

ON THE GLOBAL GAN–GROSS–PRASAD CONJECTURE FOR UNITARY GROUPS: APPROXIMATING SMOOTH TRANSFER OF JACQUET–RALLIS

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ABSTRACT. Zhang [Zha14] proved the global Gan–Gross–Prasad conjecture for $U(n+1) \times U(n)$ under some local conditions. One of the conditions is that the unitary groups are split at the archimedean places. We remove this assumption at the archimedean places in this paper.

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1. INTRODUCTION

In [Zha14], W. Zhang proved the global Gan–Gross–Prasad conjecture [GGP12, Conjecture 26.1] for the unitary groups $U(n+1) \times U(n)$ under some local conditions using a simple version of the Jacquet–Rallis relative trace formula. Zhang’s result requires (among other things) that the unitary groups are split at all archimedean places. In this paper, we remove this condition at the archimedean places.

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The Gan–Gross–Prasad conjecture. Let E/F be a quadratic extension of number fields and $W \subset V$ be a pair of hermitian spaces of dimension n and $n + 1$ over E respectively. Let $U(W)$ and $U(V)$ be the corresponding unitary groups. The embedding $W \subset V$ induces an embedding $U(W) \subset U(V)$. Let $G = U(V) \times U(W)$ and $H = U(W)$. The group H is considered as a subgroup of G via the diagonal embedding.

Let π be an irreducible cuspidal automorphic representation of $G(\mathbb{A}_F)$. Here by a cuspidal automorphic representation, we mean a subrepresentation of the space of the cuspidal automorphic forms. We put

$$P(\varphi) = \int_{H(F) \backslash H(\mathbb{A}_F)} \varphi(h) dh, \quad \varphi \in \pi.$$

This is an $H(\mathbb{A}_F)$ -invariant continuous linear form on π .

Let $G' = \text{Res}_{E/F}(\text{GL}_{n+1} \times \text{GL}_n)$. Let v a place of F . If v splits in E or the representation π_v is unramified, one can define a representation $\text{BC}(\pi_v)$ of $G'(F_v)$, called the local base change of π_v . It follows from [Mok15, Theorem 2.5.2] and [KMSW, Theorem 1.7.1] that we may associate an isobaric automorphic representation Π of $G'(\mathbb{A}_F)$ to π such that for almost all places v of F , $\Pi_v \simeq \text{BC}(\pi_v)$. We call it the weak base change of π and denote it by $\text{BC}(\pi)$. It follows from [Mok15, Proposition 10.0.4] and [KMSW, Corollary 3.3.2] that for all split places v of F , $\Pi_v \simeq \text{BC}(\pi_v)$.

Let $W' \subset V'$ be another pair of hermitian spaces of dimension n and $n + 1$ respectively. We say that $W' \subset V'$ is relevant if $V/W \simeq V'/W'$ as one dimensional hermitian spaces. We shall add the subscripts W and W' to indicate the dependance on W or W' . For instance, we write G_W for G and $G_{W'}$ for $U(V') \times U(W')$. We fix an isomorphism for almost all places v of F between $G_W(F_v)$ and $G_{W'}(F_v)$. Let π' be an irreducible cuspidal automorphic representation of $G_{W'}(\mathbb{A}_F)$. We say that π and π' are nearly equivalent if for almost all places we have $\pi_v \simeq \pi'_v$. By the strong multiplicity one theorem for GL_n , if π and π' are nearly equivalent, their weak base change must be the same. For π' , we have an analogous $H_{W'}(\mathbb{A}_F)$ -invariant continuous linear form P' on π' .

We define the L -function

$$L(s, \Pi, \text{St}) = L(s, \Pi_{n+1} \times \Pi_n),$$

where the right hand side is the Rankin–Selberg convolution of Π_{n+1} and Π_n if $\Pi = \Pi_{n+1} \boxtimes \Pi_n$ and Π_i is an irreducible cuspidal automorphic representation of $\text{GL}_i(\mathbb{A}_E)$, $i = n, n + 1$. The main result of this paper is the following theorem.

Theorem 1.1. *Let π be an irreducible cuspidal automorphic representation of $G(\mathbb{A}_F)$ and Π be its weak base change. Assume that there are two split places v_1, v_2 of F , v_1 being nonarchimedean, such that π_{v_1} is supercuspidal and π_{v_2} is tempered. Then the following are equivalent.*

- (1) *The central L -value does not vanish: $L(\frac{1}{2}, \Pi, \text{St}) \neq 0$.*

- (2) *There is a relevant pair of hermitian spaces $W' \subset V'$ and an irreducible cuspidal automorphic representation π' of $G_{W'}(\mathbb{A}_F)$ that is nearly equivalent to π , such that the linear form P' on π' is not identically zero.*

Remark 1.2. The recent work of Beuzart-Plessis [BP] allows us to further weaken the hypothesis to that “there is a place v_1 such that Π_{v_1} is supercuspidal” (v_1 need not be split and there is no temperedness assumption at another place v_2). This relies on the local spherical character identity established in [BP].

The relative trace formulae approach. We now discuss the proof of Theorem 1.1. Jacquet–Rallis [JR11] proposed an approach based on the relative trace formulae. There are three major ingredients in this relative trace formula.

- (1) The fundamental lemma.
- (2) The smooth transfer conjecture.
- (3) The fine spectral expansion.

The fundamental lemma of Jacquet–Rallis was proved by Yun [Yun11] soon after [JR11]. The fine spectral expansion seems to be a very difficult problem on its own. It is needed to remove the condition that the representation is supercuspidal at some split place in Theorem 1.1. We will not deal with this problem in this paper.

The main problem that we will focus on in this paper is the smooth transfer conjecture of Jacquet–Rallis. This is a local problem. One of the main results in [Zha14] is the proof of this conjecture in the non-archimedean case and the case when the unitary groups are split. This result leads to a simple version of the Jacquet–Rallis relative trace formula. With this simple version of the relative trace formula, Zhang proved Theorem 1.1 in [Zha14] under the following additional assumption.

All archimedean places of F split in E .

The results and the techniques in [Zha14] are significant breakthrough towards a solution of the Gan–Gross–Prasad conjecture. However, for many arithmetic applications, the assumption that E/F splits at all archimedean places seems too restrictive. The goal of this paper is to remove this assumption. We have not succeeded in proving the smooth transfer conjecture of Jacquet–Rallis. However, we are able to prove a weaker version of it which is sufficient to imply Theorem 1.1. We briefly discuss our approach.

To Π , we associate certain distribution $I_\Pi(f')$ where f' is a Schwartz test function on $G'(\mathbb{A}_F)$. It has the property that $L(\frac{1}{2}, \Pi, \text{St}) \neq 0$ if and only if I_Π is not identically zero. To each $\pi_{W'}$ that is nearly equivalent to π , we associate certain distribution $J_{\pi_{W'}}(f_{W'})$ where $f_{W'}$ is a Schwartz test function on $G_{W'}(\mathbb{A}_F)$. It has the property that P' is not identically zero on $\pi_{W'}$ if and only if $J_{\pi_{W'}}$ is not identically zero. Assume that $f' = \otimes f'_v$ and $f_{W'} = \otimes f_{W',v}$ are factorizable. We

say that f' and the collection of test functions $\{f_{W'}\}$ where W' runs over all hermitian spaces of dimension n are smooth transfer of each other if for all places v of F , f'_v and $\{f_{W',v}\}$ are smooth transfer of each other, the later meaning that the regular semisimple orbital integrals of f'_v and $\{f_{W',v}\}$ coincide (see Section 2 for the definitions). The Jacquet–Rallis relative trace formula is an identify

$$(1.1) \quad I_{\Pi}(f') = \sum_{W'} \sum_{\pi_{W'}} J_{\pi_{W'}}(f_{W'})$$

where f' and $\{f_{W'}\}$ are smooth transfer of each other, the outer sum on the right hand side runs over all hermitian spaces of dimension n and the inner sum runs over all automorphic representations $\pi_{W'}$ of $G_{W'}(\mathbb{A}_F)$ that are nearly equivalent to π .

The smooth transfer conjecture of Jacquet–Rallis asserts that for any place v of F , the smooth transfer of any f'_v (resp. $\{f_{W',v}\}$) exists. Zhang [Zha14] proved this conjecture when $E_v/F_v \neq \mathbb{C}/\mathbb{R}$. Our major innovation in this paper is the following. Assume that $E_v/F_v = \mathbb{C}/\mathbb{R}$. We say that f'_v is transferable if its smooth transfer exists. For a fixed W , we say that $f_{W,v}$ is transferable if the collection $\{f_{W,v}; 0\}$ is transferable where 0 means that $f_{W',v} = 0$ if $W' \neq W$. We prove the following statement.

(*) *Every f'_v (resp. f_W) can be approximated by transferable ones.*

Here approximation means the approximation in the the space of Schwartz functions $\mathcal{S}(G'(F_v))$ (resp. $\mathcal{S}(G_W(F_v))$). We refer the readers to Notation below for the explanation of convergence in these spaces.

Being transferable is only a restriction on the nonsplit archimedean places. At other places, all test functions are transferable. The result (*) is sufficient to imply Theorem 1.1. We sketch the argument to prove that (1) implies (2). A more detailed proof is contained in Section 12. Assume (1) of Theorem 1.1. Then we may choose a nice test function f' (this is a condition on f'_v where v is split in E , see [Zha14, Section 2.2] and Section 12 for some discussions) so that $I_{\Pi}(f') \neq 0$. Fix a nonsplit archimedean place v_0 of F . It is not hard to show that $I_{\Pi}(f')$ is a continuous linear functional on $\mathcal{S}(G'(F_{v_0}))$ if we fix all components of $f' = \otimes f'_v$ other than f'_{v_0} . Then we apply (*) to modify f' at the nonsplit archimedean places so that f' is nice and transferable and we still have $I_{\Pi}(f') \neq 0$. It follows from [Zha14] that the relative trace formula (1.1) holds for nice test functions. Then we conclude that there is at least one nonzero term on the right hand side. This proves (1) implies (2).

We remark that with the same technique, using approximating smooth transfer rather than the actual smooth transfer, the assumption in [Xue14, Theorem 1.1.1] at the archimedean places can also be removed.

Smooth transfer. We explain a little more details about the proof of (*). The proof is largely inspired by [Zha14].

We first recall how the smooth transfer conjecture was proved in the non-archimedean case in [Zha14]. Let us introduce more notation. Let F be a local field and $E = F \times F$ or a quadratic field extension of F . Let V be an n dimensional hermitian space over E and $H = \mathrm{U}(V)$. Let \mathfrak{h} be the Lie algebra of H and $\mathcal{V} = \mathfrak{h} \times V$. The group H acts on \mathcal{V} by conjugating on \mathfrak{h} and on V in the natural way. For test functions on \mathcal{V} , there are three $H(F)$ -invariant Fourier transform, namely the Fourier transform with respect to \mathfrak{h} , V and the total space \mathcal{V} , which we denote by $\mathcal{F}_{\mathfrak{h}}$, \mathcal{F}_V and $\mathcal{F}_{\mathcal{V}}$ respectively. The smooth transfer conjecture is first reduced to an analogous statement on the Lie algebra, i.e. a transfer problem for test functions on \mathcal{V} .

Now assume that F is non-archimedean. Then the space of test functions is $\mathcal{C}_c^\infty(\mathcal{V})$, the space of compactly supported locally constant functions on \mathcal{V} . Let $\mathcal{V} \rightarrow \mathcal{V} // H$ be the categorical quotient (see below *Notation* for the explanation). The categorical quotient $\mathcal{V} // H$ is isomorphic to the $2n$ -dimensional affine space over F . We let $\mathcal{N} \subset \mathcal{V}$ be the nilpotent cone in \mathcal{V} , namely the inverse image of $0 \in \mathcal{V} // H$. The smooth transfer conjecture is proved by induction on the dimension of V . There are three major steps.

- (1) If $f \in \mathcal{C}_c^\infty(\mathcal{V} \setminus \mathcal{N})$, then we apply the induction hypothesis to show that the smooth transfer of f exists. The point is that the orbital integral near a semisimple point that is not in the nilpotent cone “looks like” an orbital integral of the same type with some different hermitian space V' and $\dim V' < \dim V$. The notion of the sliced representation and semi-algebraic Luna slice are systematically used in this step.
- (2) If $f \in \mathcal{C}_c^\infty(\mathcal{V})$ is transferable, then so is $\mathcal{F}_* f$ where $*$ = \mathfrak{h}, V or \mathcal{V} . The main tool is a local relative trace formula. The essential ingredient in this local relative trace formula is the estimate of the upper bound of the orbital integrals.
- (3) Any $f \in \mathcal{C}_c^\infty(\mathcal{V})$ can be written as

$$f = f_1 + f_2 + f_3 + f_4 + f_0,$$

where $f_1, \mathcal{F}_{\mathfrak{h}} f_2, \mathcal{F}_V f_3, \mathcal{F}_{\mathcal{V}} f_4 \in \mathcal{C}_c^\infty(\mathcal{V} \setminus \mathcal{N})$ and any regular semisimple orbital integrals of f_0 vanish. This statement is deduced from the following fact.

(**) *Let D be an $H(F)$ -invariant distribution on \mathcal{V} . If D and all its $H(F)$ -invariant Fourier transforms are supported in \mathcal{N} , then $D = 0$.*

This fact is proved in [Aiz13, Theorem 6.2.1].

Now assume that F is archimedean. One would like to prove the smooth transfer conjecture along the same line as in the non-archimedean case. Then the space of test function should be replaced by $\mathcal{S}(\mathcal{V})$, the Schwartz space over \mathcal{V} . Steps (1) and (2) above can be proved by roughly the same technique as in the non-archimedean case. The fact that we are dealing with Schwartz functions brings in some technical difficulties at various points. But this is not a serious problem. The major difficulty appears in Step (3). The fact (**) also holds in the archimedean

case. However, (**) is not sufficient to imply the conclusion in Step (3), but only a weaker version of it, namely,

(***) *Any $f \in \mathcal{S}(\mathcal{V})$ can be approximated by functions of the form $f_1 + f_2 + f_3 + f_4 + f_0$, where $f_1, \mathcal{F}_{\mathfrak{h}}f_2, \mathcal{F}_V f_3, \mathcal{F}_{\mathcal{V}}f_4 \in \mathcal{S}(\mathcal{V} \setminus \mathcal{N})$ and any regular semisimple orbital integrals of f_0 vanish.*

We have not succeeded in proving Step (3) in the archimedean case, nor the smooth transfer conjecture. However, to prove (*), the statement (***) is sufficient. Like the non-archimedean case, we again use induction. The difference is that we have a weaker induction hypothesis in Step (1) and need to prove a weaker statement. Thus our proof of (*) is divided into the following steps.

- (1) Reduce the statement (*) to the Lie algebra. This is done in Section 3.
- (2) We prove Step (2) above. This is done in Section 4 to Section 9.
- (3) We use induction and the fact (***) to prove the Lie algebra version of (*). This is done in Sections 10 and 11.

Notation. Below we list some notation that will be used through out this paper.

- We usually use capital letters G, H etc. to denote groups and use the corresponding Gothic letters $\mathfrak{g}, \mathfrak{h}$ etc. to denote their Lie algebras.
- Let F be a field. Then we use F_n (resp. F^n) to denote the space of row (resp. column) vectors with n entries. We put $e = e_n = (0, \dots, 0, 1) \in F_n$. We write $\text{diag}[g_1, \dots, g_r]$ for the blocked diagonal matrix with diagonal blocks g_1, \dots, g_r .
- Let F be field and $\chi : F^\times \rightarrow \mathbb{C}^\times$ be a character. We often identify χ with a character of $\text{GL}_n(F)$, by composing it with the determinant map.
- Let F be a local field of characteristic zero and X an algebraic variety over F . Then X is endowed with the Zariski topology and $X(F)$ is endowed with the analytic topology. In particular, if F is non-archimedean, then $X(F)$ is an l -space in the sense of [BZ76]. If $F = \mathbb{R}$ and X is smooth, then $X(F)$ is a Nash manifold [AG08]. If $F = \mathbb{C}$, then $X(F)$ is a complex analytic space.
- We make use of the notion of categorical quotients. Let $X = \text{Spec } A$ be an affine variety over a field F of characteristic zero and H a reductive group acting on X . Let $\gamma \in X(F)$ and $h \in H(F)$. The action of h on γ is usually written as γ^h . Let $Y \subset X(F)$ and $U \subset H(F)$. We denote by Y^U the set $\{\gamma^h \mid \gamma \in Y, h \in U\}$. The element γ is called semisimple if the orbit of γ is Zariski closed. If F is a local field, this is equivalent to that the orbit is a closed subset of $X(F)$ in the analytic topology. The element γ is called regular if its stabilizer is of minimal dimension. We denote by X_{rs} the locus of regular semisimple elements in X . This is an Zariski open subset of X . Let A^H be the ring of invariant functions on X . The categorical quotient is a morphism

$q : X \rightarrow X//H = \text{Spec } A^H$. It parameterizes semisimple orbits in X . We usually simply call $X//H$ the categorical quotient. We write $(X//H)_{\text{rs}}$ for the regular semisimple locus. This is a Zariski open subset of $X//H$. Suppose that F is a local field. An open subset $\Omega \subset X(F)$ is called saturated if there is an open subset $U \subset (X//H)(F)$ so that $\Omega = q^{-1}(U)$.

- We make use of the notion of Schwartz functions systematically. Over a non-archimedean local field, if M is an l -space, then the Schwartz space $\mathcal{S}(M) = \mathcal{C}_c^\infty(M)$, the space of compactly supported and locally constant functions on M . Over an archimedean local field, we choose to work in the setting of Schwartz functions on Nash manifolds [AG08]. Let M be a Nash manifold and we denote by $\mathcal{S}(M)$ the space of Schwartz functions on M . These are smooth functions f on M so that $\sup_{x \in M} |Df(x)|$ is finite for any Nash differential operator D . For any Nash differential operator D on M , we define a seminorm $\|\cdot\|_D$ on $\mathcal{S}(M)$ by

$$\|f\|_D = \sup_{x \in M} |Df(x)|.$$

Then under the seminorms $\{\|\cdot\|_D\}_D$, $\mathcal{S}(M)$ is a Fréchet space [AG08, Corollary 4.1.2] and contains $\mathcal{C}_c^\infty(M)$ as a dense subspace. Suppose that $\{g_n\}$ is a sequence of Schwartz functions on M . We denote by $g_n \rightrightarrows f$ if g_n converges to f in $\mathcal{S}(M)$. More concretely, $g_n \rightrightarrows f$ means that Dg_n converges to Df uniformly for any Nash differential operator D on M .

Remarks on the measures. All integrals in this paper depend on the measures that we choose. However, as we are dealing with the nonvanishing problem, the choice of the measure usually does not matter. Thus unless otherwise specified at certain places (e.g. Kloosterman integrals), we assume that we have fixed a measure on each space over which we are integrating. On the groups, we always fix a Haar measure. On vector spaces over the archimedean local fields, we always take the Lebesgue measure.

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2. ORBITAL INTEGRALS AND THE SMOOTH TRANSFER CONJECTURE OF JACQUET–RALLIS

In this section, we define the orbital integrals and recall the smooth transfer conjecture of Jacquet–Rallis.

Let F be a local field of characteristic zero and E an quadratic étale algebra over F . Let η be the quadratic character of F^\times associated to E/F by the local class field theory.

We start from the general linear side. We shall always consider GL_n as a subgroup of GL_{n+1} via the embedding $h \mapsto \mathrm{diag}[h, 1]$. Let S_{n+1} be the subvariety of $\mathrm{Res}_{E/F} \mathrm{GL}_{n+1}$ consisting of matrices satisfying $g\bar{g} = 1$. The group GL_n acts on S_{n+1} by conjugation. Let σ be the composition of maps

$$\mathrm{GL}_{n+1}(E) \times \mathrm{GL}_n(E) \rightarrow \mathrm{GL}_{n+1}(E) \rightarrow S_{n+1}(F),$$

where the first map is given by $(g_{n+1}, g_n) \mapsto g_n^{-1}g_{n+1}$ and the second map is given by $g \mapsto g\bar{g}^{-1}$. Note that by the Hilbert Satz 90, the map σ is surjective. Any element $\gamma \in S_{n+1}(F)$ is of the form $\gamma = t\bar{t}^{-1}$ where $t \in \mathrm{GL}_{n+1}(E)$.

We fix a character $\mu : E^\times \rightarrow \mathbb{C}^\times$ so that $\mu|_{F^\times} = \eta$. We view it as a character of $\mathrm{GL}_n(E)$ by composing it with the determinant map. We define a map

$$\tilde{\sigma} : \mathcal{S}(\mathrm{GL}_{n+1}(E) \times \mathrm{GL}_n(E)) \rightarrow \mathcal{S}(S_{n+1}(F))$$

as follows. Let $f' \in \mathcal{S}(\mathrm{GL}_{n+1}(E) \times \mathrm{GL}_n(E))$. Then we put

$$\tilde{\sigma}(f')(t\bar{t}^{-1}) = \int_{\mathrm{GL}_n(E)} \int_{\mathrm{GL}_{n+1}(F)} f'(gth, g)\mu^n(th)dh dg.$$

Lemma 2.1. *The map $\tilde{\sigma}$ is continuous and surjective. Moreover, the inverse image of any dense subset L of $\mathcal{S}(S_{n+1}(F))$ is dense in $\mathcal{S}(\mathrm{GL}_{n+1}(E) \times \mathrm{GL}_n(E))$.*

Proof. We only need to prove the lemma when F is archimedean. The continuity of $\tilde{\sigma}$ is clear. The surjectivity of $\tilde{\sigma}$ follows from [AG09, Theorem B.2.4]. The last statement of the lemma follows from the Banach Open Mapping Theorem. \square

Let $f' \in \mathcal{S}(S_{n+1}(F))$ be a Schwartz test function and $\gamma \in S_{n+1}(F)$ be a regular semisimple element. We define

$$O(\gamma, f') = \int_{\mathrm{GL}_n(F)} f'(h^{-1}\gamma h)\eta(h)dh.$$

Lemma 2.2. *For any regular semisimple $\gamma \in S_{n+1}(F)$, the linear functional*

$$f' \mapsto O(\gamma, f'), \quad f' \in \mathcal{S}(S_{n+1}(F))$$

is continuous.

Proof. Since γ is regular semisimple, its orbit $\gamma^{\mathrm{GL}_n(F)}$ in $S_{n+1}(F)$ is a closed algebraic subset. In particular, if F is archimedean, it is a closed Nash submanifold. The restriction map

$$\mathcal{S}(S_{n+1}(F)) \rightarrow \mathcal{S}(\gamma^{\mathrm{GL}_n(F)})$$

is thus continuous. The continuity of $O(\gamma, f')$ then follows. \square

We fix a transfer factor Ω as follows:

$$\Omega(\gamma) = \mu((\det \gamma)^{-[(n+1)/2]} \det(e, e\gamma, e\gamma^2, \dots, e\gamma^n)), \quad \gamma \in S_{n+1}(F).$$

Now we treat the unitary group side. Let V be an $n+1$ dimensional hermitian space over E and $W \subset V$ an n dimensional subspace that admits an orthogonal decomposition $V = W \oplus Ev_0$ where v_0 is an anisotropic vector in V . We may assume that $\langle v_0, v_0 \rangle = 1$ where $\langle -, - \rangle$ is the hermitian form on V . Another pair of hermitian spaces $W' \subset V'$ of dimension n and $n+1$ respectively will be called relevant if it admits the orthogonal decomposition $V' = W' \oplus Ev_0$. Let $U(V)$ and $U(W)$ be the corresponding unitary groups. We always consider $U(W)$ as a subgroup of $U(V)$ via the embedding induced by $W \subset V$. Then $U(W)$ acts on $U(V)$ by conjugation.

For $\delta = (\delta_{n+1}, \delta_n) \in U(V)(F) \times U(W)(F)$, we put $\tau(\delta) = \delta_{n+1}\delta_n^{-1} \in U(V)(F)$. We define a map of functions $\tilde{\tau} : \mathcal{S}(U(V)(F) \times U(W)(F)) \rightarrow \mathcal{S}(U(V)(F))$ as

$$\tilde{\tau}(f)(\delta) = \int_{U(W)(F)} f(\delta h, h) dh.$$

We sometimes denote this map by $\tilde{\tau}_W$ to emphasize the dependance on the space W .

Lemma 2.3. *The map $\tilde{\tau}$ is continuous and surjective. Moreover, the inverse image of any dense subset L of $\mathcal{S}(U(V)(F))$ is a dense subset of $\mathcal{S}(U(V)(F) \times U(W)(F))$.*

Proof. This can be proved in the same way as Lemma 2.1. We omit the details. \square

Let $f \in \mathcal{S}(U(V)(F))$ be a Schwartz function and $\delta \in U(V)(F)$ be a regular semisimple element. We define

$$O(\delta, f) = \int_{U(W)(F)} f(h^{-1}\delta h) dh.$$

Lemma 2.4. *For any regular semisimple $\delta \in U(V)(F)$, the linear functional*

$$f \mapsto O(\delta, f), \quad f \in \mathcal{S}(U(V)(F))$$

is continuous.

Proof. This can be proved in the same way as Lemma 2.2. We omit the details. \square

Choose a basis of W and we obtain a basis of V by adding v_0 . Then $U(V)$ is realized as a subgroup of $\text{Res}_{E/F} \text{GL}_{n+1}$. We say that $\gamma \in S_{n+1}(F)$ and $\delta \in U(V)(F)$ match if they are conjugate by an element in $\text{GL}_n(E)$ (both considered as elements in $\text{GL}_{n+1}(E)$). This notion depends only on the orbits of γ and δ under the action of $\text{GL}_n(F)$ and $U(W)(F)$ respectively. Thus we also say that the orbit of γ and the orbit of δ match in this case. The matching of orbits defines a bijection between regular semisimple orbits in $S_{n+1}(F)$ and the disjoint union of regular semisimple orbits in all $U(V)(F)$ where $V = W \oplus Ev_0$ and W ranges over all isomorphism classes of hermitian spaces of dimension n . Let $f' \in \mathcal{S}(S_{n+1}(F))$ and $\{f_W\}$ be a collection of

functions with $f_W \in \mathcal{S}(\mathrm{U}(V)(F))$ and W ranges over all isomorphism classes of hermitian spaces of dimension n and $V = W \oplus Ev_0$. We say that f' and $\{f_W\}$ are smooth transfer of each other if for all matching regular semisimple $\gamma \in S_{n+1}(F)$ and $\delta \in \mathrm{U}(V)(F)$, we have

$$\Omega(\gamma)O(\gamma, f') = O(\delta, f_W).$$

If $f' \in \mathcal{S}(\mathrm{GL}_{n+1}(E) \times \mathrm{GL}_n(E))$ and $f_W \in \mathcal{S}(\mathrm{U}(V)(F) \times \mathrm{U}(W)(F))$ where W ranges over all isomorphism classes and $V = W \oplus Ev_0$, we say that f' and the collection $\{f_W\}$ are smooth transfer of each other if $\tilde{\sigma}(f')$ and the collection $\{\tilde{\tau}_W(f_W)\}$ are smooth transfer of each other. We say that $f' \in \mathcal{S}(S_{n+1}(F))$ (resp. $\{f_W\}$) is transferable if its smooth transfer exists. For a fixed W , we say that $f_W \in \mathcal{S}(\mathrm{U}(V)(F))$ is transferable if the collection $\{f_W, 0, \dots, 0\}$ is transferable. Here 0 means the $f_{W'} = 0$ if $W' \neq W$. We say that $f' \in \mathcal{S}(\mathrm{GL}_{n+1}(E) \times \mathrm{GL}_n(E))$ is transferable if $\tilde{\sigma}(f')$ is transferable. For a fixed W , we say that $f_W \in \mathcal{S}(\mathrm{U}(V)(F) \times \mathrm{U}(W)(F))$ is transferable if the collection $\{\tilde{\tau}_W(f_W), 0, \dots, 0\}$ is transferable. Here 0 means the $f_{W'} = 0$ if $W' \neq W$.

Conjecture 2.5. *Any test function $f' \in \mathcal{S}(S_{n+1}(F))$ (resp. $f \in \mathcal{S}(\mathrm{U}(V)(F))$) is transferable.*

Theorem 2.6. *If $E/F \neq \mathbb{C}/\mathbb{R}$, then Conjecture 2.5 holds.*

This is one of the main results of [Zha14].

The following is the main technical result of this paper.

Theorem 2.7. *Let $E/F = \mathbb{C}/\mathbb{R}$. The set of transferable functions in $\mathcal{S}(S_{n+1}(F))$ is dense. Similarly, the set of transferable functions in $\mathcal{S}(\mathrm{U}(V)(F))$ is dense.*

This will be proved in the next section.

Corollary 2.8. *Let $E/F = \mathbb{C}/\mathbb{R}$. Any $f' \in \mathcal{S}(\mathrm{GL}_{n+1}(E) \times \mathrm{GL}_n(E))$ can be approximated by transferable functions. Similarly, any $f_W \in \mathcal{S}(\mathrm{U}(V)(F) \times \mathrm{U}(W)(F))$ can be approximated by transferable ones.*

Proof. This follows from Theorem 2.7 and Lemma 2.1 and 2.3. □

3. ORBITAL INTEGRALS AND THE SMOOTH TRANSFER ON THE LIE ALGEBRA

We introduce in this section a Lie algebra analogue of the Jacquet–Rallis smooth transfer conjecture. We keep the notations from the previous section.

We start from the general linear group side. Let \mathfrak{s}_{n+1} be the “Lie algebra” of $S_{n+1}(F)$, namely

$$\mathfrak{s}_{n+1} = \{X \in M_{n+1}(E) \mid X + \overline{X} = 0\}.$$

We view \mathfrak{s}_{n+1} as an algebraic variety over F . The group GL_n acts on \mathfrak{s}_{n+1} by conjugation. Let $f' \in \mathcal{S}(\mathfrak{s}_{n+1}(F))$ be a Schwartz function and $\gamma \in \mathfrak{s}_{n+1}(F)$ a regular semisimple element. We define

$$O(\gamma, f') = \int_{\mathrm{GL}_n(F)} f'(h^{-1}\gamma h)\eta(h)dh.$$

Similarly to the group case, the integral is absolutely convergent and the linear functional $f' \mapsto O(\gamma, f')$ is continuous.

We now consider the unitary group side. We choose a basis of V so that the hermitian form on V is given by a matrix β . Let $\mathfrak{u}(V)$ be the Lie algebra of $\mathrm{U}(V)$, namely,

$$\mathfrak{u}(V) = \{X \in M_{n+1}(E) \mid \beta X + {}^t\bar{X}\beta = 0\},$$

We view $\mathfrak{u}(V)$ as an algebraic variety over F and $\mathrm{U}(W)$ acts on it by conjugation. Let $f_W \in \mathcal{S}(\mathfrak{u}(V)(F))$ be a Schwartz function and $\delta \in \mathfrak{u}(V)(F)$ be a regular semisimple element. We define

$$O(\gamma, f_W) = \int_{\mathrm{U}(W)(F)} f_W(h^{-1}\delta h)dh.$$

Similarly to the group case, the linear functional $f_W \mapsto O(\gamma, f_W)$ is continuous.

We fix a transfer factor ω on $\mathfrak{s}_{n+1}(F)$ as follows. Fix an element $\tau \in F^\times$ so that $E = F(\sqrt{\tau})$. Then we define

$$\omega(\gamma) = \eta(\det(e, e\gamma/\sqrt{\tau}, \dots, e(\gamma/\sqrt{\tau})^n)).$$

The matching of orbits and the smooth transfer conjecture maybe formulated in terms of the categorical quotients. The categorical quotients $\mathfrak{s}_{n+1}/\mathrm{GL}_n$ and $\mathfrak{u}(V)/\mathrm{U}(W)$ are naturally identified and are isomorphic to the $2n + 1$ -dimensional affine space, c.f. [Zha14, Section 3]. Denote this quotient by Q . The canonical morphism $q : \mathfrak{s}_{n+1} \rightarrow \mathfrak{s}_{n+1}/\mathrm{GL}_n$ is given by

$$\gamma \mapsto (\mathrm{Tr} \wedge^i \gamma, e\gamma^j e), \quad i = 1, \dots, n+1, \quad j = 1, \dots, n.$$

Similarly the canonical morphism $q_W : \mathfrak{u}(V) \rightarrow \mathfrak{u}(V)/\mathrm{U}(W)$ are given by the morphism

$$\delta \mapsto (\mathrm{Tr} \wedge^i \delta, \langle \delta^j v_0, v_0 \rangle), \quad i = 1, \dots, n+1, \quad j = 1, \dots, n,$$

where $\langle -, - \rangle$ stands for the hermitian form on V . We say that $\gamma \in \mathfrak{s}_{n+1}(F)$ and $\delta \in \mathfrak{u}(V)(F)$ match if their images in $Q(F)$ coincide. Similar to the group case, the matching of regular semisimple orbits defines a bijection between the regular semisimple orbits in $\mathfrak{s}_{n+1}(F)$ and disjoint union of regular semisimple orbits in $\mathfrak{u}(V)(F)$ where W runs over all isomorphism classes of hermitian spaces of dimension n and $V = W \oplus Ev_0$, i.e.

$$\mathfrak{s}_{n+1}(F)_{\mathrm{rs}}/\mathrm{GL}_n(F) \longleftrightarrow \coprod_W \mathfrak{u}(V)(F)_{\mathrm{rs}}/U(W)(F),$$

where the subscript rs stands for the subset of regular semisimple elements.

Let $f' \in \mathcal{S}(\mathfrak{s}_{n+1}(F))$ and $f_W \in \mathcal{S}(\mathfrak{u}(V)(F))$ be Schwartz test functions. We say that f' and the collection $\{f_W\}$ are smooth transfer of each other if for all matching regular semisimple orbits $\gamma \in \mathfrak{s}_{n+1}(F)$ and $\delta \in \mathfrak{u}(V)(F)$, we have

$$\omega(\gamma)O(\gamma, f') = O(\delta, f_W).$$

We say that f' (resp. $\{f_W\}$) is transferable if its smooth transfer exists. For a fixed W , we say that the smooth transfer of $f_W \in \mathcal{S}(\mathfrak{u}(V)(F))$ exists if we can find a collection of functions $\{f_{W'}; 0, \dots, 0\}$ which is transferable. Here 0 means the $f_{W'} = 0$ if $W' \neq W$.

Conjecture 3.1. *All test functions $f' \in \mathcal{S}(\mathfrak{s}_{n+1}(F))$ (resp. $f \in \mathcal{S}(\mathfrak{u}(V)(F))$) are transferable.*

Theorem 3.2. *If $E/F \neq \mathbb{C}/\mathbb{R}$, then Conjecture 3.1 holds.*

Again this is proved in [Zha14].

Theorem 3.3. *Assume that $E/F = \mathbb{C}/\mathbb{R}$. Then the set of transferable functions in $\mathcal{S}(\mathfrak{s}_{n+1}(F))$ (resp. $\mathcal{S}(\mathfrak{u}(V)(F))$) is dense.*

This will be proved in Section 11.

For the rest of this section, we will take $E/F = \mathbb{C}/\mathbb{R}$.

Lemma 3.4. *Suppose that f' and $\{f_W\}$ are smooth transfer of each other. Let $\alpha \in \mathcal{S}(Q(\mathbb{R}))$ be a Schwartz function. Let $\tilde{\alpha}' = \alpha \circ q$ and for each hermitian space W let $\widetilde{\alpha}_W = \alpha \circ q_W$. Then $f'\tilde{\alpha}' \in \mathcal{S}(\mathfrak{s}_{n+1}(\mathbb{R}))$ and $f_W\widetilde{\alpha}_W \in \mathcal{S}(\mathfrak{u}(V)(\mathbb{R}))$. Moreover, $f'\tilde{\alpha}'$ and $\{f_W\widetilde{\alpha}_W\}$ are smooth transfer of each other.*

Proof. Since α is a Schwartz function, the functions $\tilde{\alpha}'$ and $\widetilde{\alpha}_W$ are tempered functions in the sense of [AG08, Definition 4.2.1]. It follows from [AG08, Proposition 4.2.1] that $f'\tilde{\alpha}'$ and $f_W\widetilde{\alpha}_W$ are Schwartz functions. Let $\gamma \in \mathfrak{s}_{n+1}(\mathbb{R})$ and $\delta \in \mathfrak{u}(V)(\mathbb{R})$ be matching regular semisimple orbits. Then $q(\gamma) = q_W(\delta)$. We have

$$O(\gamma, f'\tilde{\alpha}') = \int_{\mathrm{GL}_n(\mathbb{R})} f'(h^{-1}\gamma h)\tilde{\alpha}'(h^{-1}\gamma h)\eta(h)dh = \alpha(q(\gamma))O(\gamma, f').$$

Similarly for $O(\delta, f_W\widetilde{\alpha}_W) = \alpha(q_W(\delta))O(\delta, f_W)$. The lemma then follows. \square

Lemma 3.5. *Let $f \in \mathcal{S}(\mathfrak{s}_{n+1}(\mathbb{R}))$. Then the function*

$$\gamma \mapsto O(\gamma, f)$$

is smooth on $Q_{\mathrm{rs}}(\mathbb{R})$. Similar result holds in the unitary case.

Proof. Suppose that γ_0 is a regular semisimple point and $U \subset Q_{\mathrm{rs}}(\mathbb{R})$ be a small Nash neighbourhood of γ_0 in $Q(\mathbb{R})$. We may assume that $f \in \mathcal{S}(q^{-1}(U))$. Since $q^{-1}(U) \rightarrow U$ is a $\mathrm{GL}_n(\mathbb{R})$ principal homogeneous space and the orbital integral is the integration along the fibers, the resulting function on U is smooth. \square

Lemma 3.6. *Let $g \in \mathcal{S}(Q_{\text{rs}}(\mathbb{R}))$ be a Schwartz function on the regular semisimple locus in $Q(\mathbb{R})$. Then there is a Schwartz function on $q^{-1}(\text{supp } g)$ so that $g = O(\cdot, f)$.*

Proof. Let U be a small open connected set in $Q_{\text{rs}}(\mathbb{R})$. We may assume that g is supported in U . Then $q^{-1}(U) \rightarrow U$ is a principal homogeneous space and the orbital integral is the integration along the fibers. The lemma then follows from [AG09, Theorem B.2.4]. \square

We show that Theorem 3.3 implies Theorem 2.7.

Proof of Theorem 2.7 assuming Theorem 3.3. We will prove it in the general linear case, the unitary case is similar and requires only minor modification of the notation. We only need to prove that any $f \in \mathcal{C}_c^\infty(S_{n+1}(\mathbb{R}))$ can be approximated by transferable functions.

Let $\nu \in E$ and

$$D_\nu = \{g \in M_{n+1} \mid \det(\nu - g) = 0\}.$$

Then by [Zha14, Lemma 3.4], for $\xi \in \mathbb{C}^1$, the morphism

$$\alpha_\xi : M_{n+1} \setminus D_1 \rightarrow \text{GL}_{n+1} \setminus D_\nu, \quad x \mapsto -\xi(1+x)(1-x)^{-1}$$

induces an $H(\mathbb{R})$ -equivariant Nash diffeomorphism $\mathfrak{s}_{n+1}(\mathbb{R}) \setminus D_1 \rightarrow S_{n+1}(\mathbb{R}) \setminus D_\xi$. Therefore composition with α_ξ gives a topological isomorphism between $\mathcal{S}(\mathfrak{s}_{n+1}(\mathbb{R}) \setminus D_1)$ and $\mathcal{S}(S_{n+1}(\mathbb{R}) \setminus D_\xi)$. Moreover, we can choose finitely many $\xi_1, \dots, \xi_r \in \mathbb{C}^1$ so that $S_{n+1}(\mathbb{R}) \setminus D_{\xi_i}$ ($i = 1, \dots, r$) form an open cover of $S_{n+1}(\mathbb{R})$. Then by the Partition of Unity, we may assume that $f \in \mathcal{C}_c^\infty(S_{n+1}(\mathbb{R}) \setminus D_\xi)$ for some $\xi \in \mathbb{C}^1$. Thus we may choose a sequence of transferable functions $g_n \in \mathcal{S}(\mathfrak{s}_{n+1}(\mathbb{R}))$ so that $g_n \rightrightarrows f \circ \alpha_\xi$.

The set $q(\text{supp}(f \circ \alpha_\xi))$ is compact in $Q(\mathbb{R}) = (\mathfrak{s}_{n+1} // H)(\mathbb{R})$. Moreover, D_1 is the inverse image of a hypersurface, denoted by C , in Q . Therefore we may choose a smooth function $\gamma \in \mathcal{C}_c^\infty(Q(\mathbb{R}))$ so that γ takes value one on $q(\text{supp}(f \circ \alpha_\xi))$ and $\text{supp } \gamma \cap C(\mathbb{R}) = \emptyset$. Let $\tilde{\gamma} = \gamma \circ q$. It is a tempered function on $\mathfrak{s}_{n+1}(\mathbb{R})$ that is supported in $\mathfrak{s}_{n+1}(\mathbb{R}) \setminus D_1$. Then $\tilde{g}_n = g_n \cdot \tilde{\gamma} \in \mathcal{S}(\mathfrak{s}_{n+1}(\mathbb{R}) \setminus D_1)$. It is still transferable by Lemma 3.4. Moreover $\tilde{g}_n \rightrightarrows (f \circ \alpha_\xi) \tilde{\gamma} = f \circ \alpha_\xi$. Let $g'_n = \tilde{g}_n \circ \alpha_\xi^{-1} \in \mathcal{S}(S_{n+1}(\mathbb{R}) \setminus D_\xi)$. Then $g'_n \rightrightarrows f$. Since the transfer factors Ω and ω are compatible with the map α_ξ by [Zha14, Lemma 3.5] (in a precise sense as described there), the function g'_n is transferable. \square

4. SETUP

We set up some notation and convention that will be kept till the last but one section. Unless otherwise stated, we assume that $E/F = \mathbb{C}/\mathbb{R}$. We consider two cases.

- General linear case. We put $H = \text{GL}_n$, $\mathfrak{h} = \mathfrak{gl}_n$, $W = \mathbb{R}_n \times \mathbb{R}^n$ and $\mathcal{V} = \mathfrak{h} \times W$. The group $H = \text{GL}_n$ acts on \mathcal{V} via

$$(X, u, v)^h = (h^{-1}Xh, uh, h^{-1}v).$$

Let $\Delta(X, u, v) = \det(uX^{i+j}v)_{0 \leq i, j \leq n-1}$. Then (X, u, v) is regular semisimple if and only if $\Delta(X, u, v) \neq 0$. We let $D(X)$ be the discriminant of X , namely

$$D(X) = (-1)^{\frac{n(n-1)}{2}} \prod_{i < j} (\lambda_i - \lambda_j)^2,$$

where $\lambda_1, \dots, \lambda_n$ are all the eigenvalues of X . We define a bilinear form on \mathcal{V} by

$$\langle (X_1, u_1, v_1), (X_2, u_2, v_2) \rangle_{\mathcal{V}} = \text{Tr } X_1 X_2 + u_1 v_2 + u_2 v_1.$$

This bilinear form is H -invariant. The subspaces \mathfrak{h} and W are H -invariant and they are orthogonal under this bilinear form. We say that an element (X, u, v) is strongly regular semisimple if X , (u, v) and (X, u, v) are all regular semisimple under the action of H . More concretely this means that $D(X) \neq 0$, $\Delta(X, u, v) \neq 0$ and $uv \neq 0$. Sometime for simplicity, we write elements in \mathcal{V} as (X, w) where $X \in \mathfrak{gl}_n(\mathbb{R})$ and $w \in W = \mathbb{R}_n \times \mathbb{R}^n$.

- Unitary case. Let W be a hermitian space of dimension n over E and $\langle -, - \rangle$ be the hermitian form on W . Let $H = \text{U}(W)$ the unitary group and $\mathfrak{h} = \mathfrak{u}(W)$ the Lie algebra of H . We put $\mathcal{V} = \mathfrak{u}(W) \times W$. The group $H = \text{U}(W)$ acts on \mathcal{V} via

$$(X, w)^h = (h^{-1} X h, h^{-1} w).$$

Let $\Delta(X, w) = \det(\langle X^i w, X^j w \rangle)_{0 \leq i, j \leq n-1}$. Then (X, w) is regular semisimple if and only if $\Delta(X, w) \neq 0$. We let $D(X)$ be the discriminant of X , namely

$$D(X) = (-1)^{\frac{n(n-1)}{2}} \prod_{i < j} (\lambda_i - \lambda_j)^2,$$

where $\lambda_1, \dots, \lambda_n$ are all the eigenvalues of X . We define a bilinear form on \mathcal{V} by

$$\langle (X_1, w_1), (X_2, w_2) \rangle_{\mathcal{V}} = \text{Tr } X_1 X_2 + \langle w_1, w_2 \rangle + \langle w_2, w_1 \rangle.$$

This bilinear form is H -invariant. The subspaces $\mathfrak{u}(W)$ and W are H -invariant and they are orthogonal under this bilinear form. We say that an element (X, w) is strongly regular semisimple if X , w and (X, w) are all regular semisimple under the action of H . More concretely, this means that $D(X) \neq 0$, $\Delta(X, w) \neq 0$ and $\langle w, w \rangle \neq 0$. We sometimes write \mathcal{V}_W and H_W for \mathcal{V} and H to emphasize the dependance on W .

In either case, if $f \in \mathcal{S}(\mathcal{V})$ is a Schwartz function and $\gamma \in \mathcal{V}$ is a regular semisimple element, then we may define the orbital integral $O(\gamma, f)$. Moreover, the categorical quotient $\mathcal{V} // H$ in both cases are naturally identified in a similar way as in Section 3. We fix a transfer factor in the general linear case by $\omega(X, u, v) = \eta(\det(u, uX, \dots, uX^{n-1}))$. We may define the notion of smooth transfer of test functions and the notion of transferable functions. As in [Zha14], this transfer problem is equivalent to the one described in Section 3. Thus we will not distinguish these two transfer problems.

5. UPPER BOUNDS OF ORBITAL INTEGRALS

We begin with some lemmas on Schwartz functions.

Lemma 5.1. *Let M be a Nash manifold and $f \in \mathcal{S}(M)$. Then there is a Schwartz function $g \in \mathcal{S}(M)$ such that $|f| \leq g$.*

Proof. By [AG09, Theorem A.1.1], there are Schwartz functions $f_1^{(i)}, f_2^{(i)} \in \mathcal{S}(M)$, $i = 1, \dots, n$, so that $f = \sum_i f_1^{(i)} \cdot f_2^{(i)}$. Then $g = \frac{1}{2} \sum_i (|f_1^{(i)}|^2 + |f_2^{(i)}|^2)$ does the job. \square

Lemma 5.2. *Let M and N be Nash manifolds and $f \in \mathcal{S}(M \times N)$. Then there are nonnegative Schwartz functions $f_1 \in \mathcal{S}(M)$ and $f_2 \in \mathcal{S}(N)$ such that $|f(m, n)| \leq f_1(m)f_2(n)$.*

Proof. Applying [AG09, Theorem A.1.1] to the natural projection map $M \times N \rightarrow M$, we see that $\mathcal{S}(M \times N) = \mathcal{S}(M)\mathcal{S}(M \times N)$, where the right hand side means multiplication of functions and a Schwartz function on M is considered as a function on $M \times N$ via pullback. Similarly, $\mathcal{S}(M \times N) = \mathcal{S}(N)\mathcal{S}(M \times N)$. Therefore $\mathcal{S}(M \times N) = \mathcal{S}(M)\mathcal{S}(N)\mathcal{S}(M \times N)$, namely given $f \in \mathcal{S}(M \times N)$, there are Schwartz functions $\alpha_i \in \mathcal{S}(M)$, $\beta_i \in \mathcal{S}(N)$, $\gamma_i \in \mathcal{S}(M \times N)$, $i = 1, \dots, k$ such that

$$f(m, n) = \sum_{i=1}^k \alpha_i(m)\beta_i(n)\gamma_i(m, n).$$

There is a constant C such that $|\gamma_i(m, n)| \leq C$ for all i . By Lemma 5.1, we may find Schwartz functions $f_1 \in \mathcal{S}(M)$ and $f_2 \in \mathcal{S}(N)$ that are nonnegative, such that $|\alpha_i| \leq f_1$ and $|\beta_i| \leq f_2$ for all i (find such a function for each individual i and then take the sum of them). Then we conclude that

$$|f(m, n)| \leq \sum_{i=1}^k |\alpha_i(m)\beta_i(n)\gamma_i(m, n)| \leq kC f_1(m)f_2(n).$$

As kCf_1 is again a nonnegative Schwartz function, the lemma follows. \square

Lemma 5.3. *Let $D = \mathbb{R}, \mathbb{C}$ or a quaternion algebra over \mathbb{R} and T be an elliptic torus in D^\times . Let $f, g \in \mathcal{S}(D)$ be Schwartz functions and $a, b, c \in D^\times$. Then there are constants $C_1, C_2, C_3 > 0$, depending on f and g only, so that*

$$\left| \int_T f(a^{-1}t^{-1}b)g(cta)dt \right| \leq \begin{cases} C_1 - C_2 \log|\nu(bc)|, & |\nu(bc)| \leq 1 \\ C_3 |\nu(bc)|^{-1}, & |\nu(bc)| > 1 \end{cases},$$

where dt is the multiplicative measure on T and $\nu : D \rightarrow \mathbb{R}$ is the reduced norm.

Proof. Assume that $|\nu(bc)| \leq 1$. We have

$$\begin{aligned} \left| \int_T f(a^{-1}t^{-1}b)g(cta)dt \right| &\leq \sup|f| \sup|g| \int_{\substack{|\nu(a^{-1}t^{-1}b)| \leq 1 \\ |\nu(cta)| \leq 1}} dt \\ &\quad + \int_{|\nu(a^{-1}t^{-1}b)| \geq 1} |f(a^{-1}t^{-1}b)g(cta)|dt + \int_{|\nu(cta)| \geq 1} |f(a^{-1}t^{-1}b)g(cta)|dt. \end{aligned}$$

The second and third terms are bounded by some constants depending on f and g only. In fact, there is a large integer r and a constant C' such that $|f(t)| \leq C'|\nu(t)|^{-r}$

$$\int_{|\nu(a^{-1}t^{-1}b)| \geq 1} |f(a^{-1}t^{-1}b)g(cta)|dt \leq C' \sup|g| \int_{|\nu(t)| \geq 1} |\nu(t)|^{-r} dt.$$

Similarly for the third term. The first term is bounded by $-C \sup|f| \sup|g| \log |\nu(bc)|$ where C is a positive constant independent of f, g and a, b, c . This proves the lemma when $|\nu(bc)| \leq 1$.

Now assume that $|\nu(bc)| > 1$. Without loss of generality, we may assume that $|\nu(a^{-1}b)| \geq 1$. Then

$$\left| \int_T f(a^{-1}t^{-1}b)g(cta)dt \right| \leq \sup|f| \int_{|\nu(t)| \geq |\nu(a^{-1}b)|} |g(cta)|dt + \int_{|\nu(t)| \leq |\nu(a^{-1}b)|} |f(a^{-1}t^{-1}b)g(cta)|dt.$$

We treat two terms separately. Since g is a Schwartz function, there is an integer k and a constant C_1 so that $|g(cta)| \leq C_1|\nu(cta)|^{-k}$. The first term is then bounded by

$$C_1 \sup|f| \int_{|\nu(t)| \geq |\nu(a^{-1}b)|} |\nu(cta)|^{-k} dt.$$

We may choose k to be large enough so that

$$\int_{|\nu(t)| \geq |\nu(a^{-1}b)|} |\nu(cta)|^{-k} dt \leq \int_{|\nu(t)| \geq |\nu(bc)|} |\nu(t)|^{-k} dt \leq C_2 |\nu(bc)|^{-1},$$

for some constant C_2 . We now treat the second term. As both f and g are Schwartz functions, we may find large integers k and r and a constant C_3 so that

$$|f(a^{-1}t^{-1}b)g(cta)| \leq C_3 |\nu(a^{-1}t^{-1}b)|^{-k} |\nu(cta)|^{-k+r} = C_3 |\nu(t)|^r |\nu(a^{-1}b)|^{-k} |\nu(ca)|^{-k+r}.$$

We may choose r so that

$$\int_{|\nu(t)| \leq |\nu(a^{-1}b)|} |\nu(t)|^r dt \leq C_4 |\nu(a^{-1}b)|^r$$

for some constant C_4 . This proves that there is a constant C_5 so that the second term is bounded by $C_5 |\nu(bc)|^{-1}$. \square

We retain the notation and settings from Section 4.

Lemma 5.4. *Let $T \subset H$ be a Cartan subgroup.*

- (1) *Suppose that we are in the general linear case. Let $f_2, f_3 \in \mathcal{S}(\mathfrak{h})$. Then there are constants $C > 0$ depending only on f_2 and f_3 and $r > 0$ depending only on n , so that for any $h, \delta_1, \delta_2 \in H(\mathbb{R})$, we have*

$$\int_{T(\mathbb{R})} |f_2(h^{-1}t^{-1}\delta_1)f_3(\delta_2th)|dt \leq C \cdot \max\{1, |\log|\det \delta_1\delta_2||\}^r.$$

- (2) *Suppose that we are in the unitary case. Let $f_2 \in \mathcal{S}(W^n)$ be a Schwartz function. Then there is an $r > 0$ depending only on n and a constant C depending only on f_2 so that for all $h \in H(\mathbb{R})$ and all $\delta = (w_1, \dots, w_n) \in W^n$ such that $\Delta = \det\langle w_i, w_j \rangle_{1 \leq i, j \leq n} \neq 0$, we have*

$$\int_{T(\mathbb{R})} |f_2(h^{-1}t^{-1}\delta)|dt \leq C \cdot \max\{1, |\log|\Delta||\}^r.$$

Proof. We prove this in the general linear case. The unitary case is similar. Let \mathfrak{t} be the Lie algebra of T . It is a Cartan subalgebra of \mathfrak{h} . We may identify \mathfrak{t} with $\prod_{i=1}^e E_i$ where E_i is either \mathbb{R} or \mathbb{C} . Then $T = \prod_{i=1}^e E_i^\times$. Let n_i be the degree of E_i over \mathbb{R} , then $\sum n_i = n$. Let P be the parabolic subgroup of H associated to the partition $\sum n_i = n$ with the Levi decomposition $P = MN$. Let $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{n}$ be the Lie algebra of P where \mathfrak{m} and \mathfrak{n} are Lie algebras of M and N respectively. We usually write $M = \prod_{i=1}^e M_i$ and $\mathfrak{m} = \bigoplus_{i=1}^e \mathfrak{m}_i$ where \mathfrak{m}_i is the Lie algebra of M_i . Note that M_i is either isomorphic to \mathbb{R}^\times if $n_i = 1$ or $\mathrm{GL}_2(\mathbb{R})$ if $n_i = 2$. We may assume that $E_i^\times \subset M_i$ hence $E_i \subset \mathfrak{m}_i$. Then T is an elliptic torus in M . Fix a maximal compact subgroup K of $H(\mathbb{R})$, e.g. the orthogonal group. We have the Iwasawa decomposition $H = NMK$ or $H = KMN$. We may write $h = nmk$, $\delta_1 = n_1m_1k_1$ and $\delta_2 = k_2m_2n_2$. Then $h^{-1}t^{-1}\delta_1 = k^{-1}m^{-1}n^{-1}tn_1m_1k_1$. Since f_2 is a Schwartz function on \mathfrak{h} , there is nonnegative Schwartz function $g_2 \in \mathcal{S}(\mathfrak{h})$ such that $|f(k_1Xk_2)| \leq g_2(X)$ for all $k_1, k_2 \in K$ and $X \in \mathfrak{h}$. Thus

$$|f(k^{-1}m^{-1}n^{-1}tn_1m_1k_1)| \leq g_2(m^{-1}n^{-1}tn_1m_1).$$

Now $m^{-1}tm_1$ is the Levi component of $m^{-1}n^{-1}tn_1m_1$ with respect to the decomposition $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{n}$. Therefore by Lemma 5.2, we may find a nonnegative Schwartz function $f'_2 \in \mathcal{S}(\mathfrak{m})$ so that

$$g_2(m^{-1}n^{-1}tn_1m_1) \leq f'_2(m^{-1}tm_1).$$

In conclusion, we have

$$|f_2(h^{-1}t^{-1}\delta_1)| \leq f'_2(m^{-1}tm_1).$$

Similarly, we may find a Schwartz function $f'_3 \in \mathcal{S}(\mathfrak{m})$ so that $|f_3(\delta_2th)| \leq f'_3(m_2tm)$. We now write $t = (t^{(i)}) \in T$ where $t^{(i)} \in E_i^\times$ and similarly for m, m_1, m_2 . By Lemma 5.2, we may assume that f'_2 and f'_3 are of the form $\prod_{i=1}^e f_2^{(i)}$ and $\prod_{i=1}^e f_3^{(i)}$ respectively where $f_2^{(i)}, f_3^{(i)} \in \mathcal{S}(\mathfrak{m}_i)$. Then

$$\int_{T(\mathbb{R})} |f_2(h^{-1}t^{-1}\delta_1)f_3(\delta_2th)|dt \leq \prod_{i=1}^e \int_{E_i^\times} f_2^{(i)}(m^{(i),-1}t^{(i),-1}m_1^{(i)})f_3^{(i)}(m_2^{(i)}t^{(i)}m^{(i)})dt^{(i)}$$

By Lemma 5.3, for $i = 1, \dots, e$, there are constants $C_1, C_2, C_3 > 0$ that depend on f'_2 and f'_3 only, so that

$$\left| \int_{E_i^\times} f_2^{(i)}(m^{(i)}, -1t^{(i)}, -1m_1^{(i)}) f_3^{(i)}(m_2^{(i)} t^{(i)} m^{(i)}) dt^{(i)} \right| \leq \begin{cases} C_1 - C_2 \log |\nu(m_1^{(i)} m_2^{(i)})|, & |\nu(m_1^{(i)} m_2^{(i)})| \leq 1 \\ C_3 |\nu(m_1^{(i)} m_2^{(i)})|^{-1}, & |\nu(m_1^{(i)} m_2^{(i)})| > 1 \end{cases},$$

where $\nu : M_i \rightarrow \mathbb{R}$ is the reduced norm, i.e. the identity map if $M_i \simeq \mathbb{R}^\times$ and the determinant map if $M_i \simeq \mathrm{GL}_2(\mathbb{R})$. We may label M_i so that for $i = 1, \dots, s$ (resp. $i = s+1, \dots, e$) we have $|\nu(m_1^{(i)} m_2^{(i)})| \leq 1$ (resp. > 1). Let

$$A = \prod_{i=1}^s |\nu(m_1^{(i)} m_2^{(i)})|, \quad B = \prod_{i=s+1}^e |\nu(m_1^{(i)} m_2^{(i)})|.$$

For $i = 1, \dots, s$, there is a constant C_4 so that

$$C_1 - C_2 \log |\nu(m_1^{(i)} m_2^{(i)})| \leq C_4 (1 - \log A).$$

It is not hard to check that the function $\alpha(x) = x(1 + |\log x|)^s$ is increasing when $x \in (0, e^{1-s}) \cup (1, \infty)$ and decreasing in $(e^{1-s}, 1)$ and $\alpha(1) = 1$. It follows that there is a constant C_5 so that $A(1 - \log A)^s \leq C_5 AB(1 + |\log AB|)^s$. Therefore $(C_4(1 - \log A))^s B^{-1} \leq C_4^s C_5 (1 + |\log AB|)^s$. The lemma then follows easily. \square

Lemma 5.5. *Let $f \in \mathcal{S}(\mathcal{V})$. There are constants $C > 0$ and $r > 0$ that depend only on f , such that for all strongly regular semisimple $(X, w) \in \mathcal{V}$, we have*

$$|O((X, w), f)| \leq C \cdot \max\{1, |\log |\Delta(X, w)||\}^r \cdot \max\{1, |D(X)|^{-\frac{1}{2}}\}.$$

Proof. We prove this in the general linear case. The unitary case is similar and easier.

By definition, we have

$$|O((X, u, v), f)| = \left| \int_{H(\mathbb{R})} f(h^{-1} X h, u h, h^{-1} v) \eta(h) dh \right| \leq \int_{H(\mathbb{R})} |f(h^{-1} X h, u h, h^{-1} v)| dh.$$

The map $\mathcal{V} \rightarrow \mathfrak{gl}_n \times \mathfrak{gl}_n \times \mathfrak{gl}_n$ given by $(X, u, v) \mapsto (X, \delta_1, \delta_2)$ where

$$\delta_1 = (X^i v)_{i=0, \dots, n-1} \in \mathfrak{gl}_n, \quad \delta_2 = (u X^i)_{i=0, \dots, n-1} \in \mathfrak{gl}_n,$$

is a closed embedding of Nash manifolds. Therefore by [AG08, Theorem 4.6.1], there is a Schwartz function $\tilde{f} \in \mathcal{S}((\mathfrak{gl}_n(\mathbb{R}))^3)$, such that

$$f(X, u, v) = \tilde{f}(X, \delta_1, \delta_2).$$

By Lemma 5.2, we need to prove that for any $f_1, f_2, f_3 \in \mathcal{S}(\mathfrak{gl}_n(\mathbb{R}))$, there are constants $C > 0$ and $r > 0$, depending on f_1, f_2 and f_3 only, so that for all strongly regular semisimple $(X, u, v) \in$

\mathcal{V} , we have

$$\int_{H(\mathbb{R})} |f_1(h^{-1}Xh)f_2(h^{-1}\delta_1)f_3(\delta_2h)|dh \leq C \cdot \max\{1, |\log|\Delta(X, w)||^r\} \cdot \max\{1, |D(X)|^{-\frac{1}{2}}\}.$$

We choose a (finite) complete set of representatives of Cartan subalgebras \mathfrak{t} of \mathfrak{h} up to H -conjugacy. We may assume that $X \in \mathfrak{t}$ is regular and let T be the stabilizer of X . Then

$$\int_{H(\mathbb{R})} |f_1(h^{-1}Xh)f_2(h^{-1}\delta_1)f_3(\delta_2h)|dh = \int_{T(\mathbb{R}) \backslash H(\mathbb{R})} |f_1(h^{-1}Xh)| \int_{T(\mathbb{R})} |f_2(h^{-1}t^{-1}\delta_1)f_3(\delta_2th)|dtdh.$$

It follows from Lemma 5.4 that there is a constant C' so that

$$\int_{H(\mathbb{R})} |f_1(h^{-1}Xh)f_2(h^{-1}\delta_1)f_3(\delta_2h)|dh \leq C' \cdot \max\{1, |\log|\Delta(X, w)||^r\} \cdot \int_{T(\mathbb{R}) \backslash H(\mathbb{R})} |f_1(h^{-1}Xh)|dh.$$

By the bound of Harish–Chandra of the (usual) orbital integral, the integral on the right hand side of the above inequality is bounded by a constant times $\max\{1, |D(X)|^{-\frac{1}{2}}\}$. Since there are only finitely many \mathfrak{t} , we may choose a uniform constant. This completes the proof. \square

Lemma 5.6. *Let $p(x) \in \mathbb{R}[x_1, \dots, x_n]$ be a homogeneous polynomial and $g \in \mathcal{S}(\mathbb{R}^n)$ be a Schwartz function. Then there is a constant $\epsilon > 0$ so that*

$$\int_{\mathbb{R}^n} |p(x)|^{-\epsilon} |g(x)| dx < \infty.$$

Proof. Firstly, by Lemma [Zha14, Lemma 4.3, Remark 12], there is an $\epsilon > 0$, such that $|p(x)|^{-\epsilon}$ is locally integrable everywhere. This in particular implies that

$$\int_{\frac{1}{2} < \|x\| < 1} |p(x)|^{-\epsilon} dx < \infty,$$

where $\|\cdot\|$ is the Euclidean norm on \mathbb{R}^n . Suppose that the degree of $p(x)$ is d , then

$$\int_{2^j < \|x\| < 2^{j+1}} |p(x)|^{-\epsilon} |g(x)| dx = 2^{j+1-(j+1)d\epsilon} \int_{\frac{1}{2} < \|x\| < 1} |p(x)|^{-\epsilon} |g(2^{j+1}x)| dx.$$

We fix a large integer r . Then there is a constant C such that if $j \geq 1$ and $\frac{1}{2} < \|x\| < 1$, we have $|g(2^{j+1}x)| \leq C \cdot 2^{-r(j+1)} \|x\|^{-r} \leq C \cdot 2^{-rj}$. Then

$$\int_{2^j < \|x\| < 2^{j+1}} |p(x)|^{-\epsilon} |g(x)| dx \leq C \cdot 2^{(j+1)-(j+1)d\epsilon-rj} \int_{\frac{1}{2} < \|x\| < 1} |p(x)|^{-\epsilon} dx.$$

We may choose a sufficiently large r so that

$$\sum_{j \geq 1} 2^{(j+1)-(j+1)d\epsilon-rj} < \infty.$$

Then

$$\begin{aligned}
& \int_{\mathbb{R}^n} |p(x)|^{-\epsilon} |g(x)| dx \\
& \leq \int_{\|x\| < 2} |p(x)|^{-\epsilon} |g(x)| dx + \sum_{j \geq 1} \int_{2^j < \|x\| < 2^{j+1}} |p(x)|^{-\epsilon} |g(x)| dx \\
& \leq \sup |g| \int_{\|x\| < 2} |p(x)|^{-\epsilon} dx + \sum_{j \geq 1} C \cdot 2^{(j+1)-(j+1)d\epsilon - rj} \int_{\frac{1}{2} < \|x\| < 1} |p(x)|^{-\epsilon} dx \\
& < \infty.
\end{aligned}$$

□

Lemma 5.7. *Let $g \in \mathcal{S}(\mathfrak{h})$ be a Schwartz function. Then there is an $\epsilon > 0$, such that*

$$\int_{\mathfrak{h}} |D(X)|^{-\frac{1}{2}-\epsilon} |g(X)| dX < \infty.$$

Proof. This can be proved by the same method as Lemma 5.6. We only need to make use of the following facts.

- $D(X)$ is a homogeneous polynomial in entries of X .
- There is an $\epsilon > 0$ such that $|D(X)|^{-\frac{1}{2}-\epsilon}$ is locally integrable everywhere. This is a theorem of Harish-Chandra.

□

Lemma 5.8. *Let $f \in \mathcal{S}(\mathcal{V})$.*

- (1) *Let $g \in \mathcal{S}(\mathcal{V})$, then the function $(X, w) \mapsto O((X, w), f)g(X, w)$ is absolutely integrable on \mathcal{V} .*
- (2) *Suppose $X \in \mathfrak{h}$ is regular semisimple and $g \in \mathcal{S}(W)$, then the function $w \mapsto O((X, w), f)g(w)$ is absolutely integrable on W .*
- (3) *Suppose that $w \in W$ is regular semisimple and $g \in \mathcal{S}(\mathfrak{h})$, then the function $X \mapsto O((X, w), f)g(X)$ is absolutely integrable on \mathfrak{h} .*

Proof. The second and third statements follow from Lemma 5.6 and Lemma 5.7 respectively. To prove the first statement, it is enough to prove that

$$\int_{\mathcal{V}} \max\{1, |D(X)|^{-\frac{1}{2}}\} \max\{1, |\log|\Delta(X, w)||\}^r |g(X, w)| dX dw < \infty.$$

It follows from Young's inequality that for any $\epsilon_1 > 0$, we have

$$|D(X)|^{-\frac{1}{2}} |\log|\Delta(X, w)||^r \leq \frac{|D(X)|^{-\frac{1}{2}(1+\epsilon_1)}}{1+\epsilon_1} + \frac{|\log|\Delta(X, w)||^{r(1+\epsilon_1)\epsilon_1^{-1}}}{(1+\epsilon_1)\epsilon_1^{-1}}.$$

It follows from Lemma 5.7 that we may choose a suitable ϵ_1 so that

$$\int_{\mathcal{V}} \frac{|D(X)|^{-\frac{1}{2}(1+\epsilon_1)}}{1+\epsilon_1} |g(X, w)| dX dw < \infty.$$

It follows from Lemma 5.6 that for any ϵ_1 we have chosen, we have

$$\int_{\mathcal{V}} \frac{|\log|\Delta(X, w)||^{r(1+\epsilon_1)\epsilon_1^{-1}}}{(1+\epsilon_1)\epsilon_1^{-1}} |g(X, w)| dX dw < \infty.$$

This proves the first statement of the lemma. \square

6. A LOCAL RELATIVE TRACE FORMULA

In this section, we prove a local relative trace formula. This is the archimedean analogue of [Zha14, Theorem 4.6].

Let $\psi : \mathbb{R} \rightarrow \mathbb{C}^\times$ be a nontrivial additive character. Let $\mathcal{V}_0 = \mathcal{V}, \mathfrak{h}$ or W and \mathcal{V}_0^\perp be its orthogonal complement. If $f \in \mathcal{S}(\mathcal{V})$, then we define its partial Fourier transform with respect to \mathcal{V}_0 by

$$\widehat{f}(y, z) = \int_{\mathcal{V}_0} f(y', z) \psi(\langle y', y \rangle_{\mathcal{V}}) dy', \quad y \in \mathcal{V}_0, \quad z \in \mathcal{V}_0^\perp.$$

Here and below in this section, we usually write an element in \mathcal{V} as (y, z) according to the decomposition $\mathcal{V} = \mathcal{V}_0 \oplus \mathcal{V}_0^\perp$.

Theorem 6.1. *Let $f_1, f_2 \in \mathcal{S}(\mathcal{V})$. Fix a regular semisimple element $z \in \mathcal{V}_0^\perp$. Define*

$$T(f_1, f_2) = \int_{\mathcal{V}_0} O((y, z), f_1) f_2(y, z) dy.$$

Then

$$T(\widehat{f}_1, f_2) = T(f_1, \widehat{f}_2).$$

Recall that $\widehat{\cdot}$ stands for the partial Fourier transform respect to \mathcal{V}_0 .

Proof. We will prove the theorem in the unitary case. The general linear case is similar.

Since z is regular semisimple, by Lemma 5.8, the double integrals

$$\int_{\mathcal{V}_0} \int_H f_1((y, z)^h) \widehat{f}_2(y, z) dh dy, \quad \int_{\mathcal{V}_0} \int_H \widehat{f}_1((y, z)^h) f_2(y, z) dh dy$$

are absolutely convergent. Thus we may change the order of integrations. Since the Fourier transform preserves the L^2 -norm of Schwartz functions, we have

$$\int_H \int_{\mathcal{V}_0} f_1((y, z)^h) \widehat{f}_2(y, z) dy dh = \int_H \int_{\mathcal{V}_0} \widehat{f}_1((y, z)^h) f_2(y, z) dy dh.$$

The theorem then follows. \square

7. KLOOSTERMAN INTEGRALS

We gather in this section some lemmas on Kloosterman integrals.

Fix the nontrivial additive character $\psi(x) = e^{2\pi\sqrt{-1}x}$ of \mathbb{R} and let $\psi_{\mathbb{C}}(z) = \psi(z + \bar{z})$ be a nontrivial additive character of \mathbb{C} . We remind the readers that the measures on \mathbb{R} and \mathbb{C} are the usual Lebesgue measures.

Let $a \in \mathbb{R}^{\times}$ and $C > 0$ Define

$$\Psi_C(a) = \int_{|a|C^{-1} < |x| < C} \psi(x + ax^{-1})\eta(x) \frac{dx}{|x|}.$$

It is not hard to see, via integration by parts, that for a fixed $a \in \mathbb{R}^{\times}$ the limit

$$\lim_{C \rightarrow \infty} \Psi_C(a)$$

exists. We denote this limit by $\Psi(a)$ and call it the Kloosterman integral. Note that if $a < 0$, then $\Psi(a) = 0$. This can be seen by making the change of variable $x \mapsto ax^{-1}$.

Now let $a > 0$ and $a = b\bar{b}$ where $b \in \mathbb{C}$. We define

$$\Phi(a) = \int_{\mathbb{C}^1} \psi_{\mathbb{C}}(ub) du,$$

where \mathbb{C}^1 stands for the set of complex numbers of norm one. The measure du is chosen so that for any $f \in \mathcal{C}_c^{\infty}(\mathbb{C}^{\times})$, we have

$$\int_{\mathbb{C}^{\times}} f(z) \frac{dz}{|z\bar{z}|} = \int_0^{\infty} \int_{\mathbb{C}^1} f(ru) \frac{dudr}{r}.$$

If $a \leq 0$, then we set $\Phi(a) = 0$. We call it the twisted Kloosterman integral.

Lemma 7.1. *There is a positive constant α that is independent of $C \in \mathbb{R} \cup \{\infty\}$, so that if $|a| > 1$, then*

$$|\Psi_C(a)| < \alpha|a|^{-\frac{1}{4}}, \quad |\Phi(a)| < \alpha|a|^{-\frac{1}{4}}.$$

Proof. We prove the lemma for Ψ_C . The same proof with simple modifications also applies to Φ .

We only need to consider the case $a > 1$ since if $a < -1$, then $\Psi_C(a) = 0$. If $a > 1$, then we let $b = \sqrt{a}$ and make the change of variable $x \mapsto xb$. Then

$$\Psi_C(a) = \int_{C^{-1}b < |x| < Cb^{-1}} e^{2\pi\sqrt{-1}b(x+x^{-1})}\eta(x) \frac{dx}{|x|}.$$

It is enough to show that there are constants A and α that are independent of C , such that

$$\left| \int_{1 < x < C} e^{2\pi\sqrt{-1}b(x+x^{-1})} \frac{dx}{x} \right| < \alpha|b|^{-\frac{1}{2}},$$

if $b > A$. Denote this integral by I . Suppose that $C - C^{-1} < b^{-1/2}$. Then

$$|I| \leq \int_{1 < x < C} dx \leq 2b^{-1/2}.$$

Assume that $C - C^{-1} > b^{-1/2}$. We split the integral into two parts

$$I = \int_{0 < x - x^{-1} < b^{-1/2}} e^{2\pi\sqrt{-1}b(x+x^{-1})} \frac{dx}{x} + \int_{b^{-1/2} < x - x^{-1} < C - C^{-1}} e^{2\pi\sqrt{-1}b(x+x^{-1})} \frac{dx}{x}.$$

Denote these integrals by I_1 and I_2 respectively. Then we have $|I_1| < 2b^{-1/2}$. We make the change of variable $t = x + x^{-1}$ in I_2 . Then

$$I_2 = \int_{\sqrt{b^{-1}+4} < t < C+C^{-1}} e^{2\pi\sqrt{-1}bt} \frac{dt}{(t^2 - 4)^{1/2}}.$$

By integration by parts, we have

$$I_2 = (2\pi\sqrt{-1}b)^{-1} \left(\frac{e^{2\pi\sqrt{-1}bt}}{(t^2 - 4)^{1/2}} \Big|_{\sqrt{b^{-1}+4}}^{C+C^{-1}} + \int_{\sqrt{b^{-1}+4} < t < C+C^{-1}} e^{2\pi\sqrt{-1}bt} \frac{tdt}{(t^2 - 4)^{3/2}} \right).$$

It is not hard to see that both terms in the parentheses are bounded by $2b^{1/2}$. The lemma then follows when $a > 1$. \square

Lemma 7.2. *There are positive constants β_1 and β_2 that are independent of $C \in \mathbb{R} \cup \{\infty\}$ so that*

$$|\Psi_C(a)| < \beta_1 |\log|a|| + \beta_2.$$

Proof. By the previous lemma, we may assume that $|a| < 1$. Suppose that $C < 1$, then

$$|\Psi_C(a)| < \int_{|a|C^{-1} < |x| < C} \frac{dx}{x} < \int_{|a| < |x| < 1} \frac{dx}{|x|} < 2|\log|a||.$$

Now suppose that $C \geq 1$. Then we break the defining integral of $\Psi_C(a)$ as

$$\int_{|a|C^{-1} < |x| < |a|} + \int_{|a| < |x| < 1} + \int_{1 < |x| < C}.$$

The first and the last integrals are equal, which can be seen via the change of variable $x \mapsto ax^{-1}$.

It is then enough to prove that there is a constant β_2 that is independent of C , so that

$$\left| \int_{1 < x < C} e^{2\pi\sqrt{-1}(x+ax^{-1})} \frac{dx}{x} \right| < \beta_2.$$

Denote this integral by I . By integration by parts, we have

$$(2\pi\sqrt{-1})I = \frac{e^{2\pi\sqrt{-1}(x+ax^{-1})}}{x} \Big|_1^C + \int_{1 < x < C} e^{2\pi\sqrt{-1}(x+ax^{-1})} \frac{2\pi\sqrt{-1}ax^{-1} + 1}{x^2} dx.$$

It is clear that both terms are bounded by a constant that is independent of C . Here we have used the fact that $|a| < 1$. The lemma then follows. \square

Lemma 7.3. *For all $a > 0$, we have $\Psi(a) = \sqrt{-1}\Phi(a)$.*

Proof. It is enough to prove that the Mellin transform of both sides are equal. Recall that if f is a smooth function on $\mathbb{R}_{>0}$, then its Mellin transform is given by

$$\int_{-\infty}^{\infty} f(x)|x|^s \frac{dx}{|x|},$$

if the integral is convergent and if $x \leq 0$, then we set $f(x) = 0$.

Suppose that $0 < \Re s < \frac{1}{4}$. Then by Lemma 7.1 and 7.2 and the Lebesgue dominated convergence theorem, we have

$$\int_{-\infty}^{\infty} \Psi(a)|a|^s \frac{da}{|a|} = \lim_{C \rightarrow \infty} \int_{-\infty}^{\infty} \int_{|a|C^{-1} < |x| < C} \psi(x + ax^{-1})\eta(x)|a|^s \frac{dx da}{|x| |a|}.$$

Making the change of variable $a \mapsto ax$, we have

$$\begin{aligned} \int_{-\infty}^{\infty} \Psi(a)|a|^s \frac{da}{|a|} &= \lim_{C \rightarrow \infty} \int_{|a| < C} \psi(a)|a|^s \frac{da}{|a|} \int_{|x| < C} \psi(x)|x|^s \eta(x) \frac{dx}{|x|} \\ &= 2\sqrt{-1} \sin \pi s (2\pi)^{-2s} \Gamma(s)^2. \end{aligned}$$

Here we have made use of the integral formula

$$\lim_{C \rightarrow \infty} \int_0^C e^{2\pi\sqrt{-1}x} x^{s-1} dx = e^{\frac{\sqrt{-1}\pi s}{2}} (2\pi)^{-2s} \Gamma(s), \quad 0 < \Re s < 1.$$

We now compute the Mellin transform of $\Phi(a)$. If $0 < \Re s < \frac{1}{4}$, then the defining integral of the Mellin transform of $\Phi(a)$ is absolutely convergent and we have

$$\int_{-\infty}^{\infty} \Phi(a)|a|^s \frac{da}{|a|} = \int_0^{\infty} \int_{\mathbb{C}^1} \psi_{\mathbb{C}}(ua^{\frac{1}{2}}) a^s du \frac{da}{a} = 2 \int_0^{\infty} \int_{\mathbb{C}^1} \psi_{\mathbb{C}}(ur) r^{2s} du \frac{dr}{r}.$$

where in the second equality we have made a change of variable $r = a^{\frac{1}{2}}$. The double integral is only convergent as an iterated integral. By the choice of the measure du , it equals

$$\lim_{C \rightarrow \infty} 2 \int_0^C \int_{\mathbb{C}^1} \psi_{\mathbb{C}}(ur) r^{2s} du \frac{dr}{r} = \lim_{C \rightarrow \infty} 2 \int_{|z\bar{z}| \leq C^2} \psi_{\mathbb{C}}(z) |z\bar{z}|^{s-1} dz = 2 \sin \pi s (2\pi)^{-2s} \Gamma(s)^2.$$

Here we have made use of the integral formula

$$\lim_{C \rightarrow \infty} \iint_{x^2 + y^2 \leq C^2} e^{4\pi\sqrt{-1}x} (x^2 + y^2)^{s-1} dx dy = \sin \pi s (2\pi)^{-2s} \Gamma(s)^2, \quad 0 < \Re s < \frac{1}{2}.$$

The lemma then follows. \square

For later use, we need to generalize the definition of Kloosterman integrals. Let $F' = \mathbb{R}^p \times \mathbb{C}^q$ with $p + 2q = m$ and $E' = F' \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{C}^m$. There is an obvious norm map $N : F'^{\times} \rightarrow \mathbb{R}^{\times}$ and the character η extends to a quadratic character η' of F' by $\eta \circ N$. Let $a \in \mathbb{C}^{\times}$, then we define

$$\Psi_C^{\mathbb{C}}(a) = \int_{|a|C^{-1} < |z| < C} \psi_{\mathbb{C}}(z + az^{-1}) \frac{dz}{|z\bar{z}|}.$$

and set $\Psi^{\mathbb{C}}(a) = \Phi^{\mathbb{C}}(a) = \lim_{C \rightarrow \infty} \Psi_C^{\mathbb{C}}(a)$. In particular, $\Psi^{\mathbb{C}}(a) = \Phi^{\mathbb{C}}(a)$. One may show, using similar methods as above, that this limit exists. Similarly, if $|a| > 1$, then $|\Psi_C^{\mathbb{C}}(a)|$ is bounded

by $\alpha|a|^{-\frac{1}{4}}$ and if $|a| < 1$, then $|\Psi_C^{\mathbb{C}}(a)|$ is bounded by $\beta_1|\log|a|| + \beta_2$ where α , β_1 and β_2 are constants which are independent of C . Then we define the Kloosterman integral

$$\Psi^{F'}(a_1, \dots, a_p; a_{p+1}, \dots, a_{p+q}) = \prod_{i=1}^p \Psi(a_i) \prod_{j=1}^q \Psi^{\mathbb{C}}(a_{p+j}).$$

Similar we define the twisted Kloosterman integral

$$\Phi^{F'}(a_1, \dots, a_p; a_{p+1}, \dots, a_{p+q}) = \prod_{i=1}^p \Phi(a_i) \prod_{j=1}^q \Phi^{\mathbb{C}}(a_{p+j}).$$

Then $\Psi^{F'} = \sqrt{-1}^p \Phi^{F'}$.

8. THE MINIMAL CASE

We study Conjecture 3.1 in the case $n = 1$ in this section.

Proposition 8.1. *Conjecture 3.1 holds when $n = 1$.*

Proof. This is proved by Jacquet in [Jac86] using the relative Shalika germ expansion. □

In the general linear case, we have $\mathcal{V} = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ and $H = \mathbb{R}^\times$ which acts on \mathcal{V} via $(g, x, y)^t = (g, xt, yt^{-1})$. The element (g, x, y) is regular semisimple if and only if $xy \neq 0$. The transfer factor is $\omega(g, x, y) = \eta(x)$. The inner product on \mathcal{V} is given by $\langle (g, x, y), (g', x', y') \rangle_{\mathcal{V}} = gg' + xy' + x'y$. Let $f \in \mathcal{S}(\mathcal{V})$ and \widehat{f} be the Fourier transform of f with respect to the second and third variable. Let $\kappa(x, y; x', y') = \eta(xy)\Psi(xy x' y')$ be a function on $\mathbb{R}^2 \times \mathbb{R}^2$ where Ψ is the Kloosterman integral in Section 7.

Lemma 8.2. *Let the notation be as above. Then for all (x, y) with $xy \neq 0$, we have*

$$\omega(g, x, y)O((g, x, y), \widehat{f}) = \int_{\mathbb{R}^2} f(g, x', y')\omega(g, x', y')\kappa(x, y; x', y')dx'dy'.$$

Proof. By definition, we have

$$\begin{aligned} & O((g, x, y), \widehat{f}) \\ &= \int_{\mathbb{R}^\times} \left(\int_{\mathbb{R}^2} f(g, x', y')\psi(x'yt + xy't^{-1})dx'dy' \right) \eta(t) \frac{dt}{|t|} \\ &= \lim_{C \rightarrow \infty} \int_{\mathbb{R}^2} f(g, x', y') \left(\int_{C^{-1} < |t| < C} \psi(x'yt + xy't^{-1})\eta(t) \frac{dt}{|t|} \right) dx'dy' \\ &= \lim_{C \rightarrow \infty} \int_{\mathbb{R}^2} f(g, x', y')\eta(x'y) \left(\int_{C^{-1}|x'y| < |t| < C|x'y|} \psi(t + xx'yy't^{-1})\eta(t) \frac{dt}{|t|} \right) dx'dy' \end{aligned}$$

Here in the second identity we may changed the order of integration because the double integral is absolutely convergent as f is a Schwartz function and the integral over t is in a compact region. It follows from Lemma 7.1 and 7.2 that the inner integral is essentially bounded by

$\log|xx'yy'|$ when $|xx'yy'|$ is small and by $|xx'yy'|^{-1/4}$ when $|xx'yy'|$ is large. The lemma then follows from Lebesgue dominated convergence theorem. \square

In the unitary case, we have that W is a one dimension hermitian space over E with the hermitian form $\langle -, - \rangle$ and $\mathcal{V} = \mathfrak{u}(W) \times W$. The group $U(W)$ is commutative and acts on \mathcal{V} via $(g, w)^u = (g, uw)$. The element (g, w) is regular semisimple if and only if $w \neq 0$. For $f \in \mathcal{S}(\mathcal{V})$ we denote by \widehat{f} its Fourier transform with respect to W . We define a function on $W \times W$ by $\kappa_W(w, w') = \Phi(\langle w, w' \rangle \overline{\langle w, w' \rangle})$ where Φ is the twisted Kloosterman integral in Section 7.

Lemma 8.3. *Let the notation be as above. Then for any (g, w) with $w \neq 0$, we have*

$$O((g, w), \widehat{f}) = \int_W f(g, w') \kappa_W(w, w') dw'.$$

Proof. This can be proved in the same way as Lemma 8.2. In fact, this is easier since the integrals are absolutely convergent. \square

Lemma 8.4. *In either the general linear case or the unitary case, if $f \in \mathcal{S}(\mathcal{V})$ is transferable, then so is \widehat{f} .*

Proof. Suppose that the function $f \in \mathcal{S}(\mathbb{R} \times \mathbb{R} \times \mathbb{R})$ and the collection $\{f_W\}$ where $f_W \in \mathcal{S}(\mathfrak{u}(W) \times W)$ match. Then by Lemma 8.2, 8.3 and Lemma 7.3, we have

$$\omega(g, x, y) O((g, x, y), \widehat{f}) = \sqrt{-1} \eta(xy) O((\sqrt{-1}g, w), \widehat{f}_W),$$

if (g, x, y) and $(\sqrt{-1}g, w)$ match, i.e. $xy = \langle w, w \rangle$. Here we have implicitly identified $\mathfrak{u}(W)$ with $\sqrt{-1}\mathbb{R}$ by identifying an element in $\mathfrak{u}(W)$ with its eigenvalue. Note that $\eta(\langle w, w \rangle) = \eta(\text{disc } W)$. Therefore \widehat{f} and $\{\sqrt{-1}\eta(\text{disc } W) \widehat{f}_W\}$ match. This proves the lemma. \square

9. FOURIER TRANSFORMS PRESERVE SMOOTH TRANSFER

Let \mathcal{V}_0 be an H -invariant subspace of \mathcal{V} and \mathcal{V}_0^\perp be its orthogonal complement. Recall that if $f \in \mathcal{S}(\mathcal{V})$, then we denote by \widehat{f} its partial Fourier transform with respect to \mathcal{V}_0 . The goal of this section is to prove the following theorem.

Theorem 9.1. *Suppose that $f \in \mathcal{S}(\mathcal{V})$ is transferable. Then so is \widehat{f} .*

We set up some notation before we delve into the proof of the theorem. We shall write

$$f \leftrightarrow \{f_W\} \quad \text{resp.} \quad (X, u, v) \leftrightarrow (X_W, w_W)$$

to indicated that the test function f on the general linear group side and the the collection of test functions $\{f_W\}$ on the unitary group side match (resp. the regular semisimple orbits (X, u, v) and (X_W, w_W) match). In both the general linear case and the unitary case, we denote by \mathcal{F}_a (resp. \mathcal{F}_b , resp. \mathcal{F}_c) the Fourier transform with respect to \mathcal{V} , (resp. \mathfrak{h} , resp. W). Put $\nu_n = \sqrt{-1}^{-n(n-1)/2}$. We consider the following three statements.

- A_n : If $f \leftrightarrow \{f_W\}$, then $\nu_{n+1}\mathcal{F}_a f \leftrightarrow \{\eta(\text{disc } W)^n \mathcal{F}_a f_W\}$.
- B_n : If $f \leftrightarrow \{f_W\}$, then $\nu_n \mathcal{F}_b f \leftrightarrow \{\eta(\text{disc } W)^{n-1} \mathcal{F}_b f_W\}$.
- C_n : If $f \leftrightarrow \{f_W\}$, then $\sqrt{-1}^{-n} \mathcal{F}_c f \leftrightarrow \{\eta(\text{disc } W) \mathcal{F}_c f_W\}$.

Lemma 9.2. $B_n + C_n \Rightarrow A_n$.

Proof. This is clear since $\mathcal{F}_a = \mathcal{F}_b \circ \mathcal{F}_c$. □

Lemma 9.3. $A_{n-1} \Rightarrow B_n$.

Proof. To simplify notation, we put $\widehat{f} = \mathcal{F}_b f$. Suppose first that we are in the general linear case. Let $O^\eta((X, u, v), f) = \omega(X, u, v)O((X, u, v), f)$. We make use of Theorem 6.1 and argue as in [Zha14, Lemma 4.18] to get

$$\int_{Q(\mathbb{R})} O^\eta((X, u, v), f) O^\eta(X, \widehat{g}) dq(X) = \int_{Q(\mathbb{R})} O^\eta((X, u, v), \widehat{f}) O^\eta(X, g) dq(X),$$

where

- $(u, v) \in \mathbb{R}_n \times \mathbb{R}^n$ and $uv \neq 0$, up to the action of $\text{GL}_n(\mathbb{R})$, we may assume that $(u, v) = (e, d^t e)$ where $d = uv \in \mathbb{R}^\times$.
- $q : \mathfrak{gl}_n \rightarrow Q = \mathfrak{gl}_n // \text{GL}_{n-1}$ is the categorical quotient where GL_{n-1} acts on Q by conjugation;
- the measure $dq(X)$ is chosen so that for any $g \in \mathcal{S}(\mathfrak{gl}_n(\mathbb{R}))$, we have
$$\int_{\mathfrak{gl}_n(\mathbb{R})} g(X) dX = \int_{Q(\mathbb{R})} \left(\int_{\text{GL}_{n-1}(\mathbb{R})} g(X^h) dh \right) dq(X).$$
- $g \in \mathcal{S}(\mathfrak{gl}_n(\mathbb{R}))$ and $O^\eta(X, g)$ (resp. $O^\eta(X, \widehat{g})$) is the $\text{GL}_{n-1}(\mathbb{R})$ orbital integral of g (resp. \widehat{g}) multiplied by the transfer factor (for the $\text{GL}_{n-1}(\mathbb{R})$ orbital integral). One should note that $\widehat{g} = \mathcal{F}_a g$.

Similarly in the unitary group case, using Theorem 6.1 again, we have

$$\int_{Q(\mathbb{R})} O((X_W, w_W), f_W) O(X_W, \widehat{g}_W) dq_W(X_W) = \int_{Q(\mathbb{R})} O((X_W, w_W), \widehat{f}_W) O(X_W, g_W) dq_W(X_W),$$

where the terms are defined in a similar way as in the general linear case, except that the stabilizer H_{W, w_W} of w_W replaces GL_{n-1} . Note that we have identified the categorical quotient of $\mathfrak{gl}_n // \text{GL}_{n-1}$ with $\mathfrak{u}(W) // H_{W, w_W}$ as before.

Suppose that $f \leftrightarrow \{f_W\}$. We would like to prove that for all matching strongly regular semisimple orbits $(X^0, u^0, v^0) \leftrightarrow (X_W^0, w_W^0)$, we have

$$(9.1) \quad O^\eta((X^0, u^0, v^0), \nu_n \widehat{f}) = O((X_W^0, w_W^0), \widehat{f}_W).$$

This implies that the same equality holds for all regular semisimple orbits since orbital integrals are smooth at regular semisimple orbits and strongly regular semisimple orbits are dense.

Suppose that for some matching $(X^0, u^0, v^0) \leftrightarrow (X_W^0, w_W^0)$, the identity (9.1) does not hold. Then we may choose a smooth function $\gamma(q)$ on $Q_{\text{rs}}(\mathbb{R})$ which is supported in a small neighbourhood of $q(X^0) = q_W(X_W^0)$ so that $\text{supp } \gamma$ is contained in the image of $U(W)(\mathbb{R})$ and

$$\int_{Q(\mathbb{R})} (O^\eta((X, u, v), \nu_n \widehat{f}) - O((X_W, w_W), \widehat{f}_W)) \gamma(q(X)) dq(X) \neq 0.$$

Since $\text{supp } \gamma \subset Q_{\text{rs}}(\mathbb{R})$, we may find g and g_W so that $O^\eta(X, g) = O(X_W, g_W) = \gamma(q(X))$. Therefore g and $\{g_W; 0, \dots, 0\}$ match. Then we have

$$\int_{Q(\mathbb{R})} O^\eta((X, u, v), \nu_n \widehat{f}) O^\eta(X, g) dq(X) \neq \int_{Q(\mathbb{R})} O((X_W, w_W), \widehat{f}_W) O(X, g_W) dq_W(X_W).$$

Therefore

$$\int_{Q(\mathbb{R})} O^\eta((X, u, v), f) O^\eta(X, \nu_n \widehat{g}) dq(X) \neq \int_{Q(\mathbb{R})} O((X_W, w_W), f_W) O(X, \widehat{g}_W) dq_W(X_W).$$

This contradicts the statement A_{n-1} . We have thus proved $\nu_n \widehat{f} \leftrightarrow \{\widehat{f}_W\}$. \square

Lemma 9.4. *The statement C_n holds.*

Proof. To simplify notation, we put $\widehat{f} = \mathcal{F}_c f$. We have shown that C_1 holds in Lemma 8.4.

Suppose that we are in the general linear case. Let $(X, u, v) \in \mathcal{V}$ be a strongly regular semisimple element. Let $F' = \mathbb{R}[X]$ be the subalgebra of \mathfrak{gl}_n generated by X and $E' = F' \otimes_{\mathbb{R}} \mathbb{C}$. Let H' be the stabilizer of X in GL_n . We may identify F' with $\mathbb{R}^p \times \mathbb{C}^r$ by sending X to $(\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_r)$ where $(\lambda_1, \dots, \lambda_p; \mu_1, \overline{\mu_1}, \dots, \mu_r, \overline{\mu_r})$ are all the eigenvalues of X , $\lambda_1, \dots, \lambda_p \in \mathbb{R}$ and μ_1, \dots, μ_r are imaginary. Then $H' \simeq \text{Res}_{F'/F} \text{GL}_1$. Then as in [Zha14, Lemma 4.19], \mathbb{R}_n (resp. \mathbb{R}^n) is a free rank one module over F' and one may define a F' -linear pairing $\mathbb{R}_n \times \mathbb{R}^n \rightarrow F'$ which we denote by $(-, -)$. It satisfies the property that $\text{Tr}_{F'/\mathbb{R}}(\lambda u, v) = u \lambda v$ for all $\lambda \in F'$, $u \in \mathbb{R}_n$, $v \in \mathbb{R}^n$. We define a function $\kappa_{F'}(u', v', u'', v'') = \eta(\text{N}(u', v')) \Psi^{F'}((u', v'')(u'', v'))$ on $\mathbb{R}_n \times \mathbb{R}^n \times \mathbb{R}_n \times \mathbb{R}^n$, where $\Psi^{F'}$ is the Kloosterman sum defined at the end of Section 7 and $\text{N} : F'^{\times} \rightarrow \mathbb{R}^{\times}$ is the norm map. Let $q : \mathbb{R}_n \times \mathbb{R}^n \rightarrow Q = (\mathbb{R}_n \times \mathbb{R}^n) // H'$. This is a one dimensional affine space over F' . Let $f \in \mathcal{S}(\mathcal{V})$ be a Schwartz functions. Then we may argue in the same way as [Zha14, Lemma 4.20] to conclude that

$$O^\eta((X, u, v), \widehat{f}) = (-1)^r \int_{Q(F')} O^\eta((X, u', v'), f) \kappa_{F'}(u, v, u', v') dq(u', v').$$

Suppose that we are in the unitary case. Let $(X_W, w_W) \in \mathcal{V}_W$ be a strongly regular semisimple element and $f_W \in \mathcal{S}(\mathcal{V}_W)$ be a Schwartz functions. Suppose that (X_W, w_W) and (X, u, v) match. Then $\mathbb{C}[X_W] \simeq E' = F' \otimes_{\mathbb{R}} \mathbb{C}$ as a \mathbb{C} -algebra. The space W is a free rank one hermitian module over E' . As in the general linear case, we may define a E' -linear pairing $W \times W \rightarrow E'$ which we denote by $(-, -)$. Define a function $\kappa_{W, F'}(w', w'') = \Phi^{F'}(\text{N}_{E'/F'}(w', w''))$ on $W \times W$ where $\Phi^{F'}$

is the twisted Kloosterman sum defined at the end of Section 7 and $N_{E'/F'}$ is the norm map. Let H'_W be the stabilizer of X_W in $U(W)$. Let $q_W : W \rightarrow Q = W//H'_W$ be the categorical quotient. Note that the categorical quotient $W//H'_W$ and $(\mathbb{R}_n \times \mathbb{R}^n)//H'$ are identified. Let $f_W \in \mathcal{S}(\mathcal{V}_W)$. Then similar to the general linear case, we have

$$O((X_W, w_W), \widehat{f_W}) = \int_{Q(F')} O((X_W, w_W), f_W) \kappa_{W, F'}(w_W, w') dq_{W'}(w').$$

For strongly regular (X, u, v) , we have $N(u, v)|D(X)| = \Delta(X, u, v)$. Indeed, both sides are invariant under the action of H , thus we may assume that X is blocked diagonal with the diagonal blocks are either 1×1 or 2×2 . The identity then follows by an easy computation. Moreover $\eta(\Delta(X, u, v)) = \eta(\text{disc } W)$ as (X, u, v) and (X_W, w_W) match. Therefore $\sqrt{-1}^{-n} \widehat{f}$ and $\{\eta(\text{disc } W) \widehat{f_W}\}$ match by Lemma 7.3. This proves C_n . \square

Proof of Theorem 9.1. Theorem 9.1 now follows from Lemma 9.2, 9.3 and 9.4. \square

10. GEOMETRIC PREPARATIONS

We recall the notion of the sliced representations. Let x be a semisimple point in \mathcal{V} and H_x its stabilizer in H . Let N be the normal space of Hx at x . Then H_x acts on N and we call it the sliced representation at x . In the unitary case, we sometimes write N_W and $H_{W,x}$ to emphasize the dependance on W . The sliced representation takes the same shape as that of \mathcal{V} . In the general linear case, the map $N(\mathbb{R}) \rightarrow (N//H_x)(\mathbb{R})$ is surjective. We refer the readers to [Zha14, Appendix B] for an explicit description of the sliced representations.

We recall the notion of étale and semi-algebraic Luna slices. In general, suppose that G is a reductive group and $X \rightarrow Y$ is an G -equivariant morphism of affine schemes. Then we say that $X \rightarrow Y$ is strongly étale if the induced morphism $X//G \rightarrow Y//G$ is étale and $X \simeq Y \times_{Y//G} X//G$. It follows that $X \rightarrow Y$ is also étale.

By an étale Luna slice at x , we mean an H_x -invariant locally closed subvariety Z of \mathcal{V} , containing x , together with an H_x -equivariant strongly étale morphism $\iota : Z \rightarrow N$ with $\iota(x) = 0$ and a strongly étale morphism $\phi : (Z \times H)//H_x \rightarrow \mathcal{V}$ induced by the H -action on \mathcal{V} . This can be summarized in the following diagram.

$$\begin{array}{ccccc}
 & & (Z \times H)//H_x & \xrightarrow{\phi} & \mathcal{V} \\
 & & \downarrow & & \downarrow \\
 N//H_x & \longleftarrow & Z//H_x & \longrightarrow & \mathcal{V}//H \\
 \uparrow & & \uparrow & & \\
 N & \xleftarrow{\iota} & Z & &
 \end{array}$$

By a semi-algebraic slice of \mathcal{V} at x , we mean a quintuple (U, p, ψ, S, N) where

- N is the sliced representation at x .
- U is an $H(\mathbb{R})$ -invariant Nash neighbourhood of x in \mathcal{V} .
- p is an $H(\mathbb{R})$ -equivariant semi-algebraic retraction $p : U \rightarrow H(\mathbb{R})x$ and $S = p^{-1}(x)$.
- ψ is an $H_x(\mathbb{R})$ -equivariant semi-algebraic embedding $S \rightarrow N(\mathbb{R})$ with saturated image and $\psi(x) = 0$.

A semi-algebraic slice can be summarized in the following diagram.

$$\begin{array}{ccc}
S \times H(\mathbb{R}) & \longrightarrow & S \xrightarrow{\psi} N, \\
\downarrow & & \\
H(\mathbb{R})x & \xleftarrow{p} U \longrightarrow & \mathcal{V}
\end{array}$$

where the vertical arrow is an $H_x(\mathbb{R})$ principal homogeneous space.

An explicit construction of étale Luna slices at any semisimple element $x \in \mathcal{V}$ has been given in [Zha14, Appendix B]. From this we may construct semi-algebraic slices at x . We describe this construction here following [AG09, Appendix A].

Let Z be an étale Luna slice at x as in [Zha14, Appendix B] and $\pi_Z : Z \rightarrow Z//H_x$ be the categorical quotient. By definition, the morphisms $Z//H_x \rightarrow \mathcal{V}//H$ and $Z//H_x \rightarrow N//H_x$ are both étale. Therefore we may choose a sufficiently small Nash neighbourhood S' of $\pi_Z(x)$ in $(Z//H_x)(\mathbb{R})$, so that the above two morphisms send S' Nash diffeomorphically to its image. Let $S \subset Z(\mathbb{R})$ be the inverse image of S' under the natural map $Z(\mathbb{R}) \rightarrow (Z//H_x)(\mathbb{R})$. Let $\psi = \iota|_S$. Let U' be the inverse image of S' in $((Z \times H)//H_x)(\mathbb{R})$.

Let $p' : U' \rightarrow (H_x \backslash H)(\mathbb{R})$ be the natural $H(\mathbb{R})$ -invariant map. Let $U'' = U' \cap p'^{-1}(H_x(\mathbb{R}) \backslash H(\mathbb{R}))$ and $U = \phi(U'') \subset \mathcal{V}$. Note that $\phi|_{U''}$ is a homeomorphism from U'' to U . Let $p = p' \circ (\phi|_{U''})^{-1}$. Then (U, p, ψ, S, N) is the desired semi-algebraic Luna slice at x .

11. THE SET OF TRANSFERABLE FUNCTIONS IS DENSE

The goal of this section is to prove Theorem 3.3.

Let \mathcal{N} be the nilpotent cone in \mathcal{V} , i.e. the inverse image of $0 \in Q$.

Proposition 11.1. *The subspace*

$$\mathcal{S}(\mathcal{V} \setminus \mathcal{N}) + \mathcal{F}_\mathfrak{h} \mathcal{S}(\mathcal{V} \setminus \mathcal{N}) + \mathcal{F}_W \mathcal{S}(\mathcal{V} \setminus \mathcal{N}) + \mathcal{F}_\mathcal{V} \mathcal{S}(\mathcal{V} \setminus \mathcal{N}) + \mathcal{S}(\mathcal{V})_{\text{null}}$$

is dense in $\mathcal{S}(\mathcal{V})$, where \mathcal{F}_* stands for the Fourier transform with respect to $*$ and $\mathcal{S}(\mathcal{V})_{\text{null}}$ is intersection of the kernels of all the (H, η) -invariant distributions. By a distribution we mean a continuous linear functional on $\mathcal{S}(\mathcal{V})$. A distribution D is called (H, η) -invariant if $D^h = \eta(h)D$.

Proof. This follows from [Aiz13, Theorem 6.2.1] for the case $\eta = 1$. As remarked by [Zha14, Theorem 4.22], the same argument works in the case η is nontrivial quadratic. \square

Proof of Theorem 3.3. We prove this by induction on $\dim W$. The case $\dim W = 1$ has been proved in Proposition 8.1. Assume that the theorem holds for all $\dim W < n$. We would like to show that the theorem holds for $\dim W = n$.

The elements in $\mathcal{S}(\mathcal{V})_{\text{null}}$ are clearly transferable. Moreover, $\mathcal{C}_c^\infty(\mathcal{V} \setminus \mathcal{N})$ is dense in $\mathcal{S}(\mathcal{V} \setminus \mathcal{N})$. By Theorem 9.1 and Proposition 11.1, it is enough to show that any $f \in \mathcal{C}_c^\infty(\mathcal{V} \setminus \mathcal{N})$ can be approximated by transferable functions.

Let $x \in \mathcal{V} \setminus \mathcal{N}$ be a semisimple element. We make use of the semi-algebraic slice (U, p, ψ, S, N) at x constructed in Section 10. In the following, without saying to the contrary, the subset, neighbourhoods, etc. are all semi-algebraic, e.g. a subset means a semi-algebraic subset. We write $q : \mathcal{V}(\mathbb{R}) \rightarrow (\mathcal{V} // H)(\mathbb{R})$ and $q_x : N(\mathbb{R}) \rightarrow (N // H_x)(\mathbb{R})$ for the natural maps. By construction, $\psi(S)$ is a saturated subsets, i.e. we can find an open subset $S' \subset (N // H_x)(\mathbb{R})$ such that $S = (q_x \circ \psi)^{-1}(S')$. Let us fix a relatively compact open subset Σ' of S' such that its closure is also contained in S' . Put $\Omega' = (q_x \circ \psi)^{-1}(\Sigma')$, $\Omega = q_x^{-1}(\Sigma') = \psi(\Omega')$. Then $\Omega'^{H(\mathbb{R})}$ is an open $H(\mathbb{R})$ -invariant neighbourhood of x in \mathcal{V} . As x ranges over all semisimple elements in $\mathcal{V} \setminus \mathcal{N}$, the neighbourhoods $\Omega'^{H(\mathbb{R})}$ form an open cover of $\mathcal{V} \setminus \mathcal{N}$. Then by the Partition of Unity, we may assume that $\text{supp } f \subset \Omega'^{H(\mathbb{R})}$ so that $f \in \mathcal{C}_c^\infty(\Omega'^{H(\mathbb{R})})$.

Since the image of Ω' in $(N // H_x)(\mathbb{R})$ is relatively compact by construction, by [Zha14, Lemma 3.10], we may find a compact subset C of $H_x(\mathbb{R}) \setminus H(\mathbb{R})$ such that $H_x(\mathbb{R})C$ contains $\{h \in H(\mathbb{R}) \mid \Omega'^h \cap \text{supp } f \neq \emptyset\}$. Let $C' \supset C$ be an open subset of $H_x(\mathbb{R}) \setminus H(\mathbb{R})$ whose closure is also compact and $\alpha \in \mathcal{C}_c^\infty(H(\mathbb{R}))$ be a function so that the function

$$g \mapsto \int_{H_x(\mathbb{R})} \alpha(hg) dh$$

takes value one on C and zero outside C' . Define a function f_x on Ω as

$$f_x(z) = \int_{H(\mathbb{R})} f(\psi^{-1}(z)^g) \alpha(g) \eta(g) dg, \quad z \in \Omega.$$

Then $f_x \in \mathcal{C}_c^\infty(\Omega)$. We may view f_x as a function on $N(\mathbb{R})$ via extension by zero.

If $y \in \Omega'$ is regular semisimple and $z = \psi(y) \in \Omega$, then $O(y, f) = O(z, f_x)$. In fact,

$$\begin{aligned} O(z, f_x) &= \int_{H_x(\mathbb{R})} f_x(z^h) \eta(h) dh = \int_{H_x(\mathbb{R})} \int_{H(\mathbb{R})} f(y^{hg}) \alpha(g) \eta(hg) dg dh \\ &= \int_{H(\mathbb{R})} f(y^g) \eta(g) \left(\int_{H_x(\mathbb{R})} \alpha(h^{-1}g) dh \right) dg. \end{aligned}$$

If $g \in H_x(\mathbb{R})C$, then the inner integral equals one. If $g \notin H_x(\mathbb{R})C$, then for any y , $y^g \notin \text{supp } f$. Thus $O(z, f_x) = O(y, f)$. If $y \notin \Omega'^{H(\mathbb{R})}$, then $O(y, f) = 0$. If $z \notin \Omega$, then $O(z, f_x) = 0$.

By the explicit description of the sliced representations [Zha14, Appendix B], the sliced representations are products of lower dimensional representations which are of the same shape as \mathcal{V} . Therefore we may speak of the smooth transfer and transferable functions for the sliced

representations. In the general linear case, by [Zha14, Lemma 3.15], we may assume that for any regular semisimple $y \in S$ and $z = \psi(y)$, the transfer factor at y is a nonzero constant times the transfer factor of the sliced representation at z .

We apply the induction hypothesis to f_x . Then we can find a sequence of transferable functions $g_n \in \mathcal{S}(N(\mathbb{R}))$, so that $g_n \rightrightarrows f_x$. The set $q_x(\text{supp } f_x)$ is compact and is contained in $q_x(\Omega)$. Therefore, we may find an open and relatively compact subset Σ of $(N//H_x)(\mathbb{R})$ such that it contains $q_x(\text{supp } f_x)$ and its closure is contained in Σ' . Let γ be a function on $(N//H_x)(\mathbb{R})$ which takes value one on $q_x(\text{supp } f_x)$ and vanishes outside Σ . Let $\tilde{\gamma} = \gamma \circ q_x$ be a function on $N(\mathbb{R})$. Then $|\tilde{\gamma}| \leq 1$ and $\text{supp } \tilde{\gamma} \subset \Omega$. Therefore $g_n \tilde{\gamma} \in \mathcal{S}(\Omega)$ and $g_n \tilde{\gamma} \rightrightarrows f_x \tilde{\gamma} = f_x$. Moreover, $g_n \tilde{\gamma}$ is also transferable by Lemma 3.4. Thus we may replace g_n by $g_n \tilde{\gamma}$ and assume that $g_n \in \mathcal{S}(\Omega)$ in the first place. We view g_n as a Schwartz function on $N(\mathbb{R})$ via extension by zero.

We now fix a $\beta \in \mathcal{C}_c^\infty(H(\mathbb{R}))$ such that

$$\int_{H(\mathbb{R})} \beta(h) \eta(h) dh = 1.$$

Define a function $f_n \in \mathcal{S}(U)$ by

$$f_n(y^h) = \int_{H_x(\mathbb{R})} g_n(\psi(y)^g) \beta(g^{-1}h) dg, \quad y \in S, \quad h \in H(\mathbb{R}).$$

Note that $f_n \in \mathcal{S}(\Omega^{H(\mathbb{R})})$. Then if $y \in S$ is regular semisimple and $z = \psi(y)$, then

$$\begin{aligned} O(y, f_n) &= \int_{H(\mathbb{R})} f_n(y^h) \eta(h) dh = \int_{H(\mathbb{R})} \int_{H_x(\mathbb{R})} g_n(z^g) \beta(g^{-1}h) \eta(h) dg dh \\ &= \int_{H_x(\mathbb{R})} g_n(z^g) \eta(g) dg = O(z, g_n). \end{aligned}$$

If $y \notin \Omega^{H(\mathbb{R})}$, then $O(y, f_n) = 0$. If $z \notin \Omega$, then $O(z, g_n) = 0$.

Claim: f_n is transferable.

We prove this in the general linear case. The unitary case is similar. Consider $y \mapsto \omega(y)O(y, f_n)$ as a function on $\mathcal{V}_{\text{rs}}(\mathbb{R})/H(\mathbb{R})$. Denote this function by ϕ . Then ϕ vanishes outside a compact subset of $(\mathcal{V}/H)(\mathbb{R})$. In fact, ϕ vanishes outside $q(\Omega')$ in $(\mathcal{V}/H)(\mathbb{R})$ which is relatively compact. Let $y' \in U$ be regular semisimple and $y'' = q(y')$. Without loss of generality, we may assume that $y' \in S$. Up to some fixed constant which depends only on x (coming from the transfer factor), $\phi(y'') = \omega(y')O(y', f_n) = \omega(\psi(y'))O(\psi(y'), g_n)$. Since g_n is transferable (for the sliced representation), by [Zha14, Proposition 3.16], $\phi|_{q(U)}$ is an orbital integral on the unitary group side. More precisely, by the explicit description of the étale and semi-algebraic Luna slices in [Zha14, Appendix B], for each hermitian space W , we may find a semisimple element $x_W \in \mathcal{V}_W$ and construct a semi-algebraic Luna slice $(U_W, p_W, \psi_W, S_W, N_W)$ at x_W , and a test function $g_W \in \mathcal{S}(N_W)$, such that g_n and $\{g_W\}$ are smooth transfer of each other (for the sliced representation). Moreover, the image of S in $(N//H_x)(\mathbb{R})$ and the union of the images of

S_W in $(N_W//H_{x_W})(\mathbb{R})$ are identified under the natural isomorphism $N//H_x \simeq N_W//H_{x_W}$. We may construct $f_W \in \mathcal{S}(U_W)$ from g_W in the same way as from g_n to f_n . Then it is clear that $\phi|_{q(U)}(y'')$ equals the $O(z_W, f_W)$ if y'' is the image of z_W in $\mathcal{V}_W(\mathbb{R})_{\text{rs}}/H_W(\mathbb{R})$. If $y'' \notin q(\Omega')$, then $\phi(y'') = 0$. It then follows from [Zha14, Proposition 3.8] that ϕ is an orbital integral on the unitary group side. This proves the claim.

We note that the sequence $\{f_n\}$ is convergent in $\mathcal{S}(\mathcal{V}(\mathbb{R}))$. In fact, the map $H(\mathbb{R}) \times S \rightarrow U$ is an $H_x(\mathbb{R})$ -principal homogeneous space and is submersive. The sequence $\beta \otimes (g_n \circ \psi)$ is convergent in $\mathcal{S}(H(\mathbb{R}) \times S)$ and f_n is obtained from $\beta \otimes (g_n \circ \psi)$ by integrating along the fiber. The map “integrating along the fibers” is a continuous map from $\mathcal{S}(H(\mathbb{R}) \times S)$ to $\mathcal{S}(U)$. It follows that f_n is convergent in $\mathcal{S}(U)$. Therefore it is convergent in $\mathcal{S}(\mathcal{V}(\mathbb{R}))$. Suppose that $f_n \rightrightarrows \tilde{f}$ for some $\tilde{f} \in \mathcal{S}(U) \subset \mathcal{S}(\mathcal{V}(\mathbb{R}))$. Note that $f_n \in \mathcal{S}(\Omega^{H(\mathbb{R})})$ for all n . Therefore $\tilde{f} \in \mathcal{S}(\Omega^{H(\mathbb{R})})$.

We claim that for any regular semisimple $y \in \mathcal{V}(\mathbb{R})$, we have $O(y, f) = O(y, \tilde{f})$. If $y \notin \Omega^{H(\mathbb{R})}$, then both orbital integrals vanish. If $y \in \Omega^{H(\mathbb{R})}$, then we may assume that $y \in \Omega'$. By construction, on the one hand, we have $O(y, f) = O(\psi(y), f_x)$. On the other hand, since $f \mapsto O(y, f)$ is a continuous linear functional, we have

$$O(y, \tilde{f}) = \lim_{n \rightarrow \infty} O(y, f_n) = \lim_{n \rightarrow \infty} O(\psi(y), g_n) = O(\psi(y), f_x).$$

This proves the claim.

Finally, let $f'_n = f_n - \tilde{f} + f$. Then $f'_n \rightrightarrows f$ and f'_n is transferable since all regular semisimple orbital integrals of $f - \tilde{f}$ vanish. This completes the induction and finishes the proof of the Theorem. \square

12. THE RELATIVE TRACE FORMULAE OF JACQUET–RALLIS

In this section, we prove Theorem 1.1 using a simple version of Jacquet–Rallis relative trace formulae. The argument is essentially the same as [Zha14, Section 2]. We will highlight the use of Theorem 2.7 and only sketch the arguments that are identical to [Zha14].

We recall some notations and settings from Theorem 1.1. Let E/F be a quadratic extension of number fields and $W \subset V$ be a pair of hermitian spaces of dimension n and $n+1$ respectively. Let $U(W)$ and $U(V)$ be the corresponding unitary groups. The embedding $W \subset V$ induces an embedding $U(W) \subset U(V)$. We shall always consider $U(W)$ as a subgroup of $U(V)$ via this embedding. We take this convention for other relevant pairs of hermitian spaces. Recall that $W' \subset V'$ is called relevant if V/W and V'/W' are isomorphic as one dimensional hermitian spaces. Let $G = U(V) \times U(W)$ and $H = U(W)$. The group H is considered as a subgroup of G via the diagonal embedding.

Let π be an irreducible cuspidal automorphic representation of $G(\mathbb{A}_F)$ and $\varphi \in \pi$. Here by a cuspidal automorphic representation, we mean a subrepresentation of the space of the cuspidal

automorphic forms. We put

$$P(\varphi) = \int_{H(F)\backslash H(\mathbb{A}_F)} \varphi(h)dh, \quad \varphi \in \pi.$$

This is an $H(\mathbb{A}_F)$ -invariant continuous linear form on π . Let $f \in \mathcal{S}(G(\mathbb{A}_F))$, we define

$$J_\pi(f) = \sum_{\varphi} P(\pi(f)\varphi)\overline{P(\varphi)}.$$

where the summation runs over an orthonormal basis of π . We may assume that each φ appearing in the sum is K -finite where K is a fixed maximal compact subgroup of $G(\mathbb{A}_F)$. This sum is absolutely convergent.

We would like to define a local analogue of J_π . For this, let us be slightly more careful about the local components of π , especially at the archimedean places. We have the decomposition $\pi = \otimes' \pi_v$. If v is a non-archimedean place, then π_v is an irreducible admissible representation of $G(F_v)$ that is realized on some Hilbert space \mathcal{H}_v . If v is archimedean, then π_v is a continuous representation of $G(F_v)$ that is realized on some Hilbert spaces \mathcal{H}_v with the norm $\|\cdot\|$. Let \mathcal{H}_v^∞ be the space of smooth vectors in \mathcal{H}_v . Then \mathcal{H}_v^∞ is a Casselman–Wallach representation of $G(F_v)$ in the sense of [Jac09], namely a smooth Fréchet representation of moderate growth. The topology on \mathcal{H}_v^∞ is induced by the collection of semi-norms $v \mapsto \|X.v\|$ where X ranges over a (countable) basis of the complexified universal enveloping algebra of the Lie algebra of $G(F_v)$. It is finer than the topology induced from \mathcal{H}_v . The Schwartz space $\mathcal{S}(G(F_v))$ acts on \mathcal{H}_v^∞ . There is an orthonormal basis of \mathcal{H}_v consisting of K_v finite vectors where K_v is a fixed maximal compact subgroup of $G(F_v)$. A continuous linear functional on \mathcal{H}_v^∞ will simply be called a continuous linear functional on π_v .

Suppose that P is not identically zero. Let v be a place of F . Then the space of $H(F_v)$ invariant continuous linear functional on π_v is one dimensional, c.f. [AGRS10, SZ12]. Fix such a linear functional $\ell_v \neq 0$. Let $f'_v \in \mathcal{S}(G(F_v))$. Put

$$J_{\pi_v}(f_v) = \sum_{\varphi_v} \ell_v(\pi_v(f_v)\varphi_v)\overline{\ell_v(\varphi_v)},$$

where the sum runs over an orthonormal basis of π_v . We may assume that each φ_v in the sum is K_v finite where K_v is a fixed maximal compact subgroup of $G(F_v)$. We also normalize, so that if π_v is unramified and φ_v is a $G(\mathfrak{o}_{F,v})$ -fixed vector of norm one, then $\ell_v(\varphi_v) = 1$. The convergence of this sum will be proved below. Then if $f = \otimes f_v \in \mathcal{S}(G(\mathbb{A}_F))$ is factorizable, there is a constant C such that

$$J_\pi(f) = C \prod_v J_{\pi_v}(f_v).$$

Let $G' = \text{Res}_{E/F}(\text{GL}_{n+1} \times \text{GL}_n)$. The group GL_n is considered as a subgroup of GL_{n+1} via the embedding $g \mapsto \text{diag}[g, 1]$. Put $H'_1 = \text{Res}_{E/F} \text{GL}_n$, $H'_2 = \text{GL}_{n+1} \times \text{GL}_n$. The group H'_1 is

considered as a subgroup of G' by the diagonal embedding while H'_2 embeds in G' in the obvious way. Let $Z_{H'_2}$ be the center of H'_2 . Let η be the quadratic character of \mathbb{A}_F^\times associated to the extension E/F via the global class field theory. We extend it to a character of $H'_2(\mathbb{A}_F)$ in the following way. Let $h = (h_{n+1}, h_n) \in H'_2(\mathbb{A}_F)$. Put

$$\eta(h) = \begin{cases} \eta(\det h_{n+1}), & n \text{ odd}; \\ \eta(\det h_n), & n \text{ even}. \end{cases}$$

Let $\Pi = \text{BC}(\pi)$ be the weak base change of π . Assume that Π is cuspidal. This will be the case in the setting of Theorem 1.1. Let $f' \in \mathcal{S}(G'(\mathbb{A}_F))$ and we define

$$I_\Pi(f') = \sum_{\varphi} \int_{H'_1(F) \backslash H'_1(\mathbb{A}_F)} \Pi(f') \varphi(h) dh \int_{Z_{H'_2}(\mathbb{A}_F) H'_2(F) \backslash H'_2(\mathbb{A}_F)} \overline{\varphi(h) \eta(h)} dh,$$

where the sum is over an orthonormal basis of Π . We may assume that each φ in the sum is K' -finite where K' is a fixed maximal compact subgroup of $G'(\mathbb{A}_F)$. This sum is absolutely convergent.

Fix a nontrivial additive character $\psi = \otimes \psi_v : F \backslash \mathbb{A}_F \rightarrow \mathbb{C}^\times$. We recall the Rankin–Selberg convolution. Let $N' = \text{Res}_{E/F} N'_{n+1} \times N'_n$ be the standard upper triangular unipotent subgroup of G' . We extend ψ to a generic character of $N'(\mathbb{A}_F)$ which we still denote by ψ . Let v be a place of F and λ_v be a ψ_v -Whittaker functional on Π_v . We fix a λ_v for each place v so that we have a decomposition

$$\int_{N'(\mathbb{A}_F)} \varphi(n) \overline{\psi(n)} dn = \prod_v \lambda_v(\varphi_v), \quad \varphi = \otimes \varphi_v \in \Pi.$$

For any $\varphi_v \in \pi_v$, we put $W_{\varphi_v}(g) = \lambda_v(\Pi_v(g) \varphi_v)$. Let $\mathcal{W}_{\pi_v} = \{W_{\varphi_v} \mid \varphi_v \in \Pi_v\}$ be the Whittaker model of Π_v . Define

$$Z(s, \varphi_v) = \int_{H'_1(F_v)} W_{\varphi_v}(h) |\det h|^{s-\frac{1}{2}} dh.$$

This integral is absolutely convergent for $\Re s \gg 0$ and has a meromorphic continuation to the whole complex plane. Let r be the order of poles of $L(s, \Pi_v, \text{St})$ at $s = \frac{1}{2}$ and we put

$$\Lambda(\varphi_v) = \lim_{s \rightarrow \frac{1}{2}} (s - \frac{1}{2})^r Z(s, \varphi_v).$$

This defines a nonzero continuous linear functional on π_v when v is archimedean, c.f. [Jac09, Theorem 2.3].

Suppose now that the linear form

$$\varphi \mapsto \beta(\varphi) = \int_{Z_{H'_2}(\mathbb{A}_F) H'_2(F) \backslash H'_2(\mathbb{A}_F)} \varphi(h) \eta(h) dh$$

is not identically zero. By the uniqueness theorem of [Fli91] or the explicit computation in [GJR01, Section 2], we fix a nonzero continuous $(H_2'(F_v), \eta_v)$ -invariant linear functional β_v on Π_v for each place v , such that $\beta = \otimes \beta_v$. Let $f'_v \in \mathcal{S}(G'(F_v))$ and we define

$$I_{\Pi_v}(f'_v) = \sum_{\varphi_v} \Lambda(\Pi_v(f'_v)\varphi_v) \overline{\beta_v(\varphi_v)},$$

where the sum runs over an orthonormal basis of Π_v . We may assume that each φ_v in the sum is K'_v finite where K'_v is a fixed maximal compact subgroup of $G'(F_v)$. The convergence of this sum will be proved below. Then there is a constant C' such that for all factorizable $f' = \otimes f'_v \in \mathcal{S}(G'(\mathbb{A}_F))$, we have

$$I_{\Pi}(f') = C' \prod_v I_{\Pi_v}(f'_v).$$

Lemma 12.1. *Assume that v is archimedean. The defining sum of I_{Π_v} is absolutely convergent. The linear functional*

$$f'_v \mapsto I_{\Pi_v}(f'_v), \quad f'_v \in \mathcal{S}(G'(F_v))$$

is continuous. Similar results hold for J_{π_v} .

Proof. For simplicity, we drop the subscripts v in the proof of the lemma. Let us consider the case I_{Π} . The case of J_{π} can be proved in the same way. Suppose that Π is realized on some Hilbert space \mathcal{H} with a norm $\|\cdot\|$ and let \mathcal{H}^{∞} be the space of smooth vectors in \mathcal{H} . Since Λ and β are both continuous (with respect to the topology on \mathcal{H}^{∞}), there is a constant $C > 0$ such that

$$|\Lambda(\Pi(f')\varphi)\beta(\varphi)| \leq C \times \|\Pi(f')\varphi\|,$$

for all $\varphi \in \mathcal{H}^{\infty}$. By [Wal88, Lemma 8.1.1], there is a continuous seminorm p on $\mathcal{S}(G'(F))$ so that

$$\sum_{\varphi} \|\Pi(f')\varphi\| \leq p(f'),$$

where φ ranges over an orthonormal basis and each φ is K' -finite. The absolute convergence and continuity then follow. \square

We say that the decomposable test functions $f' \in \mathcal{S}(G'(\mathbb{A}_F))$ and $\{f_W\}_W$ where $f_W \in \mathcal{S}(G_W(\mathbb{A}_F))$ and W runs over all hermitian spaces of dimension n are smooth transfer of each other if for all places v of F , the test functions f'_v and $\{f_{W,v}\}$ are smooth transfer of each other. We say that a test function $f' = \otimes f'_v \in \mathcal{S}(G'(\mathbb{A}_F))$ (resp. $f = \otimes f_v \in \mathcal{S}(G_W(\mathbb{A}_F))$ for a fixed W) is transferable if f'_v (resp. f_v) is transferable for all v .

Proposition 12.2. *Suppose that I_{Π} is not identically zero. Then there is a transferable test function $f' \in \mathcal{S}(G'(\mathbb{A}_F))$ so that $I_{\Pi}(f') \neq 0$. Similar assertion holds for J_{π} .*

Proof. This follows from Theorem 2.7 and Lemma 12.1. \square

We now record the following relative trace identity from [Zha14, Proposition 2.11]. We note that since our test functions are not necessarily compactly supported, there are some additional difficulties in establishing this relative trace identity. These difficulties are solved by Beuzart-Plessis [BP, Appendix A].

Proposition 12.3. *Let f' and $\{f_W\}$ be decomposable test functions. Let π be an irreducible cuspidal automorphic representation of $G_W(\mathbb{A}_F)$ such that it is supercuspidal at some split place v_0 of F . Suppose that f' and $\{f_W\}$ are nice test functions and satisfy the conditions of [Zha14, Proposition 2.10]. We are not going to record all the conditions. Suffices to say is that it is a collection of conditions on f'_v and $\{f_{W,v}\}$ where v is split. The actual conditions will not bother us. Assume that f' and $\{f_W\}$ are smooth transfer of each other. Then*

$$(12.1) \quad I_{\Pi}(f') = \sum_W \sum_{\pi_W} J_{\pi_W}(f_W),$$

where the outer sum on the right hand runs over all hermitian spaces of dimension n and the inner sum runs over all automorphic representations π_W of $G_W(\mathbb{A}_F)$ that are nearly equivalent to π .

Proposition 12.4. *We keep the assumptions from Theorem 1.1. Then if P is not identically zero on π , then $L(\frac{1}{2}, \Pi, \text{St}) \neq 0$ and there is an $\varphi \in \Pi$ so that*

$$\int_{Z_2(\mathbb{A}_F)H'_2(F)\backslash H'_2(\mathbb{A}_F)} \varphi(h)\eta(\det h)dh \neq 0.$$

In particular, in Theorem 1.1, (2) implies (1).

Proof. It suffices to prove that there is an $f' \in \mathcal{S}(G'(\mathbb{A}_F))$ so that $I_{\Pi}(f') \neq 0$.

Let $f = \otimes f_v \in \mathcal{S}(G(\mathbb{A}_F))$. We say that f is of positive type if there is a $f_1 = \otimes f_{1,v} \in \mathcal{S}(G(\mathbb{A}_F))$ so that $f = f_1 * f_1^*$ where $f_1^*(g) = f_1(g^{-1})$. Then if f is of positive, then $J_{\pi}(f) \geq 0$.

By assumption, the linear form P is not identically zero on π . Then we may choose an $f_2 = \otimes f_{2,v} \in \mathcal{S}(G(\mathbb{A}_F))$ that is of positive type and $J_{\pi}(f_2) > 0$ and for any π_W that is nearly equivalent to π , we have $J_{\pi_W}(f_2) \geq 0$. In particular,

$$(12.2) \quad \sum_{\pi_W} J_{\pi_W}(f_2) \neq 0.$$

We then modify f_2 at the places v_1 and v_2 as in [Zha14, Proof of Proposition 2.13] so that f_2 is a nice test function and satisfies the conditions of Proposition 12.3.

The only problem is that f_2 might not be transferable. Now we make use of Theorem 2.7 to modify f_2 at the nonsplit archimedean places so that f_2 is transferable and we still have (12.2).

Let $f' \in \mathcal{S}(G'(\mathbb{A}_F))$ be the smooth transfer of $\{f_2, 0, \dots, 0\}$. The relative trace identity (12.1) reduces to

$$I_{\Pi}(f') = \sum_{\pi_W} J_{\pi_W}(f_2) \neq 0.$$

This proves the proposition. \square

Proposition 12.5. *We keep the assumptions from Theorem 1.1. There is an $\varphi \in \Pi$ so that*

$$\int_{Z_2(\mathbb{A}_F)H'_2(F)\backslash H'_2(\mathbb{A}_F)} \varphi(h)\eta(h)dh \neq 0.$$

Proof. The proof is identical to [Zha14, Theorem 1.4]. The only difference is that we make use of Proposition 12.4 instead of [Zha14, Proposition 2.13]. We omit the details. \square

Proof of Theorem 1.1. The implication (2) \Rightarrow (1) is proved in Proposition 12.4. We now prove (1) \Rightarrow (2). Assume assertion (1) of Theorem 1.1. By Proposition 12.5, this implies that there is a test function $f' \in \mathcal{S}(G'(\mathbb{A}_F))$ such that $I_{\Pi'}(f') \neq 0$.

We may modify f' at the places v_1 and v_2 (note that they are split in E) as in [Zha14, Section 2.7] so that f' is nice and $I_{\Pi}(f') \neq 0$. We then make use of Theorem 2.7 to modify f' at the nonsplit archimedean places v so that f' is transferable and $I_{\Pi}(f') \neq 0$. Then we apply Proposition 12.3 to conclude that the right hand side of (12.1) is not zero. Therefore there is at least one W' and $\pi_{W'}$ that is nearly equivalent to π , so that $J_{\pi_{W'}}$ is not identically zero. This completes the proof of Theorem 1.1. \square

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