liu11b] is the following.

Theorem 4.7. *This conjecture holds when $r = 1$.*

In this case, $\mathbb{V}$ is of rank two, X_K is a Shimura curve, and the Beilinson–Bloch height pairing is the Neron–Tate height pairing. Therefore it is unconditionally defined. Moreover Liu's result holds even when the Shimura varieties are not projective.

We prove the following variant of Theorem 4.7. Assume from now till the end of this subsection that $r = 1$.

Proposition 4.8. *Let π be the abstract theta lift of σ to $\mathbb{G}(\mathbb{A}_F)$. Let $K \subset \mathbb{G}(\mathbb{A}_{F,f})$ be an open compact group. Let $\varphi, \varphi' \in \pi$ and $\phi, \phi' \in \mathcal{S}(\mathbb{V})$. Assume that they are all fixed by K. Denote by $\mathrm{d}h$ the Tamagawa measure on $H(\mathbb{A}_F)$ and we choose measures $\mathrm{d}g_v$ in the definition of $Z_v^\natural$ so that $\prod \mathrm{d}g_v$ is the Tamagawa measure on $\mathbb{G}(\mathbb{A}_F)$. We fix another measure on $\mathbb{G}(\mathbb{A}_{F,f})$ and use it to define the function $t_{\varphi,\varphi'}$. The volume of K under this measure is denoted by $\mathrm{vol}' K$.*

(1) *The function in $(h, h') \in H_1(\mathbb{A}_F) \times H_1(\mathbb{A}_F)$ given by*

$$(h, h') \mapsto \mathrm{vol}\, K\, \mathrm{vol}'\, K \times \langle T(t_{\varphi,\varphi'})_{K,*}\Delta_{X_K,1,*}Z(h,\phi)_K, \overline{\Delta_{X_K,1,*}Z(h',\phi')_K}\rangle_{\mathrm{NT}}$$

defines an automorphic form in $\sigma \otimes \overline{\sigma}$, which is a cusp form. It is independent of the choice of K and the choice of the measure used to define $t_{\varphi,\varphi'}$.

(2) *We have*

$$\begin{aligned} &\mathrm{vol}\, K\, \mathrm{vol}'\, K \times \int_{[H_1]} \langle T(t_{\varphi,\varphi'})_{K,*}\Delta_{X_K,1,*}Z(h,\phi)_K, \overline{\Delta_{X_K,1,*}Z(h,\phi')_K}\rangle_{\mathrm{NT}}\mathrm{d}h \\ =&\frac{L'(\frac{1}{2}, \sigma \times \chi)}{L(1,\eta)\zeta(2)} \prod_v Z_v^\natural(\varphi, \varphi', \phi, \phi'). \end{aligned} \tag{4.3}$$

Proof. The first statement is clear. Note that we have assumed throughout that σ is a cuspidal automorphic representations. Let us prove the second one. Too ease notation, for $\varphi, \varphi' \in \pi$, $\phi, \phi' \in \mathcal{S}(\mathbb{V}^r)$, we put

$$A_{\varphi,\varphi',\phi,\phi'}(h, h') = \langle T(t_{\varphi,\varphi'})_{K,*}\Delta_{X_K,1,*}Z(h,\phi)_K, \overline{\Delta_{X_K,1,*}Z(h',\phi')_K}\rangle_{\mathrm{NT}}.$$

There is a constant c so that

$$\int_{[H_1]} A_{\varphi,\varphi',\phi,\phi'}(h,h)\mathrm{d}h = c \times \prod_v Z_v^\natural(\varphi, \varphi', \phi, \phi'),$$

as both sides define $H_1(\mathbb{A}_F) \times H_1(\mathbb{A}_F) \times \mathrm{U}(\mathbb{V})(\mathbb{A}_F)$ equivariant maps $\omega_\chi \otimes \overline{\omega_\chi} \otimes \overline{\pi} \otimes \pi \to \mathbb{C}$ and the space of such maps is one dimensional by (3.1). We only need to compute this constant c for some choices ϕ, φ. For this we may assume that ϕ is K_0-finite, where K_0 refers to a maximal compact subgroup of $H_1(\mathbb{A}_F) \times \mathbb{G}(\mathbb{A}_F)$. We consider

$$I = \mathrm{vol}\, K \sum_f \sum_\varphi \iint_{[H_1]^2} \overline{f(h)} f(h') A_{\varphi,\varphi,\phi,\phi'}(h,h')\mathrm{d}h\mathrm{d}h',$$

where f (resp. φ) ranges over an orthonormal basis of σ (resp. π). Since ϕ is K_0 finite, we may choose the orthonormal bases of σ and π so that there are only finitely many nonzero terms in this sum.

Summing over f first, we have

$$\begin{aligned} I &= \operatorname{vol} K \times \sum_{\varphi} \int_{[H_1]} A_{\varphi,\varphi,\phi,\phi}(h,h)\mathrm{d}h \\ &= c \times \operatorname{vol} K \times \prod_v \sum_{\varphi_v} Z_v^{\natural}(\varphi_v, \varphi_v, \phi_v, \phi_v), \end{aligned}$$

where φ_v runs over an orthonormal basis of π_v. Summing over φ first, by Theorem 4.7, we have

$$\begin{aligned} I &= \sum_f \iint_{[H_1]^2} \overline{f(h)} f(h') \langle \Delta_{X_K,1,*} Z(h,\phi)_K, \overline{\Delta_{X_K,1,*} Z(h',\phi)_K} \rangle_{\mathrm{NT}} \mathrm{d}h \mathrm{d}h' \\ &= \frac{L'(\frac{1}{2}, \sigma \times \chi)}{L(1,\eta)\zeta(2)} (\operatorname{vol}' K)^{-1} \prod_v \sum_{f_v} Z_v^{\natural}(f_v, f_v, \phi_v, \phi_v). \end{aligned}$$

where f_v runs over an orthonormal basis of σ_v.

It follows from Proposition 3.4 that

$$\prod_v \sum_{\varphi_v} Z_v^{\natural}(\varphi_v, \varphi_v, \phi_v, \phi_v) = \prod_v \sum_{f_v} Z_v^{\natural}(f_v, f_v, \phi_v, \phi_v).$$

Indeed, Proposition 3.4 shows that this equality holds for each individual v if the measures on both sides are the normalized local measures. The Tamagawa measures on $H(\mathbb{A}_F)$ and $\mathbb{G}(\mathbb{A}_F)$ are the same multiple of the product of normalized local measures. The desired equality then follows.

Note that for any v we may choose ϕ_v so that both sides of the above equality are nonzero. We thus conclude that

$$c = (\operatorname{vol} K \operatorname{vol}' K)^{-1} \frac{L'(\frac{1}{2}, \sigma \times \chi)}{L(1,\eta)\zeta(2)}.$$

This proves the proposition. □

Remark 4.9. We may have a more flexible choice of the measures. Suppose that in the definition of $Z_v^{\natural}$, we use the measure $\mathrm{d}g_v$ on $\mathrm{U}(\mathbb{V})(F_v)$. If $\prod_v \mathrm{d}g_v$ and $\mathrm{d}h$ are the same multiple of the Tamagawa measure on the respective group, then the formula still holds.

4.6. **Hecke correspondences as arithmetic theta lifts.** Assume in this subsection that $m = 2r+1$. Let π be an irreducible admissible tempered representation of $\mathbb{G}(\mathbb{A}_F)$ which contributes to the cohomology $\mathrm{H}^{2r}(X)$. Note that this implies that $\pi_\infty = \mathbf{1}_{\mathbb{G}_\infty}$. Let us assume that there is an irreducible cuspidal tempered automorphic representation of σ of $H_r(\mathbb{A}_F)$, such that π is the abstract theta lift of σ.

Let us fix an $H_r(\mathbb{A}_F) \times \mathbb{G}(\mathbb{A}_F)$ equivariant map $p : \overline{\sigma} \otimes \omega_\chi \to \theta_\chi(\sigma)$. Assume that $f_1, f_2 \in \sigma$, $\phi_1, \phi_2 \in \mathcal{S}(\mathbb{V}^r)$ and $\varphi_1 = p(f_1, \phi_1)$, $\varphi_2 = p(f_2, \phi_2)$. We normalize the map p place by place so that

$$\langle \varphi_{1,v}, \varphi_{2,v} \rangle = Z_v^{\natural}(f_{1,v}, f_{2,v}, \phi_{1,v}, \phi_{2,v}). \tag{4.4}$$

Proposition 4.10. *We have*

$$\mathrm{cl}(\Theta_{f_1}^{\phi_1} \times \overline{\Theta_{f_2}^{\phi_2}}) = \mathrm{vol}\, X_K \,\mathrm{vol}'\, K \times \frac{L(1, \sigma \times \chi)}{\prod_{i=2}^m L(i, \eta^i)} \mathrm{cl}(T(t_{\varphi_1,\varphi_2})_K) \in \mathrm{H}^{4r}(X_K \times X_K). \tag{4.5}$$

We fix a measure on $\mathbb{G}(\mathbb{A}_{F,f})$ to define t_{φ_1,φ_2} and use this measure to compute $\mathrm{vol}'\, K$. The right hand side is independent of the choice of this measure.

Proof. Let us first show that two sides of (4.5) differ only by some constant. Indeed, on the one hand, the map

$$\pi^K \otimes \overline{\pi^K} \to \mathrm{H}^{4r}(X_K \times X_K), \quad (\varphi_1, \varphi_2) \mapsto \mathrm{cl}(T(t_{\varphi_1,\varphi_2})_K)$$

lands in one of the Künneth components of $\mathrm{H}^{4r}(X_K \times X_K)$. Since π is tempered by assumption, it contributes only to the middle cohomology, i.e. $\mathrm{H}^{2r}(X_K) \otimes \mathrm{H}^{2r}(X_K)$. On the other hand, the map

$$\overline{\sigma} \otimes \omega_\chi \otimes \sigma \otimes \overline{\omega_\chi} \to \mathrm{H}^{2r}(X_K) \times \mathrm{H}^{2r}(X_K), \quad (f_1, f_2, \phi_1, \phi_2) \mapsto \mathrm{cl}(\Theta_{f_1}^{\phi_1} \times \overline{\Theta_{f_2}^{\phi_2}})$$

factors through $\pi \otimes \overline{\pi}$. It follows that both sides of (4.5) give Hecke equivariant maps

$$\pi^K \otimes \overline{\pi^K} \to \mathrm{H}^{2r}(X_K) \times \mathrm{H}^{2r}(X_K).$$

By (3.1), and the fact that the tempered part of $\mathrm{H}^{2r}(X_K)$ is multiplicity free as a Hecke module (cf. [Mok15, KMSW]), these two maps must be proportional.

It remains to compute this constant. To prove the proposition, it is enough to show that we may choose data so that

$$(\Theta_{f_1}^{\phi_1} \times \overline{\Theta_{f_2}^{\phi_2}}) \cdot \Delta_K = \mathrm{vol}\, X_K \,\mathrm{vol}'\, K \times \frac{L(1, \sigma \times \chi)}{\prod_{i=2}^m L(i, \eta^i)} \times (T(t_{\varphi_1,\varphi_2})_K \cdot \Delta_K) \neq 0,$$

where Δ_K is the diagonal cycle of $X_K \times X_K$ and the intersection number is taken on X_K.

The left hand side is computed as follows. We have

$$\begin{aligned}(\Theta_{f_1}^{\phi_1} \times \overline{\Theta_{f_2}^{\phi_2}}) \cdot \Delta_K &= \Theta_{f_1}^{\phi_1} \cdot \overline{\Theta_{f_2}^{\phi_2}} \\ &= \int_{[H_r]^2} \overline{f_1(h_1)} f_2(h_2) (Z_1(h, \phi_1)_K \cdot \overline{Z(h_2, \phi_2)_K}) \mathrm{d}h_1 \mathrm{d}h_2.\end{aligned}$$

By Proposition 4.4, we have

$$Z(h_1, \phi_1)_K \cdot \overline{Z(h_2, \phi_2)_K} = Z(i(h_1, h_2), \phi_1 \otimes \overline{\phi_2})_K \in \mathrm{Ch}^{2r}(X_K),$$

where $i : H_r \times H_r \to H_{2r}$ is given by (3.2). By Proposition 4.3,

$$\deg Z(i(h_1, h_2), \phi_1 \otimes \phi_2)_K = \mathrm{vol}\, X_K \times E(g, f_s^{\phi_1 \otimes \overline{\phi_2}})|_{s=\frac{1}{2}}.$$

We thus conclude that the left hand side equals

$$\mathrm{vol}\, X_K \times \iint_{[H_r]^2} \overline{f_1(h_1)} f_2(h_2) E(i(h_1, h_2), f_s^{\phi_1 \otimes \overline{\phi_2}}) \mathrm{d}h_1 \mathrm{d}h_2.$$

It follows from the theory of doubling zeta integrals that this equals

$$\mathrm{vol}\, X_K \times \frac{L(1, \sigma \times \chi)}{\prod_{i=2}^m L(i, \eta^i)} \prod_v Z_v^\natural(f_{1,v}, f_{2,v}, \phi_{1,v}, \phi_{2,v}),$$

The right hand side can be computed using the Lefschetz trace formula. We have

$$T(t_{\varphi_1,\varphi_2})_K \cdot \Delta_K = \sum_{i=0}^{4}(-1)^i \operatorname{Tr}(T(t_{\varphi_1,\varphi_2})_K^* \mid H^i(X_K)).$$

It follows from the definition of $T(t_{\varphi_1,\varphi_2})_K$ that the trace equals $(\mathrm{vol}' K)^{-1}\langle\varphi_1,\varphi_2\rangle$, which by our normalization equals

$$(\mathrm{vol}' K)^{-1}\prod_v Z_v^\natural(f_{1,v}, f_{2,v}, \phi_{1,v}, \phi_{2,v}).$$

It is clear that we may choose data so that this number is nonzero. Therefore we conclude that

$$\mathrm{cl}(\Theta_{f_1}^{\phi_1}\times\overline{\Theta_{f_2}^{\phi_2}}) = \mathrm{vol}\, X_K \,\mathrm{vol}' K \times \frac{L(1,\sigma\times\chi)}{\prod_{i=2}^m L(i,\eta^i)}\,\mathrm{cl}(T(t_{\varphi_1,\varphi_2})_K).$$

This is precisely what we want to prove. ☐

Remark 4.11. Based on Proposition 4.10 and Hypothesis 1.1, we expect that

$$\Theta_{f_1}^{\phi_1}\times\overline{\Theta_{f_2}^{\phi_2}} = \mathrm{vol}\, X_K \,\mathrm{vol}' K \times \frac{L(1,\sigma\times\chi)}{\prod_{i=2}^m L(i,\eta^i)} \times \Delta_{X_K\times X_K,4,*} T(t_{\varphi_1,\varphi_2})_K$$

as formal series valued in the Chow group of $X_K \times X_K$, where $\Delta_{X_K\times X_K,4,*}$ stands for the fourth Künneth–Chow component of $X_K\times X_K$. However, we do not have a proof of this result nor will use it in the following discussion.

5. The arithmetic GGP conjecture

5.1. **The conjecture.** We state the arithmetic GGP conjecture in this subsection. We state it in the form of an identity between the height pairing and the central derivative of certain L-functions. This is slightly more precise than the statement given in [GGP12, Zha12]. This precise form of the conjecture was first speculated in [Zha, Zha], with less care on the measures.

Let $\mathbb{W}$ be an incoherent Hermitian space over $\mathbb{A}_E$ of dimension n which is positive definite at all infinite places. Let $\mathbb{V} = \mathbb{W} + \mathbb{A}_E$. Then $\mathbb{V}$ is an incoherent Hermitian space over $\mathbb{A}_E$ of dimension $n+1$ which is positive definite at all infinite places. We have an inclusion $\mathbb{W}\subset\mathbb{V}$ which induces an embedding $\mathrm{U}(\mathbb{W})\to\mathrm{U}(\mathbb{V})$. Put $\mathbb{H}=\mathrm{U}(\mathbb{W})$, $\mathbb{G}=\mathrm{U}(\mathbb{W})\times\mathrm{U}(\mathbb{V})$. We have (projective systems) of Shimura varieties Y, X and $M = Y\times X$ attached to $\mathrm{U}(\mathbb{W})$, $\mathrm{U}(\mathbb{V})$ and $\mathbb{G}$ respectively. Let $K'\subset\mathbb{G}(\mathbb{A}_{F,f})$ be an open compact subgroup and $K = K'\cap\mathbb{H}(\mathbb{A}_{F,f})$. Then there is an embedding $Y_K\to M_{K'}$. The image of Y_K in $M_{K'}$ defines an element in $\mathrm{Ch}^*(M_{K'})$ which we denote by y_K. Its Hecke equivariant projection (assume it exists) to $\mathrm{Ch}^*(M_K)_0$ is denote by $y_{K,0}$.

Let $\pi=\otimes\pi_v$ be an irreducible admissible tempered representation of $\mathbb{G}(\mathbb{A}_F)$ which appears in the cohomology $\mathrm{H}^{2n-1}(M)$. Let $\varphi,\varphi'\in\pi$. Let us fix a measure $\mathrm{d}g$ on $\mathbb{G}(\mathbb{A}_{F,f})$ and choose a function $t_{\varphi,\varphi'}\in C_c^\infty(\mathbb{G})$ which maps to $\varphi\otimes\overline{\varphi'}$ under the map (4.1).

Let us now introduce the L-functions and the local linear forms. Write $\pi=\pi_{n+1}\boxtimes\pi_n$ where π_{n+1} (resp. π_n) is an irreducible admissible tempered representation of $\mathrm{U}(\mathbb{V})(\mathbb{A}_F)$ (resp. $\mathrm{U}(\mathbb{W})(\mathbb{A}_F)$).

Let $\Pi_i = \mathrm{BC}(\pi_i)$ be the base change of π_i, which is an irreducible tempered representation of $\mathrm{GL}_i(\mathbb{A}_E)$. We have the L-functions

$$L(s, \Pi_{n+1} \times \Pi_n), \quad L(s, \pi_{n+1}, \mathrm{Ad}), \quad L(s, \pi_n, \mathrm{Ad}).$$

In fact, as the usual GGP conjecture, we only need the definition of these L-functions at almost all places, namely the partial L-functions, cf. [Zha14a, Remark 1]. In the same fashion, the arithmetic GGP conjecture can be formulated in terms of these partial L-functions. The actual definition of these L-functions at finitely many (bad) places will not enter into the discussion below.

Let us put

$$\Delta_{n+1} = \prod_{i=1}^{n+1} L(i, \eta^i) = L(1,\eta)\zeta_F(2)L(3,\eta)\zeta_F(4)\cdots L(n+1, \eta^{n+1}).$$

In general, Π_i may not be a cuspidal automorphic representation of $\mathrm{GL}_i(\mathbb{A}_E)$. Suppose that for $i = n, n+1$, Π_i is written as an isobaric sum

$$\Pi_i = \Pi_i^{(1)} \boxplus \cdots \boxplus \Pi_i^{(\beta_i)}, \quad \Pi_i^{(j)} \text{ is cuspidal}, \quad j = 1, \cdots, \beta_i.$$

We put $\beta = \beta_n + \beta_{n+1}$.

We now define the local linear forms. We put

$$\alpha_v(\varphi_v, \varphi'_v) = \int_{\mathbb{H}(F_v)} \langle \pi_v(h)\varphi_v, \varphi'_v \rangle \mathrm{d}h.$$

If π_v is unramified and φ_v, φ'_v are $\mathbb{G}(\mathfrak{o}_{F,v})$-fixed, $\langle \varphi_v, \varphi'_v \rangle = 1$ and the measure $\mathrm{d}h$ is chosen so that $\mathrm{vol}\,\mathbb{H}(\mathfrak{o}_{F,v}) = 1$, then we have

$$\alpha_v(\varphi_v, \varphi'_v) = \Delta_{n+1,v} \frac{L(\frac{1}{2}, \Pi_{n+1,v} \times \Pi_{n,v})}{L(1, \pi_{n+1,v}, \mathrm{Ad})L(1, \pi_{n,v}, \mathrm{Ad})}.$$

Therefore we put for all place v of F,

$$\alpha_v^\natural(\varphi_v, \varphi'_v) = \left(\Delta_{n+1,v} \frac{L(\frac{1}{2}, \Pi_{n+1,v} \times \Pi_{n,v})}{L(1, \pi_{n+1,v}, \mathrm{Ad})L(1, \pi_{n,v}, \mathrm{Ad})} \right)^{-1} \alpha_v(\varphi_v, \varphi'_v).$$

We now state a refinement of the arithmetic GGP conjecture.

Conjecture 5.1. *Let the notation be as above. For all $\varphi = \otimes\varphi_v$, $\varphi' = \otimes\varphi'_v \in \pi$, we have*

$$\begin{aligned} &(\mathrm{vol}\,K)^2(\mathrm{vol}\,M_{K'}\,\mathrm{vol}'\,K')\langle T(t_{\varphi,\varphi'})_{K',*}y_{K,0}, y_{K,0}\rangle_{\mathrm{BB}} \\ &= \frac{\mathrm{vol}\,K\,\mathrm{vol}\,Y_K}{2^{\beta-1}} \Delta_{n+1} \frac{L'(\frac{1}{2}, \Pi_{n+1} \times \Pi_n)}{L(1, \pi_{n+1,v}, \mathrm{Ad})L(1, \pi_{n,v}, \mathrm{Ad})} \prod_v \alpha_v^\natural(\varphi_v, \varphi'_v), \end{aligned} \tag{5.1}$$

where $\langle -, - \rangle_{\mathrm{BB}}$ is the height pairing on $M_{K'}$ and the measures are as follows.

- *In the definition of α_v, we choose a measure $\mathrm{d}g_v$ for each v so that $\prod \mathrm{d}g_v$ equals the Tamagawa measure of $\mathbb{H}(\mathbb{A}_F)$.*

– *The volume* $\mathrm{vol}\, K$ *is computed using the measure specified in Subsection 1.5. The volume* $\mathrm{vol}\, Y_K$ *is computed using the volume form given by the top wedge product of the Chern form of the Hodge bundle, cf. Subsection 4.1.*
– *We choose a measure* $\mathrm{d}g$ *on* $\mathbb{G}(\mathbb{A}_{F,f})$ *and the map* $C_c^\infty(\mathbb{G}(\mathbb{A}_{F,f})) \to \bigoplus \pi \otimes \overline{\pi}$ *is defined using this measure. The volume* $\mathrm{vol}'\, K'$ *is computed using this measure.*

It is not hard to check that the left hand side does not depend on the choice of the open compact subgroup K' and the measure used to compute $\mathrm{vol}'\, K'$.

The case $n = 1$ is equivalent to the Gross–Zagier formulae, in the generality of [YZZ13]. Note that this is not a complete triviality due to different formulation of our conjecture and the main result of [YZZ13]. We will explain this in the appendix.

The main theorem of this paper is to prove certain endoscopic cases of the conjecture when $n = 2$. Before we state the result, let us fix some choice of the characters in various Weil representations and theta lifts. We fix a character $\mu : E^\times \backslash \mathbb{A}_E^\times \to \mathbb{C}^\times$ so that $\mu|_{\mathbb{A}_F^\times} = \eta$. We always use the character $\chi_{\mathbb{V}} = \mu^{\dim \mathbb{V}}$ to define the Weil representation of $H_r(\mathbb{A}_F) \times \mathrm{U}(\mathbb{V})(\mathbb{A}_F)$ for any Hermitian module $\mathbb{V}$ over $\mathbb{A}_E$ (incoherent or not). In what follows we simply speak of the Weil representations and the theta lifts, without mentioning the characters.

Theorem 5.2. *Suppose that* $n = 2$ *and the Hypothesis 1.1 holds. Assume the following two conditions.*

(1) *The Shimura varieties we consider are all projective.*
(2) *There are irreducible cuspidal automorphic representations* σ_1 *and* σ_2 *of* $H_1(\mathbb{A}_F)$ *so that* π_3 *(resp.* π_2*) is the abstract theta lift of* σ_2 *(resp.* σ_1*).*

Then for all $\varphi = \otimes \varphi_v \in \pi$*, we have*

$$\begin{aligned}&(\mathrm{vol}\, K)^2 (\mathrm{vol}\, M_{K'}\, \mathrm{vol}'\, K') \times \langle T(t_{\varphi,\varphi'})_{K,*} \Delta_{M,3,*} y_K, \Delta_{M,3,*} y_K \rangle \\ =&\frac{\mathrm{vol}\, K\, \mathrm{vol}\, Y_K}{4} \times \Delta_3 \frac{L'(\frac{1}{2}, \Pi_3 \times \Pi_2)}{L(1, \pi_{3,v}, \mathrm{Ad}) L(1, \pi_{2,v}, \mathrm{Ad})} \prod_v \alpha_v^\natural(\varphi_v, \varphi_v').\end{aligned}$$

Namely, under the assumption of the theorem, Conjecture 5.1 holds with $\beta = 3$*. Here* $\Delta_{M,3}$ *is the Künneth–Chow component of* M *constructed using the Hodge class.*

One drawback of this theorem is that it depends on Hypothesis 1.1, especially (2), which is impossible to check even for some very simple 3-folds. It is for this reason that we will formulate this result in a different but conjecturally equivalent form in the next subsection.

5.2. **Projectors.** For the rest of the section, we assume $n = 2$. To unconditionally formulate our main result, we use arithmetic theta lifts to define the projectors instead of the Hecke operators.

Let us choose the open compact subgroups as follows. Let $K'' \subset \mathrm{U}(\mathbb{V})(\mathbb{A}_{F,f})$ be a sufficiently small open compact subgroup. Put $K = K'' \cap \mathrm{U}(\mathbb{W})(\mathbb{A}_{F,f})$ and $K' = K'' \times K \subset \mathbb{G}(\mathbb{A}_{F,f})$. Then we have embeddings $Y_K \to X_{K''}$ and $Y_K \to M_{K'}$.

Let $\pi = \pi_3 \boxtimes \pi_2$ be an irreducible admissible representation of $\mathbb{G}(\mathbb{A}_F)$ as in Conjecture 5.1 and $\varphi, \varphi' \in \pi$. Assume that there are irreducible cuspidal automorphic representations σ_1 and σ_2 of $H_1(\mathbb{A}_F)$ so that π_3 (resp. π_2) is the abstract theta lift of σ_2 (resp. σ_1). Let us define a map

$$\mathsf{T}(\varphi, \varphi')_{K'} : \mathrm{Ch}^2(M_{K'})_0 \to \mathrm{Ch}^2(M_{K'})_0$$

as follows. Write $\varphi = \varphi_3 \otimes \phi_2$, $\varphi' = \varphi'_3 \otimes \varphi'_2$. We choose $f_2, f'_2 \in \sigma_2$, $\phi_3, \phi'_3 \in \mathcal{S}(\mathbb{V})$ and an $H(\mathbb{A}_F) \times \mathrm{U}(\mathbb{V})(\mathbb{A}_F)$ equivariant map $p : \overline{\sigma_2} \otimes \omega_\chi \to \pi_3$ so that $\varphi_3 = p(f_2, \phi_3)$, $\varphi'_3 = p(f'_2, \phi'_3)$. We assume that the map p is chosen so that

$$\langle \varphi_{3,v}, \varphi'_{3,v} \rangle = Z_v^\natural(f_{2,v}, f'_{2,v}, \phi_{3,v}, \phi'_{3,v})$$

for all places v of F, where $Z_v^\natural$ is the normalized local doubling zeta integral. For later use, we further assume that $\phi_3 = \phi_2 \otimes \phi_1$ and $\phi'_3 = \phi'_2 \otimes \phi'_1$ where $\phi_2, \phi'_2 \in \mathcal{S}(\mathbb{W})$ and $\phi_1, \phi'_1 \in \mathcal{S}(\mathbb{A}_E)$. Let us fix a $t_{\varphi_2,\varphi'_2} \in C_c^\infty(\mathrm{U}(\mathbb{W})(\mathbb{A}_{F,f}))$. We define

$$\mathsf{T}(\varphi, \varphi')_{K'} = (\Theta_{f_2}^{\phi_3} \times \overline{\Theta_{f'_2}^{\phi'_3}} \times T(t_{\varphi_2,\varphi'_2})_K)_*,$$

where $\Theta_{f_2}^{\phi_3} \times \overline{\Theta_{f'_2}^{\phi'_3}} \times T(t_{\varphi_2,\varphi'_2})$ is viewed as a correspondence on $M_{K'}$. It is not hard to check that as K' varies, the operators $\mathsf{T}(\varphi, \varphi')_{K'}$ define a map

$$\mathsf{T}(\varphi, \varphi') : \varprojlim_{K'} \mathrm{Ch}^2(M_{K'})_0 \to \varinjlim_{K'} \mathrm{Ch}^2(M_{K'})_0.$$

Note that as before, this depends on a measure we choose on $\mathrm{U}(\mathbb{W})(\mathbb{A}_{F,f})$ which is used to define t_{φ_2,φ'_2}. If we assume Hypothesis 1.1 (2) for the 3-fold $M_{K'}$, then this endomorphism depends only on φ, φ', but not on the choices of $f_2, f'_2, \phi_3, \phi'_3$ and t_{φ_2,φ'_2}. Moverover up to some constant, this map is given by $T(t_{\varphi,\varphi'})_*$, cf. Lemma 5.5 below.

In the next theorem, we do *not* assume Hypothesis 1.1.

Theorem 5.3. *Assume that both $X_{K''}$ and Y_K have regular models $\mathcal{X}$ and $\mathcal{Y}$ respectively and we can find a regular model of $X_{K''} \times Y_K$ dominating $\mathcal{X} \times \mathcal{Y}$.*

(1) *The height pairing*

$$\mathrm{vol}\, K\, \mathrm{vol}'\, K \times \langle \mathsf{T}(\varphi, \varphi')_{K'} \Delta_{M,3,*} y_K, \Delta_{M,3,*} y_K \rangle_{\mathrm{BB}}$$

is well defined and is independent of the choice of K.

(2) *With the same choice of the measures as Conjecture 5.1, we have*

$$\begin{aligned} &\mathrm{vol}\, K\, \mathrm{vol}'\, K \times \langle \mathsf{T}(\varphi, \varphi')_{K'} \Delta_{M,3,*} y_K, \Delta_{M,3,*} y_K \rangle_{\mathrm{BB}} \\ =&\frac{1}{4} \times \frac{L(\frac{1}{2}, \mathrm{BC}(\sigma_1) \times \mathrm{BC}(\sigma_2) \otimes \mu^{-1}) L'(\frac{1}{2}, \sigma_2 \times \mu^2)}{L(1, \sigma_1, \mathrm{Ad}) L(1, \sigma_2, \mathrm{Ad})} \times \prod_v \alpha_v^\natural(\varphi_v, \varphi'_v). \end{aligned}$$

In particular, the above height pairing is independent of the choice of $f_2, f'_2, \phi_3, \phi'_3$ and t_{φ_2,φ'_2}, but only on φ and φ'.

(3) *We have*

$$\langle \mathsf{T}(\varphi,\varphi)_{K'}\Delta_{M,3,*}y_K, \Delta_{M,3,*}y_K\rangle_{\mathrm{BB}} \geq 0,$$

$$\textit{and } \langle \mathsf{T}(\varphi,\varphi)_{K'}\Delta_{M,3,*}y_K, \Delta_{M,3,*}y_K\rangle_{\mathrm{BB}} = 0 \textit{ if and only if } \mathsf{T}(\varphi,\varphi)_{K'}\Delta_{M,3,*}y_K = 0.$$

Remark 5.4. The condition of the theorem will be satisfied if $X_{K''}$ has a good model in the sense of [GS95]. This means that irreducible components in the closed fibers of $\mathcal{X}$ are all smooth and the only singularities in the closed fibers of $\mathcal{X}$ are ordinary double points. This is expected to hold for all surfaces. Even though it is hard to prove this in general, it can be verified when the level of the Picard modular surface is simple. By the semistable reduction theorem of algebraic curves, Y has a good model $\mathcal{Y}$. A good model of $X \times Y$ is obtained by applying an explicit desingularization procedure to $\mathcal{X} \times \mathcal{Y}$, cf. [GS95, Proposition 6.3].

The proof of Theorem 5.3 will be given in the next two subsections. We show here that this theorem and Theorem 5.2 are equivalent under the working hypothesis.

Lemma 5.5. *Assume Hypothesis 1.1. Then Theorem 5.3 (2) and Theorem 5.2 are equivalent.*

Proof. It follows from Proposition 4.10 that

$$\mathrm{vol}\, X_{K''}\, \mathrm{vol}'\, K'' \times \frac{L(1,\sigma_2\times\mu^2)}{\zeta(2)L(3,\eta)} \times T(t_{\varphi_3,\varphi_3'})$$

and $\Theta_{f_2}^{\phi_3} \times \overline{\Theta_{f_2'}^{\phi_3'}}$ define the same cohomology class in $\mathrm{H}^4(X_{K''}\times X_{K''})$. Therefore by Hypothesis 1.1, we have

$$\begin{aligned}&\mathrm{vol}\, X_{K''}\, \mathrm{vol}'\, K'' \times \frac{L(1,\sigma_2\times\mu^2)}{\zeta(2)L(3,\eta)} \times \langle (T(t_{\varphi_3,\varphi_3'})\times T(t_{\varphi_2,\varphi_2'}))_*\Delta_{M,3,*}y, \Delta_{M,3,*}y\rangle \\ =&\langle \mathsf{T}(\varphi,\varphi')\Delta_{M,3,*}y, \Delta_{M,3,*}y\rangle.\end{aligned}$$

It is well-known that $L(s,\pi_3,\mathrm{Ad}) = L(s,\eta)L(s,\sigma_2\times\mu^2)L(s,\sigma_2,\mathrm{Ad})$ and $L(s,\Pi_3\times\Pi_2) = L(s,\mathrm{BC}(\sigma_1)\times \mathrm{BC}(\sigma_2)\times\mu^{-1})L(s,\sigma_1\times\mu^2)$. We have $L(\frac{1}{2},\sigma_1\times\mu^2)=0$ since its abstract theta lift is π_2, which is a representation of an incoherent unitary group. Thus $L'(\frac{1}{2},\Pi_3\times\Pi_2) = L(\frac{1}{2},\mathrm{BC}(\sigma_1)\times\mathrm{BC}(\sigma_2)\times \mu^{-1})L'(\frac{1}{2},\sigma_1\times\mu^2)$. The equivalence of Theorem 5.2 and 5.3 (2) then follows. □

5.3. **Proof of Theorem 5.3: geometry.** In what follows, to simplify notation, we drop the subscripts K, K' and K''. We should however bear in mind that we work on a fixed level. We also write only $\langle -,-\rangle$ for the height pairing, instead of adding the subscripts BB or NT.

Lemma 5.6. *We have* $\Delta_{X,1,*}\Theta_{f_2}^{\phi_3} = 0$ *and* $\Delta_{X,2,*}\Theta_{f_2}^{\phi_3} = \Theta_{f_2}^{\phi_3}$.

Proof. Let us prove $\Delta_{X,1,*}\Theta_{f_2}^{\phi_3}=0$. The other identity follows easily.

Let $\mathrm{AJ}:\mathrm{Pic}^0(X)\to \mathrm{H}^1(E,\mathrm{H}^1_{\mathrm{et}}(X))$ be the Abel–Jacobi map, where $\mathrm{H}^1_{\mathrm{et}}(X)$ is the étale cohomology of $X_{\overline{E}}$ on which the Galois group of E acts. The map

$$\overline{\sigma_2}\otimes\omega \to \mathrm{H}^1(E,\mathrm{H}^1_{\mathrm{et}}(X)), \quad (f_2,\phi_3)\mapsto \mathrm{AJ}(\Delta_{X,1,*}\Theta_{f_2}^{\phi_3})$$

factors through π_3. Thus we have a map $\pi_3 \to \mathrm{H}^1(E, \mathrm{H}^1_{\mathrm{et}}(X))$, which is Hecke equivariant. We claim that it is the zero map. Suppose that this is not the case. Fix a large finite set S of places of F and let $\mathcal{H}^S$ be the Hecke algebra of $\mathrm{U}(\mathbb{V})$ away from S. The representation π_3 defines a character of the Hecke algebra $\mathcal{H}^S \to \overline{\mathbb{Q}}^\times$ and let $\mathfrak{m}$ be its kernel. Then π_3 defines a nonzero $\mathcal{H}^S_{\mathfrak{m}}$-submodule of $\mathrm{H}^1(E, \mathrm{H}^1(X))_{\mathfrak{m}}$. We have $\mathrm{H}^1(E, \mathrm{H}^1_{\mathrm{et}}(X))_{\mathfrak{m}} \simeq \mathrm{H}^1(E, \mathrm{H}^1_{\mathrm{et}}(X)_{\mathfrak{m}})$. But π_3 is tempered, so it does not contribute to $\mathrm{H}^1_{\mathrm{et}}(X)$. Therefore $\mathrm{H}^1_{\mathrm{et}}(X)_{\mathfrak{m}} = 0$, hence $\mathrm{H}^1(E, \mathrm{H}^1_{\mathrm{et}}(X))_{\mathfrak{m}} = 0$. This is a contradiction. This proves our claim.

Therefore for any $f_2 \in \sigma_2$ and $\phi_3 \in \mathcal{S}(\mathbb{V})$, the image of $\Delta_{X,1,*}\Theta^{\phi_3}_{f_2}$ in $\mathrm{H}^1(E, \mathrm{H}^1_{\mathrm{et}}(X))$ is zero. It follows that $\Delta_{X,1,*}\Theta^{\phi_3}_{f_2}$ itself is trivial since $\mathrm{Pic}^0(X) \to \mathrm{H}^1(E, \mathrm{H}^1_{\mathrm{et}}(X))$ is injective. This is true since we are in the case of divisors. ☐

Lemma 5.7. *We have* $\mathsf{T}(\varphi, \varphi')\Delta_{M,3,*}y = \overline{\Theta^{\phi'_3}_{f'_2}} \times T(t_{\varphi_2,\varphi'_2})_*\Delta_{Y,1,*}\Theta^{\phi_3}_{f_2}|_Y$.

Proof. It is not hard to see that

$$(\Theta^{\phi_3}_{f_2} \times \overline{\Theta^{\phi'_3}_{f'_2}} \times T(t_{\varphi_2,\varphi'_2}))_*\Delta_{M,3,*}y = \overline{\Theta^{\phi'_3}_{f'_2}} \times T(t_{\varphi_2,\varphi'_2})_*((\Theta^{\phi_3}_{f_2} \times \Delta_Y)_*\Delta_{M,3,*}y),$$

where $\Theta^{\phi_3}_{f_2} \times \Delta_Y$ is cycle on $X \times Y \times Y$ and is viewed as a correspondence between $X \times Y$ and Y. It is then enough to compute $(\Theta^{\phi_3}_{f_2} \times \Delta_Y)_*\Delta_{M,3,*}y$. To simplify notation, we write Θ for $\Theta^{\phi_3}_{f_2}$.

Let $p : X \times Y \times Y \to X \times Y$ and $q : X \times Y \times Y \to Y$ be two projections. By definition

$$(\Theta \times \Delta_Y)_*\Delta_{M,3,*}y = q_*((\Delta_{M,3,*}y \times Y) \cdot (\Theta \times \Delta_Y)).$$

Note that $\Delta_{M,3,*}y \times Y = (\Delta_{M,3} \times \Delta_Y)_*(y \times Y)$, where $\Delta_{M,3} \times \Delta_Y$ is a correspondence on $X \times Y \times Y$. It follows that

$$(\Delta_{M,3,*}y \times Y) \cdot (\Theta \times \Delta_Y) = (y \times Y) \cdot (\Delta_{M,3} \times \Delta_Y)^*(\Theta \times \Delta_Y).$$

We have

$$\Delta_{M,3} = \Delta_{X,3} \times \Delta_{Y,0} + \Delta_{X,2} \times \Delta_{Y,1} + \Delta_{X,1} \times \Delta_{Y,2}.$$

We note that

$$(\Delta_{X,1} \times \Delta_{Y,2} \times \Delta_Y)^*(\Theta \times \Delta_Y) = 0$$

since it equals $\Delta^*_{X,1}\Theta \times \cdots$ and $\Delta^*_{X,1}\Theta = 0$. We claim that

$$q_*((y \times Y) \cdot (\Delta_{X,3} \times \Delta_{Y,0} \times \Delta_Y)^*(\Theta \times \Delta_Y)) = 0.$$

Indeed, it is not hard to see that

$$(\Delta_{X,3} \times \Delta_{Y,0} \times \Delta_Y)^*(\Theta \times \Delta_Y) = \Delta^*_{X,3}\Theta \times (\Delta_{Y,0} \times \Delta_Y)^*\Delta_Y.$$

By Lemma 5.6, $\Delta^*_{X,3}\Theta = \Delta_{X,1,*}\Theta = 0$. We have

$$(\Delta_{X,2} \times \Delta_{Y,1} \times \Delta_Y)^*(\Theta \times \Delta_Y) = \Delta^*_{X,2}\Theta \times \Delta_{Y,1}.$$

Finally

$$q_*((y \times Y) \cdot (\Delta^*_{X,2}\Theta \times \Delta_{Y,1})) = \Delta_{Y,1,*}(\Delta_{X,2,*}\Theta|_Y) = \Delta_{Y,1,*}\Theta|_Y.$$

In the last inequality we have made use of Lemma 5.6 and the fact that $\Delta_{X,2}$ is symmetric by construction. □

5.4. **Proof of Theorem 5.3: arithmetic seesaw.** We are going to make use of the GGP conjecture for $\mathrm{U}(2) \times \mathrm{U}(2)$. Let us briefly recall it here. We temporarily denote by σ_1 and σ_2 two arbitrary irreducible cuspidal tempered automorphic representations of $H(\mathbb{A}_F)$ and choose $f_1, f_1' \in \sigma_1$, $f_2, f_2' \in \sigma_2$. Let $\phi_1, \phi_1' \in \mathcal{S}(\mathbb{A}_E)$ be Schwartz functions and $\theta(h, \phi_1)$, $\theta(h, \phi_1')$ the theta functions on $H(\mathbb{A}_F)$ attached to ϕ_1 and ϕ_1' respectively. Assume that f_1, f_1', f_2, f_2' and ϕ_1, ϕ_1' are all factorizable. Fix a place v of F and define the following local linear form

$$\beta_v(f_{1,v}, f_{1,v}', f_{2,v}, f_{2,v}', \phi_{1,v}, \phi_{1,v}') = \int_{H(F_v)} \langle \sigma_{1,v}(h) f_{1,v}, f_{1,v}' \rangle \langle \sigma_{2,v}(h) f_{2,v}, f_{2,v}' \rangle \overline{\langle \omega(h)\phi_{1,v}, \phi_{1,v}' \rangle} \mathrm{d}h.$$

Let us also put

$$\beta_v^\natural = \left(\Delta_{2,v} \frac{L(\frac{1}{2}, \mathrm{BC}(\sigma_{1,v}) \times \mathrm{BC}(\sigma_{2,v}) \otimes \mu_v^{-1})}{L(1, \sigma_{1,v}, \mathrm{Ad}) L(1, \sigma_{2,v}, \mathrm{Ad})} \right)^{-1} \beta_v.$$

The GGP conjecture for $\mathrm{U}(2) \times \mathrm{U}(2)$ states the following.

Proposition 5.8 ([Har14, Corollary 4.7]). *Let* $\mathrm{d}h$ *be the Tamagawa measure on* $H(\mathbb{A}_F)$ *and we choose a measure* $\mathrm{d}h_v$ *for each place* v *in the definition* β_v *so that* $\mathrm{d}h = \prod \mathrm{d}h_v$*. Then we have*

$$\int_{[H]} f_1(h) f_2(h) \overline{\theta(h, \phi_1)} \mathrm{d}h \cdot \overline{\int_{[H]} f_1'(h) f_2'(h) \overline{\theta(h, \phi_1')} \mathrm{d}h}$$
$$= \frac{1}{2^\beta} \Delta_2 \frac{L(\frac{1}{2}, \mathrm{BC}(\sigma_1) \times \mathrm{BC}(\sigma_2) \otimes \mu^{-1})}{L(1, \sigma_1, \mathrm{Ad}) L(1, \sigma_2, \mathrm{Ad})} \prod_v \beta_v^\natural(f_{1,v}, f_{1,v}', f_{2,v}, f_{2,v}', \phi_{1,v}, \phi_{1,v}'),$$

where β *is defined for* σ_1 *and* σ_2 *in the same way as in the formulation of the arithmetic GGP conjecture. The product on the right hand side is over all places of* F *and for almost all* v *the terms* $\beta_v^\natural(\cdots)$ *equals one.*

Proof of Theorem 5.3. By Lemma 5.7, we have

$$\langle \mathsf{T}(\varphi, \varphi') \Delta_{M,3,*} y, \Delta_{M,3,*} y \rangle = \langle \overline{\Theta_{f_2'}^{\phi_3'}} \times T(t_{\varphi_2, \varphi_2'})_* \Delta_{Y,1,*} \Theta_{f_2}^{\phi_3}|_Y, \Delta_{M,3,*} y \rangle.$$

By Proposition 2.2, this height pairing is well-defined and equals

$$\langle T(t_{\varphi_2, \varphi_2'})_* (\Delta_{Y,1,*} \Theta_{f_2}^{\phi_3}|_Y), \overline{\Delta_{Y,1,*}(\Delta_{X,2,*} \Theta_{f_2'}^{\phi_3'})|_Y} \rangle.$$

Note that this proves the first assertion of Theorem 5.3. Making use of Lemma 5.6 again, we have

$$\langle \mathsf{T}(\varphi, \varphi') \Delta_{M,3,*} y, \Delta_{M,3,*} y \rangle = \langle T(t_{\varphi_2, \varphi_2'})_* (\Delta_{Y,1,*} \Theta_{f_2}^{\phi_3}|_Y), \overline{\Delta_{Y,1,*} \Theta_{f_2'}^{\phi_3'}|_Y} \rangle. \tag{5.2}$$

The third assertion of Theorem 5.3 follows from this identity and the positivity and nondegeneracy of the Neron–Tate height pairing.

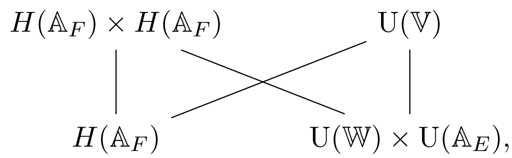
To prove the second assertion, it remains to calculate this height pairing. By Proposition 4.5, we have

$$\Theta_{f_2}^{\phi_3}|_Y = \int_{[H]} \overline{f_2(h)} Z(h, \phi_2) \theta(h, \phi_1) \mathrm{d}h.$$

Similarly for $\Theta_{f_2'}^{\phi_3'}$. Therefore

$$
\begin{aligned}
&\langle T(t_{\varphi_2,\varphi_2'})_*(\Delta_{Y,1,*}\Theta_{f_2}^{\phi_3})|_Y, \overline{\Delta_{Y,1,*}\Theta_{f_2'}^{\phi_3'}|_Y}\rangle \\
&= \iint_{[H]^2} \overline{f_2(h)} f_2'(h') \langle T(t_{\varphi_2,\varphi_2'})_* \Delta_{Y,1,*} Z(h,\phi_2), \overline{\Delta_{Y,1,*}Z(h',\phi_2)}\rangle \theta(h,\phi_1)\overline{\theta(h',\phi_1')}\mathrm{d}h\mathrm{d}h'.
\end{aligned} \tag{5.3}
$$

This is the "arithmetic seesaw" alluded in the title of this subsection. Classically, the seesaw argument amounts to changing order of integrations. In the current situation, the "arithmetic seesaw" is changing the order of integration and the height pairing. The seesaw diagram we use here is a "doubled" version of the following one



and the right hand side corresponds to the arithmetic intersection. Thus a diagram was first used in [GGP12, Section 14] and then used in [Har14, Xue16] to relate the GGP conjecture (not the arithmetic one) for $\mathrm{U}(n)\times\mathrm{U}(n+1)$ and for $\mathrm{U}(n)\times\mathrm{U}(n)$.

We make use of Proposition 5.8 to calculate the integral (5.3). We first note that since π_3 and π_2 are both tempered, σ_1 and σ_2 are also tempered. Moreover, by looking at the archimedean components of σ_1 and σ_2, cf. Lemma 3.1, we see that they do not come from theta lifts of unitary groups of smaller size. Thus both σ_1 and σ_2 are stable, i.e. the base change of σ_1 and σ_2 are cuspidal. Therefore when the GGP conjecture is applied to σ_1 and σ_2, we have $\beta = 2$. Combining the GGP conjecture and Proposition 4.8, we obtain that

$$
\begin{aligned}
(5.3) = &\left(\frac{\Delta_2}{4}\frac{L(\frac12,\mathrm{BC}(\sigma_1)\times\mathrm{BC}(\sigma_2)\otimes\mu^{-1})}{L(1,\sigma_1,\mathrm{Ad})L(1,\sigma_2,\mathrm{Ad})}\right)\times\left((\mathrm{vol}\,K\,\mathrm{vol}'\,K)^{-1}\times\frac{L'(\frac12,\sigma_1\times\mu^2)}{L(1,\eta)\zeta(2)}\right)\\
&\times\prod_v\left(\Delta_{2,v}\frac{L(\frac12,\mathrm{BC}(\sigma_{1,v})\times\mathrm{BC}(\sigma_{2,v})\otimes\mu_v^{-1})}{L(1,\sigma_{1,v},\mathrm{Ad})L(1,\sigma_{2,v},\mathrm{Ad})}\right)^{-1}\\
&\int_{H(F_v)}\overline{\langle\sigma_2(h)f_{2,v},f_{2,v}'\rangle}Z_v^\natural(\varphi_{2,v},\varphi_{2,v}',\omega(h)\phi_{2,v},\phi_{2,v}')\langle\omega_v(h)\phi_1,\phi_1'\rangle\mathrm{d}h.
\end{aligned}
$$

Note that we have $\phi_{3,v}=\phi_{2,v}\otimes\phi_{1,v}$, $\phi_{3,v}'=\phi_{2,v}'\otimes\phi_{1,v}'$ and $\langle\varphi_{3,v},\varphi_{3,v}'\rangle = Z_v^\natural(f_{2,v},f_{2,v}',\phi_{3,v},\phi_{3,v}')$. With these choices, we have proved in [Xue16, Appendix A] that the double integral in the product is absolutely convergent and equals $\alpha_v^\natural(\varphi_{3,v},\varphi_{3,v}',\varphi_{2,v},\varphi_{2,v}')$. This is referred to as the local seesaw identity in [Xue16]. Therefore by the identity (5.3), we conclude that

$$
\begin{aligned}
&\mathrm{vol}\,K\,\mathrm{vol}'\,K\times\langle(\mathsf{T}(\varphi,\varphi')_{K'}\Delta_{M_{K'},3,*}y_K,\Delta_{M_{K'},3,*}y_K\rangle_{\mathrm{BB}}\\
&=\left(\frac{\Delta_2}{4}\frac{L(\frac12,\mathrm{BC}(\sigma_1)\times\mathrm{BC}(\sigma_2)\otimes\mu^{-1})}{L(1,\sigma_1,\mathrm{Ad})L(1,\sigma_2,\mathrm{Ad})}\right)\left(\frac{L'(\frac12,\sigma_1\times\mu^2)}{L(1,\eta)\zeta(2)}\right)\prod_v\alpha_v^\natural(\varphi_{3,v},\varphi_{3,v}',\varphi_{2,v},\varphi_{2,v}').
\end{aligned}
$$

It is clear that the right hand side depends only on φ, φ', but not on the choices of f_2, f_2', ϕ_3, ϕ_3' and $t_{\varphi_2,\varphi_2'}$. This proves the second assertion of Theorem 5.3. □

Appendix A. The Gross–Zagier formula

The goal of this appendix is to deduce Conjecture 5.1 in the case of $n = 1$ from the Gross–Zagier formula in the generality of [YZZ13]. For simplicity, we do this only in a special case, namely, $F \neq \mathbb{Q}$ so that the Shimura curve is proper, π_2 is an automorphic representation of $\mathrm{U}(2)(\mathbb{A}_F)$ with trivial central character and π_1 is the trivial representation of $\mathrm{U}(1)(\mathbb{A}_F)$.

A.1. **Formulae on quaternion algebras.** Let us recall the results from [YZZ13] in this subsection. Due to the lack of space, we would not be able to explain all the definitions. The readers should consult [YZZ13] for a detailed discussion. Let $\mathbb{B}$ be an incoherent quaternion algebra over $\mathbb{A}_F$ which ramifies at all archimedean places, i.e. $\mathbb{B} = \otimes_v B_v$ where B_v is a quaternion algebra of F_v such that B_v is the Hamiltonian division algebra over $\mathbb{R}$ if v is archimedean and $\mathbb{B}$ is not of the form $B \otimes \mathbb{A}_F$ where B is a quaternion algebra over F. Attached to this $\mathbb{B}^\times$ is a (projective system of) Shimura curve(s) defined over F, denoted by $\mathrm{Sh}_{\mathbb{B}^\times}$. We fix an embedding $\mathbb{A}_E \to \mathbb{B}$. The group $\mathbb{A}_E^\times$ acts on $\mathrm{Sh}_{\mathbb{B}^\times}$ by the right translation. Let $\mathrm{Sh}_{\mathbb{B}^\times}^{E^\times}$ be the subscheme fixed by $E^\times$ (again defined over F) and $P \in \mathrm{Sh}_{\mathbb{B}^\times}^{E^\times}(E^{\mathrm{ab}})$ where E^{ab} stands for the maximal abelian extension of E. Let ξ_P to be the normalized Hodge bundle (cf. [YZZ13, p. 65]). We put

$$\overline{P} = \frac{1}{\mathrm{vol}(E^\times \backslash \mathbb{A}_E^\times / \mathbb{A}_F^\times)} \int_{E^\times \backslash \mathbb{A}_E^\times / \mathbb{A}_F^\times} T_t(P - \xi_P) \mathrm{d}t \in \mathrm{Alb}(\mathrm{Sh}_{\mathbb{B}^\times})(\overline{F}),$$

where T_t is the Hecke operator defined by the "right multiplication by t" (cf. [YZZ13, p. 62], our notation in Subsection 4.2 is γ_t). This is independent of the measure $\mathrm{d}t$ on $\mathbb{A}_E^\times / \mathbb{A}_F^\times$.

Let π be an irreducible admissible automorphic representation of $\mathbb{B}^\times$ appearing in $\mathrm{H}^1(\mathrm{Sh}_{\mathbb{B}^\times})$ (this is called the automorphic representation of $\mathbb{B}^\times$ of weight zero in [YZZ13, Section 3.2.2]). We assume that the central character of π is trivial. Note that π_∞ is the trivial representation. Let $\varphi, \varphi' \in \pi$. A projector

$$T(\varphi \otimes \varphi') : \mathrm{Alb}(\mathrm{Sh}_{\mathbb{B}^\times}) \to \mathrm{Jac}(\mathrm{Sh}_{\mathbb{B}^\times}).$$

is defined in [YZZ13, Subsection 3.3.1]. We briefly explain it here as follows. Fix an embedding $\iota : F \to \mathbb{C}$ and an open compact subgroup K of $\mathbb{B}^\times(\mathbb{A}_{F,f})$. Fix an embedding $\iota : F \to \mathbb{C}$. We have $\mathrm{H}^{1,0}(\mathrm{Sh}_{\mathbb{B}^\times,\iota,K}(\mathbb{C})) \simeq \oplus_\sigma \sigma^K$ where σ stands for all the irreducible admissible representations of $\mathbb{B}^\times$ which appears in the cohomology of $\mathrm{Sh}_{\mathbb{B}^\times}$. In particular any such a σ can be realized as a subspace of $\mathrm{H}^{1,0}(\mathrm{Sh}_{\mathbb{B}^\times,\iota}(\mathbb{C}))$. We fix a pairing on $\mathrm{H}^{1,0}(\mathrm{Sh}_{\mathbb{B}^\times,\iota}(\mathbb{C}))$. Let $\varphi, \varphi' \in \mathrm{H}^{1,0}(\mathrm{Sh}_{\mathbb{B}^\times,\iota K}(\mathbb{C}))$. Then put

$$\langle \varphi, \varphi' \rangle = (\mathrm{vol}\, \mathrm{Sh}_{\mathbb{B}^\times,\iota,K}(\mathbb{C}))^{-1} \int_{\mathrm{Sh}_{\mathbb{B}^\times,\iota,K}(\mathbb{C})} \varphi \wedge \overline{\varphi'}.$$

This is independent of the choice of K and Hecke invariant. It thus gives an inner product on σ and identifies $\overline{\sigma}$ with $\widetilde{\sigma}$. There is an embedding

$$\bigoplus \sigma^K \otimes \overline{\sigma}^K \to \mathrm{Hom}(\mathrm{H}^{1,0}(\mathrm{Sh}_{\mathbb{B}^\times,\iota,K}(\mathbb{C})), \mathrm{H}^{1,0}(\mathrm{Sh}_{\mathbb{B}^\times,\iota,K}(\mathbb{C}))),$$

and it is explained in [YZZ13, Proposition 3.14] that this embedding factors through the natural map

$$\mathrm{Hom}(\mathrm{Alb}(\mathrm{Sh}_{\mathbb{B}^\times,\iota,K}), \mathrm{Jac}(\mathrm{Sh}_{\mathbb{B}^\times,\iota,K})) \to \mathrm{Hom}(\mathrm{H}^{1,0}(\mathrm{Sh}_{\mathbb{B}^\times,\iota,K}(\mathbb{C})), \mathrm{H}^{1,0}(\mathrm{Sh}_{\mathbb{B}^\times,\iota,K}(\mathbb{C}))).$$

This defines the projector $T(\varphi \otimes \varphi')_K$. As K varies, the system $\{\mathrm{vol}\,\mathrm{Sh}_{\mathbb{B}^\times,\iota,K}(\mathbb{C}) \cdot T(\varphi \otimes \varphi')_K\}$ is independent of K. This is the projector $T(\varphi \otimes \varphi')$. The main result of [YZZ13] is the following.

Theorem A.1. *We have*

$$\text{(A.1)} \qquad \langle T(\varphi \otimes \varphi')\overline{P}, \overline{P}\rangle_{\mathrm{NT}} = \frac{\zeta_F(2) L'(\frac{1}{2}, \pi_E)}{4L(1,\eta)^2 L(1,\pi,\mathrm{Ad})} \prod_v \widetilde{\alpha}_v^\natural(\varphi_v, \varphi'_v).$$

The notation needs explanation.

- The height pairing is between $\mathrm{Jac}(\mathrm{Sh}_{\mathbb{B}^\times})(\overline{F})$ and $\mathrm{Alb}(\mathrm{Sh}_{\mathbb{B}^\times})(\overline{F})$. For any $x \in \mathrm{Jac}(\mathrm{Sh}_{\mathbb{B}^\times})(L)$ and $y \in \mathrm{Alb}(\mathrm{Sh}_{\mathbb{B}^\times})(L)$, where L is a finite extension of F, we have the usual Neron–Tate height pairing $\langle x, y\rangle_{\mathrm{NT}}$ and the height pairing in the theorem is given by $(L : F)^{-1}\langle x, y\rangle_{\mathrm{NT}}$. It has the property that it is independent of the field L in which x and y lie.
- The representation π_E is the base change of π to $\mathrm{GL}_2(\mathbb{A}_E)$. The adjoint L-function is the one for π as a representation of $\mathrm{P}\mathbb{B}^\times = \mathbb{B}^\times/\mathbb{A}_F^\times$. It is of degree three.
- The local linear form $\widetilde{\alpha}_v^\natural$ is defined by
$$\frac{L(1,\eta_v) L(1,\pi_v,\mathrm{Ad})}{\zeta_{F_v}(2) L(\frac{1}{2}, \pi_{v,E_v})} \int_{E_v^\times/F_v^\times} \langle \pi_v(t)\varphi_v, \varphi'_v\rangle \mathrm{d}t$$
where the measure $\mathrm{d}t_v$ is specified in [YZZ13, Subsection 1.6]. What we need about these measures is that the product $\prod_v \mathrm{d}t_v$ is not the Tamagawa measure of $\mathbb{A}_E^\times/\mathbb{A}_F^\times$, but rather the one that gives $\mathrm{vol}(E^\times\backslash\mathbb{A}_E^\times/\mathbb{A}_F^\times) = 2L(1,\eta)$.

For our purposes, we need such a formula for the Shimura curves attached to $\mathrm{P}\mathbb{B}^\times$. We have a Shimura curve $\mathrm{Sh}_{\mathrm{P}\mathbb{B}^\times}$ defined over F, and a morphism $p : \mathrm{Sh}_{\mathbb{B}^\times} \to \mathrm{Sh}_{\mathrm{P}\mathbb{B}^\times}$. The embedding $\mathbb{A}^\times \to \mathbb{B}^\times$ induces an embedding $\mathbb{A}_E^\times/\mathbb{A}_F^\times \to \mathrm{P}\mathbb{B}^\times$. The group $\mathbb{A}_E^\times/\mathbb{A}_F^\times$ acts on $\mathrm{Sh}_{\mathrm{P}\mathbb{B}^\times}$ by right translation. We let $P' = p(P) \in \mathrm{Sh}_{\mathrm{P}\mathbb{B}^\times}^{E^\times/F^\times}(E^{\mathrm{ab}})$. We define

$$\overline{P'} = \frac{1}{\mathrm{vol}(E^\times\backslash\mathbb{A}_E^\times/\mathbb{A}_F^\times)} \int_{E^\times\backslash\mathbb{A}_E^\times/\mathbb{A}_F^\times} T_t(P' - \xi_{P'})\mathrm{d}t \in \mathrm{Alb}(\mathrm{Sh}_{\mathrm{P}\mathbb{B}^\times})(\overline{F}),$$

where T_t is again the "right multiplication by t". It is clear that $p_*(\overline{P}) = \overline{P'}$.

The representation π is naturally a representation of $\mathrm{P}\mathbb{B}^\times$. Thus for $\varphi, \varphi' \in \pi$, we may again define a projector

$$S(\varphi \otimes \varphi') : \mathrm{Alb}(\mathrm{Sh}_{\mathrm{P}\mathbb{B}^\times}) \to \mathrm{Jac}(\mathrm{Sh}_{\mathrm{P}\mathbb{B}^\times}).$$

Then we have

Lemma A.2. *We have*

$$\langle S(\varphi\otimes\varphi')\overline{P'},\overline{P'}\rangle_{\mathrm{NT}}=\frac{\zeta_F(2)L'(\frac{1}{2},\pi_E)}{4L(1,\eta)^2L(1,\pi,\mathrm{Ad})}\prod_v\widetilde{\alpha}_v^\natural(\varphi_v,\varphi'_v),$$

where the height pairing is defined in the same way as in Theorem A.1 and the right hand side is the same as that of (A.1).

Proof. Since $p_*\overline{P}=\overline{P'}$, by the projection formula, we only have to prove that

$$p^*S(\varphi\otimes\varphi')p_*=T(\varphi\otimes\varphi').$$

This can be proved in the same way as [YZZ13, Proposition 3.14(2)]. We omit the details. □

A.2. **Unitary groups in two variables.** The discussion in this subsection is a very special case of [HS12, Chapter 4]. Due to the lack of space, we would not be able to explain all the details. The readers should consult [HS12] for a detailed exposition.

In the following we denote by $\nu:\mathbb{B}^\times\to\mathbb{A}_F^\times$ the reduced norm of $\mathbb{B}$. With the embedding $\mathbb{A}_E\to\mathbb{B}$, the incoherent quaternion algebra $\mathbb{B}$ is a two dimensional incoherent Hermitian space over $\mathbb{A}_E$. It is well-known that $\mathrm{U}(\mathbb{B})\simeq(\mathbb{B}^\times\times\mathbb{A}_E^\times)^1/\mathbb{A}_F^\times$ where the superscript 1 means that it consists of elements (b,e) such that $\nu(b)=N_{E/F}(e)$ and $\mathbb{A}_F^\times$ embeds diagonally. We also have the similitude unitary group $\mathrm{GU}(\mathbb{B})\simeq(\mathbb{B}^\times\times\mathbb{A}_E^\times)/\mathbb{A}_F^\times$ and the similitude factor is given by $(b,e)\mapsto\nu(b)N_{E/F}(e)^{-1}$. Note that the center of $\mathrm{GU}(\mathbb{B})$ isomorphic to $\mathbb{A}_E^\times$, embedded in $\mathrm{GU}(\mathbb{B})$ as $(1,e)$, $e\in\mathbb{A}_E^\times$. We have morphisms $\mathrm{U}(\mathbb{B})\to\mathrm{GU}(\mathbb{B})\to\mathrm{P}\mathbb{B}^\times$ which is the natural inclusion followed by projecting to the first factor. The image of this composition consists of elements whose reduced norm lie in $N\mathbb{A}_E^\times$ which we denote by $\mathrm{P}\mathbb{B}^{\times,\dagger}$. We have a morphism $\mathrm{Sh}_{\mathrm{U}(\mathbb{B})}\to\mathrm{Sh}_{\mathrm{P}\mathbb{B}^\times}$ whose image is $\mathrm{Sh}_{\mathrm{P}\mathbb{B}^{\times,\dagger}}$, the Shimura variety attached to $\mathrm{P}\mathbb{B}^{\times,\dagger}$.

Recall that we have a representation π of $\mathrm{P}\mathbb{B}^\times$. Let us fix an embedding $\iota:F\to\mathbb{C}$ and an embedding $\iota':E\to\mathbb{C}$ above it. Then π can be realized as a subrepresentation of $\mathrm{H}^{1,0}(\mathrm{Sh}_{\mathrm{P}\mathbb{B}^\times,\iota}(\mathbb{C}))$ under the action of $\mathrm{P}\mathbb{B}^\times$. We fix this realization of π. We let π^+ be the restriction of π to $\mathrm{U}(\mathbb{B})$ along the morphism $\mathrm{Sh}_{\mathrm{U}(\mathbb{B}),\iota'}(\mathbb{C})\to\mathrm{Sh}_{\mathrm{P}\mathbb{B}^\times,\iota}(\mathbb{C})$. Note that this is the restriction of the differential forms, not the the restriction of the representation π to the group $\mathrm{U}(\mathbb{B})$. Then π^+ is a subspace of $\mathrm{H}^{1,0}(\mathrm{Sh}_{\mathrm{U}(\mathbb{B}),\iota'}(\mathbb{C}))$.

We now relate our situation to that in [HS12, Chapter 4]. Let $B(\iota)$ be the nearby quaternion algebra of $\mathbb{B}$ at ι, i.e. a quaternion algebra over F which differs from $\mathbb{B}$ only at the place ι. Let $\pi(\iota)$ be the Jacquet–Langlands transfer of π to $B(\iota)(\mathbb{A}_F)$. This is an automorphic representation of $B(\iota)(\mathbb{A}_F)$. We have $\pi(\iota)_v\simeq\pi_v$ if $v\neq\iota$ and $\pi(\iota)_\iota$ is the discrete series representation of $B(\iota)(F_\iota)$ of weight two. Then π can be identified with a subspace of $\pi(\iota)$ consisting of automorphic forms on $B(\iota)(\mathbb{A}_F)$ that are holomorphic at the place ι, i.e. the ι component is a lowest weight vector. In fact, there is a standard way to pass from a holomorphic differential form on $\mathrm{Sh}_{\mathrm{P}\mathbb{B}^\times,\iota}(\mathbb{C})$ to an automorphic form on $B(\iota)(\mathbb{A}_F)$ which is holomorphic at ι. The most general discussion is in [Har84, Section 2.1, 2.2]. But in our special situation, the usual passage "differential forms" →

"modular forms" $\to$ "automorphic forms" suffices. With a similar discussion as above, one has an automorphic representation $\pi^+(\iota)$ of $\mathrm{U}(B(\iota))(\mathbb{A}_F)$ and π^+ is can be identified with a subspace of $\pi^+(\iota)$ consisting of automorphic forms on $\mathrm{U}(B(\iota))(\mathbb{A}_F)$ that are holomorphic at the place ι. The restriction of differential forms along the morphism $\mathrm{Sh}_{\mathrm{U}(\mathbb{B}),\iota'}(\mathbb{C}) \to \mathrm{Sh}_{\mathrm{P}\mathbb{B}^\times,\iota}(\mathbb{C})$ corresponds to the restriction of automorphic forms along the natural map $\mathrm{U}(B(\iota))(\mathbb{A}_F) \to B(\iota)(\mathbb{A}_F)$. The relation between π^+ and π is thus transformed to that of $\pi^+(\iota)$ and $\pi(\iota)$, the later being discussed in detail in [HS12, Chapter 4]. In the notation there, $G = \mathrm{U}(B(\iota))$ and $\widetilde{G} = \mathrm{GU}(B(\iota)) = (B(\iota)^\times \times E^\times)/F^\times$, and they have the same derived group.

We now explaining the relation between π^+ and π. There are two cases.

(1) $\pi \otimes \eta \circ \nu \not\simeq \pi$. In this case, by [HS12, Theorem 4.14], the natural map $\pi \to \pi^+$ is a bijection. Moreover all irreducible constituents of the restriction of π to $\mathrm{U}(\mathbb{B})$, as abstract representations, are automorphic. By [HS12, Proposition 4.21], these constituents are mutually inequivalent (as abstract representations). We denote by π^0 one of the irreducible constituent of π^+. We have that π_E (which is the same as $\mathrm{BC}(\pi^0)$) is a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_E)$. As π^0 is realized naturally on some subspace of π, we put an inner product on π^0 by restricting that from π. Thus suppose we have $\varphi, \varphi' \in \pi$ and φ^0, φ'^0 their restrictions to $\mathrm{Sh}_{\mathrm{U}(\mathbb{B}),\iota'}$, then $\langle \varphi, \varphi' \rangle_\pi = \langle \varphi^0, \varphi'^0 \rangle_{\pi^0}$.

Now suppose that $\pi = \otimes \pi_v$. Then $\pi^0 = \otimes \pi_v^0$ where π_v^0 is an irreducible constituent of π_v, as an abstract representation of $\mathrm{U}(\mathbb{B})(F_v)$. Note that by [HS12, Lemma 2.4, 2.12], π_v is irreducible (resp. direct sum of two inequivalent irreducible representations) if $\pi_v \otimes \eta_v \circ \nu_v \not\simeq \pi_v$ (resp. $\simeq$). Let us fix an inner product $\langle -,- \rangle_{\pi_v}$ on π_v for each v so that $\langle -,- \rangle_\pi = \prod_v \langle -,- \rangle_{\pi_v}$. By restricting the inner product on π_v to its irreducible constituents, we obtain an inner product on each π_v^0. We then have $\langle -,- \rangle_{\pi^0} = \prod_v \langle -,- \rangle_{\pi_v^0}$.

(2) $\pi \otimes \eta \circ \nu \simeq \pi$. Then for places v, $\pi_v \otimes \eta_v \circ \nu_v \simeq \pi_v$ we have $\pi_v = \pi_v^+ \oplus \pi_v^-$ where $\pi_v^\pm$ is an irreducible representation of $\mathrm{U}(\mathbb{B})(F_v)$ (if v is archimedean, then we put $\pi_v^- = 0$). We choose the labeling so that if π_v is an unramified representation of $\mathrm{P}\mathbb{B}^\times(F_v)$, then π_v^+ is an unramified representation of $\mathrm{U}(\mathbb{B})(F_v)$. We may moreover assume that $\otimes \pi_v^+$ is an irreducible constituent of π^0, i.e. it is automorphic as a representation of $\mathrm{U}(\mathbb{B})$. By [HS12, Proposition 4.21], we have that $\otimes \pi_v^{\epsilon_v}$ where $\epsilon_v = \pm$ is automorphic if and only if there are even number of minus signs in ϵ_v, in other words,

$$\pi^+ \simeq \bigoplus_{\prod_v \epsilon_v = 1} \bigotimes \pi_v^{\epsilon_v}.$$

Let π^1 (resp. π^2) be the subspace of π consisting of φ's that are (resp. are not) supported in the image of $\mathrm{Sh}_{\mathrm{U}(\mathbb{B}),\iota'}(\mathbb{C})$ in $\mathrm{Sh}_{\mathrm{P}\mathbb{B}^\times,\iota}(\mathbb{C})$. Then by [HS12, Lemma 4.14, Theorem 4.14], π^1 and π^2 are representations of $\mathrm{U}(\mathbb{B})$, the natural restriction map $\pi \to \pi^+$ (restriction as functions) identifies π^1 with π^+ as $\mathrm{U}(\mathbb{B})$ representations and the kernel is π^2.

Let $\pi^0 = \otimes \pi_v^{\epsilon_v}$ be an irreducible constituent of π^+. Since $\pi^1 \to \pi^+$ is an isomorphism, we see that π^0 is again naturally realized on a subspace of π. Thus we may define inner products on π^0 and π_v^0 as in the previous case, i.e. by restricting inner products from π and π_v respectively. In this way, we again have

$$\langle \varphi, \varphi' \rangle_\pi = \langle \varphi'^0, \varphi'^0 \rangle_{\pi^0}, \quad \langle \varphi^0, \varphi'^0 \rangle_{\pi^0} = \prod_v \langle \varphi_v^0, \varphi_v'^0 \rangle,$$

where $\varphi, \varphi' \in \pi^1$ and φ^0, φ'^0 their restrictions to $\mathrm{U}(\mathbb{B})$ respectively.

A.3. **The formula on the unitary groups.** Let us get back to the setup of Conjecture 5.1 in the case of $n = 1$. Recall that we have put $\mathbb{G} = \mathrm{U}(\mathbb{B}) \times \mathrm{U}(\mathbb{A}_E)$ and $\mathbb{H} = \mathrm{U}(\mathbb{A}_E)$. We have the Shimura curve M attached to $\mathbb{G}$ defined over E. The group $\mathbb{H}$ embeds in $\mathbb{G}$ diagonally and acts by right translation on M. Let K' be an open compact subgroup of $\mathbb{G}(\mathbb{A}_{F,f})$ and $K = K' \cap \mathbb{H}(\mathbb{A}_{F,f})$. Then we put

$$\overline{Q}_{K'} = (\#Y_K)^{-1} y_{K,0} \in \mathrm{Alb}(M_{K'})(E),$$

where $y_{K,0} = \Delta_{M_{K'},1,*} y_K$ and the $\Delta_{M_{K'},1}$ is the first Künneth–Chow component of $M_{K'}$ defined using the Hodge bundle. As K varies, we have a system $\overline{Q} \in \mathrm{Alb}(M)(E)$.

We have a representation π of $\mathrm{P}\mathbb{B}^\times$ (with the realization discussed in the previous subsection) and its restriction to $\mathrm{U}(\mathbb{B})$ (the restriction of differential forms as discussed before). We let π^0 be an irreducible constituent of this restriction. By extending it trivially on $\mathbb{A}_E^\times$, we view π^0 as a representation of $\mathbb{G}$.

If $\pi \otimes \eta \circ \nu \not\simeq \pi$ (resp. $\simeq$), we put $\pi^1 = \pi$ (resp. π^1 as in the previous subsection). Let $\varphi^0, \varphi'^0 \in \pi^0$ and choose $\varphi, \varphi' \in \pi^1$ so that their restrictions are φ^0 and φ'^0 respectively. Similarly to the case of Shimura varieties attached to quaternion algebras, we have a projector $R(\varphi^0 \otimes \varphi'^0)$.

Lemma A.3. *We have*

$$\langle R(\varphi^0 \otimes \varphi'^0)\overline{Q}, \overline{Q} \rangle_{\mathrm{NT}} = \frac{\zeta_F(2) L'(\frac{1}{2}, \pi_E)}{2^s L(1, \eta) L(1, \pi^0, \mathrm{Ad})} \prod_v \widetilde{\alpha}_v^\natural(\varphi_v, \varphi_v'),$$

where $s = 2$ or 1 according to whether $\pi \otimes (\eta \circ \nu) \simeq \pi$ or not, the local linear forms $\widetilde{\alpha}_v^\natural$ on the right hand side are the same as that of (A.1), the adjoint L-function is the adjoint L-function for π^0 as a representation of $\mathrm{U}(\mathbb{B})$*, and the height pairing is the usual height pairing between* $\mathrm{Alb}(M)(E)$ *and* $\mathrm{Jac}(M)(E)$.

Proof. Let us denote by $p : M \to \mathrm{Sh}_{\mathrm{P}\mathbb{B}^\times}$ the natural morphism. We fix a an embedding $\iota : F \to \mathbb{C}$ and an embedding $\iota' : E \to \mathbb{C}$ over it. Choose open compact subgroups K of $\mathrm{P}\mathbb{B}^\times(\mathbb{A}_{F,f})$ and K' of $\mathbb{G}(\mathbb{A}_{F,f})$ which maps into K. The morphism p induces the natural morphisms p_* and p^* on the level of differential forms. The map p^* is simply the pullback map while p_* is more complicated. If $f \in \mathrm{H}^{1,0}(\mathrm{Sh}_{\mathrm{P}\mathbb{B}^\times,\iota,K}(\mathbb{C}))$, then we have $p_* p^* f = \deg p \cdot f'$ where $\deg p$ stands for the degree of the morphism from $M_{\iota',K'}(\mathbb{C})$ to its image in $\mathrm{Sh}_{\mathrm{P}\mathbb{B}^\times,\iota,K}(\mathbb{C})$ and f' stands for the restriction of f to the image of p. We claim that

$$R(\varphi^0\otimes\varphi'^0)=2^{s-1}p^*T(\varphi\otimes\varphi')p_*.$$

Assume this for the moment, we see that Lemma A.3 follows from Lemma A.2. This is because of $p_*\overline{Q}=P'\in\mathrm{Sh}_{\mathrm{P}\mathbb{B}^\times}(\overline{F})$ and the projection formula. We only need to note that the height pairing in Lemma A.2 is over F while it is over E in Lemma A.3. Moreover we have $L(1,\pi^0,\mathrm{Ad})=L(1,\eta)L(1,\pi,\mathrm{Ad})$.

It remains to prove the claim. We only have to prove it in $\mathrm{Hom}(\mathrm{H}^{1,0}(M_\iota(\mathbb{C})),\mathrm{H}^{1,0}(M_\iota(\mathbb{C})))$. Let $f^0\in\pi^0$ and choose an element $f\in\pi^1$ whose restriction is f^0. We have

$$R(\varphi^0\otimes\varphi'^0)f^0=\mathrm{vol}\,\mathrm{Sh}_{\mathrm{U}(\mathbb{B}),\iota',K'}(\mathbb{C})\cdot\langle f,\varphi'\rangle\varphi^0,$$

and

$$p^*T(\varphi\otimes\varphi')p_*f^0=\mathrm{vol}\,\mathrm{Sh}_{\mathrm{P}\mathbb{B}^\times,\iota,K}(\mathbb{C})\cdot\langle p_*p^*f,\varphi'\rangle\varphi^0.$$

Here the pairings are the ones on $\mathrm{H}^{1,0}(\mathrm{Sh}_{\mathrm{P}\mathbb{B}^\times,\iota,K}(\mathbb{C}))$. The point is that p_*p^*f is not simply a multiple of f in the case $\pi\otimes\eta\circ\nu\not\simeq\pi$. Indeed, if $\pi\otimes\eta\circ\nu\simeq\pi$, then $p_*p^*f=\deg p\cdot f$, Thus $\langle p_*p^*f,\varphi'\rangle=\langle f,\varphi'\rangle$. If $\pi\otimes\eta\circ\nu\not\simeq\pi$, we however have

$$2\langle p_*p^*f,\varphi'\rangle=\deg p\cdot\langle f,\varphi'\rangle.$$

We also have $2\,\mathrm{vol}\,\mathrm{Sh}_{\mathrm{U}(\mathbb{B}),\iota',K'}(\mathbb{C})=\deg p\cdot\mathrm{vol}\,\mathrm{Sh}_{\mathrm{P}\mathbb{B}^\times,\iota,K}(\mathbb{C})$. The claim then follows. □

With all these preparations, we can now prove the following result.

Proposition A.4. *Let K' be an open compact subgroup of $\mathbb{G}(\mathbb{A}_{F,f})$. We keep the notation in Conjecture 5.1. Then we have*

$$(\mathrm{vol}\,K)^2(\mathrm{vol}\,M_{K'}\,\mathrm{vol}'\,K')\langle T(t_{\varphi^0,\varphi'^0})_{K',*}y_{K,0},y_{K,0}\rangle_{\mathrm{NT}}=\frac{\zeta_F(2)L'(\frac{1}{2},\pi_E)}{2^sL(1,\pi^0,\mathrm{Ad})}\prod_v\alpha_v^\natural(\varphi_v^0,\varphi_v'^0),$$

where the height pairing is between $\mathrm{Alb}(M_{K'})(E)$ and $\mathrm{Jac}(M_{K'})(E)$, the adjoint L-function on the right hand side is the adjoint L-function of π as a representation of $\mathrm{U}(\mathbb{B})$, $s=2$ or 1 according to whether $\pi\otimes(\eta\circ\nu)\simeq\pi$ or not. Under our choice of the measures we have $\mathrm{vol}\,K\#Y_K=1$.

Remark A.5. In terms of the notation in Conjecture 5.1, $\beta=s+1$. Moreover $\pi_E=\mathrm{BC}(\pi^0)$. In particular, Conjecture 5.1 holds in this case.

Proof. Let us first explain that the projector can be alternatively defined in terms of the Hecke operators as follows. Let K' be an open compact subgroup of $\mathbb{G}(\mathbb{A}_{F,f})$. Fix a measure on $\mathbb{G}(\mathbb{A}_{F,f})$ and denote the volume of any open compact by vol'. Then as in Subsection 4.2, we have a Hecke correspondence $T(t_{\varphi^0,\varphi'^0})_{K'}$ on $M_{K'}\times M_{K'}$. It defines a map

$$T(t_{\varphi^0,\varphi'^0})_{K',*}:\mathrm{Alb}(M_{K'})\to\mathrm{Jac}(M_{K'}).$$

As K' varies, the system $\{\mathrm{vol}' K' \mathrm{vol}\, M_{K'} T(t_{\varphi^0,\varphi'^0})_{K',*}\}$ defines a map $\mathrm{Alb}(M) \to \mathrm{Jac}(M)$, which is the desired projector $R(\varphi^0 \otimes \varphi'^0)$ (cf. [YZZ13, Proposition 3.14]).

Let us choose $\varphi, \varphi' \in \pi^1$ so that there restriction to π^0 is φ^0 and φ'^0 respectively. By the above explanation of the projectors in terms of the Hecke operators, the left hand side equals

$$(\mathrm{vol}\, K \# Y_K)^2 \langle R(\varphi^0 \otimes \varphi'^0)\overline{Q}, \overline{Q}\rangle_{\mathrm{NT}}.$$

By Lemma A.3, this equals

$$(\mathrm{vol}\, K \# Y_K)^2 \frac{\zeta_F(2) L'(\frac{1}{2}, \pi_E)}{2^s L(1,\eta) L(1, \pi, \mathrm{Ad})} \prod_v \widetilde{\alpha}_v^\natural(\varphi_v, \varphi_v').$$

In Conjecture 5.1, the product of the measures in defining $\alpha_v^\natural$ is the Tamagawa measure of $\mathbb{A}_E^\times/\mathbb{A}_F^\times$ while the product of the measures in defining $\widetilde{\alpha}_v^\natural$ is $L(1,\eta)$ times the Tamagawa measure. It follows that the left hand side equals

$$(\mathrm{vol}\, K \# Y_K)^2 \frac{\zeta_F(2) L'(\frac{1}{2}, \pi_E)}{2^s L(1, \pi, \mathrm{Ad})} \prod_v \alpha_v^\natural(\varphi_v^0, \varphi_v'^0).$$

It remains to prove that $\mathrm{vol}\, K \# Y_K = 1$ when K is small. Note that $E^\times/F^\times$ is discrete in $\mathbb{A}_{E,f}^\times/\mathbb{A}_{F,f}^\times$. It follows from the definition of Y_K that

$$\mathrm{vol}\, K \# Y_K = \mathrm{vol}\, E^\times \backslash \mathbb{A}_{E,f}^\times / \mathbb{A}_{F,f}^\times.$$

Moreover under the normalized local measure, we have $\mathrm{vol}\, \mathbb{C}^\times/\mathbb{R}^\times = 1$. Therefore we have

$$\mathrm{vol}\, E^\times \backslash \mathbb{A}_{E,f}^\times / \mathbb{A}_{F,f}^\times = \mathrm{vol}\, E^\times \backslash \mathbb{A}_E^\times / \mathbb{A}_F^\times,$$

where the vol on the right hand side is computed using the product the normalized local measures divided by $2L(1,\eta)$. The product of normalized local measures (over all places) is $L(1,\eta)$ times the Tamagawa measure of $\mathbb{A}_E^\times/\mathbb{A}_F^\times$. Thus we have $\mathrm{vol}\, E^\times \backslash \mathbb{A}_E^\times/\mathbb{A}_F^\times = 1$ and hence $\mathrm{vol}\, K \# Y_K = 1$. This proves the proposition. □

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