REFINED GLOBAL GAN–GROSS–PRASAD CONJECTURE FOR FOURIER–JACOBI PERIODS ON SYMPLECTIC GROUPS

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Abstract. In this paper, we propose a conjectural identity between the Fourier–Jacobi periods on symplectic groups and the central value of certain Rankin–Selberg L-functions. This identity can be viewed as a refinement to the global Gan–Gross–Prasad conjecture for $\mathrm{Sp}(2n)\times\mathrm{Mp}(2m)$. To support this conjectural identity, we show that when $n=m$ and $n=m\pm1$, it can be deduced from the Ichino–Ikeda's conjecture in some cases via theta correspondences. As a corollary, the conjectural identity holds when $n=m=1$ or when $n=2$, $m=1$ and the automorphic representation on the bigger group is endoscopic.

Contents

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1. Introduction

In this paper, we propose a conjectural identity between the Fourier–Jacobi periods on symplectic groups and the central value of certain Rankin–Selberg L-functions. This identity can be viewed as a refinement to the (global) Gan–Gross–Prasad conjecture [8] for $\mathrm{Sp}(2n) \times \mathrm{Mp}(2m)$.

The Gan–Gross–Prasad conjecture predicts that the nonvanishing of certain periods is equivalent to the nonvanishing of the central value of certain L-functions. There are two types of periods: Bessel periods and Fourier–Jacobi periods. Bessel periods are periods of automorphic forms on orthogonal groups or hermitian unitary groups. A lot of work has been devoted to the study of Bessel periods, starting from the pioneering work of Waldspurger [47]. In their seminal work [24], based on an extensive study of the known low rank examples, Ichino and Ikeda proposed a precise formula relating the Bessel periods on $\mathrm{SO}(n+1) \times \mathrm{SO}(n)$ and the central value of some Rankin–Selberg L-functions. The analogous formula for Bessel periods on the hermitian unitary groups $\mathrm{U}(n+1) \times \mathrm{U}(n)$ has been worked out by N. Harris in his thesis [18] . W. Zhang [51, 54] then proved a large part of the conjectural formula for $\mathrm{U}(n+1) \times \mathrm{U}(n)$, using the relative trace formulae proposed by Jacquet–Rallis [25]. This has been further improved by Beuzart-Plessis [6]. Recently, Liu [36] proposed a conjectural formula for Bessel periods in general, i.e. the Bessel periods on $\mathrm{SO}(n+2r+1) \times \mathrm{SO}(n)$ or $\mathrm{U}(n+2r+1) \times \mathrm{U}(n)$. Some low rank cases have also been considered in [36].

There is a parallel theory for the Fourier–Jacobi periods. They are the periods of automorphic forms on $\mathrm{Mp}(2n+2r) \times \mathrm{Sp}(2n)$ or $\mathrm{U}(n+2r) \times \mathrm{U}(n)$. The case of Fourier–Jacobi periods on $\mathrm{U}(n) \times \mathrm{U}(n)$ has been considered in [49, 50]. We proposed a conjectural formula relating the Fourier–Jacobi periods on $\mathrm{U}(n) \times \mathrm{U}(n)$ and the central value of some L-functions. We proved this conjectural formula in some cases, using the relative trace formula proposed by Liu [35]. In the other extreme case, where one of the groups is trivial, the Fourier–Jacobi periods is simply the Whittaker–Fourier coefficients. In this situation, Lapid–Mao [29] proposed a formula computing the norm of the Whittaker–Fourier coefficients. In a series of papers [30–32], they proved the formula for Whittaker–Fourier coefficients on $\mathrm{Mp}(2n)$, under some simplifying conditions at the archimedean places.

The goal of this paper is to formulate a conjectural identity between the Fourier–Jacobi periods and the central value of some Rankin–Selberg L-functions for symplectic groups. We also verify that this conjecture is compatible with Ichino–Ikeda's conjecture in some cases. As a corollary, the conjectural identity holds in some low rank cases. We now describe our results in more detail.

For simplicity, in the introduction, we consider only the Fourier–Jacobi periods on $\mathrm{Sp}(2n+2r) \times \mathrm{Mp}(2n)$ $(r \geq 0)$. The case $r < 0$ will be explained in the main context of the paper. Let F be a number field and $\psi : F\backslash\mathbb{A}_F \to \mathbb{C}^\times$ be a nontrivial additive character. Let (W_2, q_2) be the symplectic space over F with an orthogonal decomposition W_0+R+R^* where R and R^* are isotropic subspaces and $R+R^*$ is the direct sum of $r-1$ hyperbolic planes. We fix a complete filtration of R and let N_{r-1} be the unipotent radical of the parabolic subgroup of G_2 fixing the complete filtration.

Let $G_2=\mathrm{Sp}(W_2)$, $G_0=\mathrm{Sp}(W_0)$ and $\widetilde{G_0}=\mathrm{Mp}(W_0)$ (the metaplectic double cover). Let π_2 (resp. π_0) be an irreducible cuspidal tempered (resp. genuine) automorphic representation of $G_2(\mathbb{A}_F)$ (resp. $\widetilde{G_0}(\mathbb{A}_F)$). Let $\varphi_2\in\pi_2$ and $\varphi_0\in\pi_0$. Let $H=W_0\ltimes F$ be the Heisenberg group attached to W_0 and ω_ψ be the Weil representation of $H(\mathbb{A}_F)\rtimes\widetilde{G_0}(\mathbb{A}_F)$ which is realized on the Schwartz space $\mathcal{S}(\mathbb{A}_F^n)$. Let $\phi\in\mathcal{S}(\mathbb{A}_F^n)$ be a Schwartz function and $\theta_\psi(\cdot,\phi)$ be a theta series on $H(\mathbb{A}_F)\rtimes\widetilde{G_0}(\mathbb{A}_F)$. Let ψ_{r-1} be an automorphic generic character of $N_{r-1}(\mathbb{A}_F)$ which is stable under the conjugation action of $H(\mathbb{A}_F)\rtimes G_0(\mathbb{A}_F)$. The Fourier–Jacobi period of $(\varphi_2,\varphi_0,\phi)$ is the following integral

(1.0.1)
$$\begin{aligned}&\mathcal{FJ}_\psi(\varphi_2,\varphi_0,\phi)\\ &=\int_{G_0(F)\backslash G_0(\mathbb{A}_F)}\int_{H(F)\backslash H(\mathbb{A}_F)}\int_{N_{r-1}(F)\backslash N_{r-1}(\mathbb{A}_F)}\varphi_2(uhg_0)\varphi_0(g_0)\overline{\psi_{r-1}(u)\theta_\psi(hg_0,\phi)}\mathrm{d}u\mathrm{d}h\mathrm{d}g_0.\end{aligned}$$

This integral is absolutely convergent since φ_2 and φ_0 are both cuspidal. It defines an element in

$$\mathrm{Hom}_{N_{r-1}(\mathbb{A}_F)\rtimes(H(\mathbb{A}_F)\rtimes G_0(\mathbb{A}_F))}(\pi_2\otimes\pi_0\otimes\overline{\omega_\psi\otimes\psi_{r-1}},\mathbb{C}).$$

This space is at most one dimensional [37,45].

The Gan–Gross–Prasad conjecture predicts [8, Conjecture 26.1] that if the above Hom-space is not zero, then the integral (1.0.1) does not vanish identically if and only if $L_\psi^S(\frac{1}{2},\pi_2\times\pi_0)$ is nonvanishing, where S is a sufficiently large finite set of places of F and $L_\psi^S(s,\pi_2\times\pi_0)$ is the tensor product L-function of π_2 and π_0 (note that this L-function depends on ψ).

The conjectural identity that we propose is

$$|\mathcal{FJ}_\psi(\varphi_2,\varphi_0,\phi)|^2=\frac{\Delta_{G_2}^S}{|S_{\pi_2}||S_{\pi_0}|}\frac{L_\psi^S(\frac{1}{2},\pi_2\times\pi_0)}{L^S(1,\pi_2,\mathrm{Ad})L_\psi^S(1,\pi_0,\mathrm{Ad})}\times\prod_{v\in S}\alpha_v(\varphi_{2,v},\varphi_{0,v},\phi_v),\tag{1.0.2}$$

where

- $\varphi_2=\otimes\varphi_{2,v}$, $\varphi_{0,v}=\otimes\varphi_{0,v}$, $\phi=\otimes\phi_v$.
- $\Delta_{G_2}^S=\prod_{i=1}^{n+r}\zeta_F^S(2i)$;
- $L_\psi^S(s,\pi_2\times\pi_0)$ is the tensor product L-function and $L^S(s,\pi_2,\mathrm{Ad})$, $L_\psi^S(1,\pi_0,\mathrm{Ad})$ are adjoint L-functions;
- α_v is a local linear form defined by integration of matrix coefficients (see Section 2.2 for the definition). It is expected that $\alpha_v\neq0$ if and only if $\mathrm{Hom}_{N_{r-1}(F_v)\rtimes(H(F_v)\rtimes G_0(F_v))}(\pi_{2,v}\otimes\pi_{0,v}\otimes\overline{\psi_{r-1,v}\otimes\omega_{\psi_v}},\mathbb{C})\neq0$.

– $\mathrm{d}g_0$ in the definition of $\mathcal{FJ}_\psi$ is the Tamagawa measure on $G_0(\mathbb{A}_F)$, $\mathrm{d}u$ and $\mathrm{d}h$ are the self-dual measures on $N_{r-1}(\mathbb{A}_F)$ and $H(\mathbb{A}_F)$ respecively;
– S_{π_2} and S_{π_0} are centralizers of the L-parameters of π_2 and π_0 respectively. They are abelian 2-groups (see Section 2.3 for a discussion).

This conjectural identity can be viewed as a refinement to the Gan–Gross–Prasad conjecture. It is motivated by the existing conjectural identities of this type [18, 24, 36, 50]. The conjectural identity claims that we should expect the same for both the Bessel periods and the Fourier–Jacobi periods. In the first part of this paper, we show that the conjectural identity (1.0.2) is well-defined, i.e. the local linear form α_v is well-defined and the right hand side of (1.0.2) is independent of the set S. In the definition of the local linear form α_v, we introduce a new way to regularize a divergent oscillating integral over a unipotent group. This gives the same results as the existing regularizations [29, 36], but has the advantage of being elementary, purely function theoretic and uniform for both archimedean and non-archimedean places.

One might be asking what happens for the Fourier–Jacobi periods on skew-hermitian unitary groups. An identity similar to 1.0.2 should also hold. We exclude that in the present paper for two reasons. First, sticking to the symplectic groups greatly simplifies the notation. More importantly, in showing that the right hand side of (1.0.2) is independent of S, we make use of some results in [14]. The analogue results for unitary groups have not appeared in print yet. D. Jiang has informed the author that X. Shen and L. Zhang are working on a more general version of the results in [14], which should cover Fourier–Jacobi periods for both symplectic groups and skew-hermitian unitary groups. Once such results are available, one can then formulate the refined Gan–Gross–Prasad conjecture in the context of skew-hermitian unitary groups.

To support our conjecture, in the second part of this paper, we show, under some hypothesis on the local and global Langlands correspondences which we will state in Section 5, that our conjecture is compatible with Ichino–Ikeda's conjecture in some cases. Thus (1.0.2) holds in some low rank cases when the Ichino–Ikeda's conjecture is known. We have the following cases.

(1) If $n = 1$ and $r = 0$, then (1.0.2) has been proved in [39, Theorem 4.5].
(2) If $r = 0$ and π_2 is a theta lift of some irreducible cuspidal tempered automorphic representation of $\mathrm{O}(2n)$, then (1.0.2) can be deduced from Ichino–Ikeda's conjecture for $\mathrm{SO}(2n+1) \times \mathrm{SO}(2n)$. In this case, if π_0 is not a theta lift from any $\mathrm{O}(2n+1)$, then both sides of (1.0.2) vanish.
(3) If $r = 1$ and π_2 is a theta lift of some irreducible cuspidal tempered automorphic representation of $\mathrm{O}(2n+2)$, then (1.0.2) can be deduced from Ichino–Ikeda's conjecture for $\mathrm{SO}(2n+2) \times \mathrm{SO}(2n+1)$. In this case, if π_0 is not a theta lift from $\mathrm{O}(2n+1)$, then both sides of (1.0.2) vanish. In particular, when $n = 1$, (1.0.2) holds for $\mathrm{Sp}(4) \times \mathrm{Mp}(2)$, if the automorphic representation on $\mathrm{Sp}(4)$ is a theta lift from $\mathrm{O}(4)$.

See Theorem 7.1.1 and 8.1.1 for the precise statments. See also Theorem 8.6.1 for an analogous statement in the case $r = -1$. In the course of proving these results, we derive a variant for the Ichino–Ikeda's conjecture for the full orthogonal group, c.f. Conjecture 6.3.1 and Proposition 6.3.3. I hope that this variant is of some independent interest. See [10] for the case of the triple product formula on $\mathrm{GO}(4)$.

Ichino informed the author that there is some minor inaccuracies in the original formulation of Ichino–Ikeda's conjecture [24, Conjecture 2.1] when the automorphic representation on the even orthogonal group appears with multiplicity two in the discrete automorphic spectrum. In this case, one needs to specify an automorphic realization. Moreover, the size of the centralizer of the Arthur parameter needs to be modified accordingly. We will take care of this modification in Section 6.

It is expected that our conjecture is compatible with the refined Gan–Gross–Prasad conjecture for $\mathrm{SO}(2n + 2r + 1) \times \mathrm{SO}(2n)$ proposed by Liu [36]. To keep this paper within a reasonable length, we postpone to check this more general compatibility in a future paper.

This paper is organized as follows. The first part of the paper consists of Sections 2, 3 and 4. In Section 2, we first define the Fourier–Jacobi periods and the local linear form α_v. Then we state the conjectural formula for the Fourier–Jacobi periods. In Section 3, we show that the local linear form α_v is well-defined, i.e. its defining integral is either absolutely convergent or can be regularized. We also prove a positivity result for α_v. In Section 4, we compute α_v when all the data involved are unramified. The argument is mostly adapted from [36]. The second part of this paper consists of Sections 5, 6, 7 and 8. In Section 5, we state some working hypotheses on the local and global Langlands correspondences and make some remarks on the theta correspondences. For orthogonal groups and symplectic groups, these hypotheses should follow from the work of Arthur [2]. For metaplectic groups, they should eventually follow from the on-going work of Wen-Wei Li (e.g. [34]). In section 6, we review the Ichino–Ikeda's conjecture and derive a variant of it for the full orthogonal group. In Section 7, we study the conjecture in the case $\mathrm{Mp}(2n) \times \mathrm{Sp}(2n)$ via a seesaw argument. This type of argument has also been used in [3, 11, 50]. In Section 8, we study the conjecture in the case $\mathrm{Sp}(2n + 2) \times \mathrm{Mp}(2n)$. For the convenience of the readers, we remark that Section 3, 4 and the second part of the paper are logically independent. Section 7 and 8 are also logically independent. They can be read in any order.

Notation and convention

The following notation will be used throughout this paper. Let F be a number field, $\mathfrak{o}_F$ the ring of integers and $\mathbb{A}_F$ the ring of adeles. For any finite place v, let $\mathfrak{o}_{F,v}$ be the ring of integers of F_v and ϖ_v a uniformizer. Let $q_v = |\mathfrak{o}_{F,v}/\varpi_v|$ be the number of elements in the residue field of

v. We fix a nontrivial additive character $\psi = \otimes\psi_v : F\backslash\mathbb{A}_F \to \mathbb{C}^\times$. We assume that ψ is unitary, thus $\psi^{-1} = \overline{\psi}$. For any $a \in F^\times$, we define an additive character ψ_a of $F\backslash\mathbb{A}_F$ by $\psi_a(x) = \psi(ax)$. For any place v of F, let $(\cdot,\cdot)_{F_v}$ be the Hilbert symbol of F_v and γ_{ψ_v} the Weil index, which is an eighth root of unity. Note that $\prod_v \gamma_{\psi_v} = 1$.

Suppose that V is a vector space and $v_1, \cdots, v_r \in V$. Then we denote by $\langle v_1, \cdots, v_r\rangle$ the subspace of V generated by $v_1, \cdots, v_r$. We write $\mathcal{S}(V)$ for the space of Schwartz functions on V.

Let (V, q_V) be a quadratic space of dimension n over F where V is the underlying vector space and q_V is the quadratic form. We can choose a basis of V so that its quadratic form is represented by a diagonal matrix with entries $a_1, \cdots, a_n$. We define the discriminant $\operatorname{disc} V$ of V by

$$\operatorname{disc} V = (-1)^{\frac{n(n-1)}{2}} a_1 \cdots a_n \in F^\times/F^{\times,2}.$$

Define a quadratic character $\chi_V : F^\times\backslash\mathbb{A}_F^\times \to \{\pm 1\}$ by $\chi_V(x) = (x, \operatorname{disc} V)_F$.

Let (W, q_W) be a symplectic space of dimension $2n$ over F where W is the underlying vector space and q_W is the symplectic form. Then we denote by $\mathrm{Sp}(W)$ or $\mathrm{Sp}(2n)$ the symplectic group attached to W and $\mathrm{Mp}(W)$ or $\mathrm{Mp}(2n)$ the metaplectic double cover. By definition, if v is a place of F, then $\mathrm{Mp}(W)(F_v) = \mathrm{Sp}(W)(F_v) \ltimes \{\pm 1\}$ and the multiplication is given by

$$(g_1, \epsilon_1)(g_2, \epsilon_2) = (g_1 g_2, \epsilon_1\epsilon_2 c(g_1, g_2)),$$

where $c(g_1, g_2)$ is some 2-cocycle on $\mathrm{Sp}(W)$ valued in $\{\pm 1\}$ [41]. Moreover

$$\mathrm{Mp}(W)(\mathbb{A}_F) = \prod_v{}' \mathrm{Mp}(W)(F_v)/\{(1, \epsilon_v)_v \mid \prod_v \epsilon_v = 1\}.$$

If $g \in \mathrm{Sp}(W)(\mathbb{A}_F)$ (resp. $\mathrm{Sp}(W)(F_v)$), then we define $\iota(g) = (g, 1) \in \mathrm{Mp}(W)(\mathbb{A}_F)$ (resp. $\mathrm{Mp}(F_v)$). Note that $g \mapsto \iota(g)$ is not a group homomorphism.

By a genuine function on $\mathrm{Mp}(W)(F_v)$, we mean a function on $\mathrm{Mp}(W)(F_v)$ which is not the pullback of a function on $\mathrm{Sp}(W)(F_v)$. We always identify a function on $\mathrm{Sp}(W)(F_v)$ with a non-genuine function on $\mathrm{Mp}(W)(F_v)$. Suppose that $f_1, \cdots, f_r$ are genuine functions on $\mathrm{Mp}(W)(F_v)$ and $h_1, \cdots h_s$ are functions on $\mathrm{Sp}(W)(F_v)$ such that the product $f_1 \cdots f_r$ is not genuine. Then we write

$$\int_{\mathrm{Sp}(W)(F_v)} f_1(g) \cdots f_r(g) h_1(g) \cdots h_s(g) \mathrm{d}g = \int_{\mathrm{Sp}(W)(F_v)} f_1(\iota(g)) \cdots f_r(\iota(g)) h_1(g) \cdots h_s(g) \mathrm{d}g.$$

An irreducible representation of $\mathrm{Mp}(W)(F_v)$ is said to be genuine if the element $(1, \epsilon)$ acts by ϵ. We always identify an irreducible representation of $\mathrm{Sp}(W)(F_v)$ with a non-genuine representation of $\mathrm{Mp}(W)(F_v)$. We make similar definitions for genuine functions and representations of $\mathrm{Mp}(W)(\mathbb{A}_F)$.

Suppose v is a non-archimedean place of F whose residue characteristic is not two. Let $B = TU$ is a Borel subgroup of $\mathrm{Sp}(2n)$ and $\widetilde{B} = \widetilde{T}U$ the inverse image of B in $\mathrm{Mp}(2n)(F_v)$. Then $\widetilde{T} \simeq (F_v^\times)^n \ltimes \{\pm 1\}$. We define a genuine character $\chi_\psi(t)$ of $\widetilde{T}$ by

$$\chi_{\psi_v}((t_1, \cdots, t_n), \epsilon) = \epsilon \gamma_{\psi_v} \gamma^{-1}_{\psi_{v, t_1 \cdots t_n}}.$$

Suppose that the conductor of ψ_v is $\mathfrak{o}_{F,v}$. By an unramified principal series representation of $\mathrm{Mp}(2n)(F_v)$, we mean the induced representation $I(\chi) = \mathrm{Ind}_{\widetilde{B}}^{\mathrm{Mp}(2n)(F_v)} \chi_{\psi_v}\chi$, where χ be a character of $T \simeq F_v^n$ defined by $\chi(t_1, \cdots, t_n) = |t_1|^{\alpha_1} \cdots |t_n|^{\alpha_n}$, $\alpha_1, \cdots, \alpha_n \in \mathbb{C}$. This convention of parabolic inductions of the metaplectic group is the one in [13]. If π_v is an unramified representation of $\mathrm{Mp}(2n)(F_v)$, then we can find an unramified character χ of T as above and $\pi_v \subset I(\chi)$. The complex numbers $(\alpha_1, \cdots, \alpha_n)$ are called the Satake parameters of π_v. Note that the Satake parameters of π_v depend also on ψ_v.

We write 1_r for the $r \times r$ identity matrix. We recursively define $\mathsf{w}_1 = \{1\}$ and $\mathsf{w}_r = \begin{pmatrix} & \mathsf{w}_{r-1} \\ 1 & \end{pmatrix}$. Suppose $a = (a_1, \cdots, a_r) \in (F^\times)^r$. We let $\mathrm{diag}[a_1, \cdots, a_r]$ be the diagonal matrix with diagonal entries $a_1, \cdots, a_r$.

Suppose that G is a unimodular locally compact topological group and $\mathrm{d}g$ a Haar measure. Suppose that π is a representation of G, realized on some space V. Let f be a continuous function on G. Then we put (whenever it makes sense, e.g. f is compactly supported and locally constant)

$$\pi(f)v = \int_G f(g)\pi(g).v \mathrm{d}g.$$

Let S be a finite set of places of F. We define a constant Δ_G^S as follows. If $G = \mathrm{Mp}(2n)$ or $\mathrm{Sp}(2n)$, we define $\Delta_G^S = \prod_{i=1}^n \zeta_F^S(2i)$. If $G = \mathrm{O}(V)$ or $\mathrm{SO}(V)$ when $n = \dim V \geq 3$, then we define

$$\Delta_G^S = \begin{cases} \zeta_F^S(2)\zeta_F^S(4) \cdots \zeta_F^S(n-1), & \text{if } n \text{ is odd} \\ \zeta_F^S(2)\zeta_F^S(4) \cdots \zeta_F^S(n-2) L^S(\frac{n}{2}, \chi_V), & \text{if } n \text{ is even,} \end{cases}$$

Suppose that v is a place F, then we define $\Delta_{G,v}$ in an analogous way, replacing the partial L-functions by the local Euler factors at v. In this case, if T is a split maximal torus in $\mathrm{Sp}(2n)$ and $\widetilde{T}$ is the inverse image of T in $\mathrm{Mp}(2n)$, then we define $\Delta_{\widetilde{T},v} = \Delta_{T,v} = (1 - q_v^{-1})^{-n}$.

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Part 1. Conjectures

2. Conjectures for the Fourier–Jacobi periods

2.1. **Global Fourier–Jacobi periods.** Let (W_2, q_2) be a $2m$ dimensional symplectic space over F. We choose a basis $\{e_m^*, \cdots, e_1^*, e_1, \cdots, e_m\}$ of W_2 so that $q_2(e_i^*, e_j) = \delta_{ij}$. For $1 \le i \le m$, let $R_i = \langle e_{m-i+1}, \cdots, e_m\rangle$ and $R_i^* = \langle e_m^*, \cdots, e_{m-i+1}^*\rangle$ be isotropic subspaces of W_2. Put $R_0 = R_0^* = \{0\}$. Let $0 \le r \le m$ be an integer and put $n = m - r$ and (W_0, q_0) the orthogonal complement of $R_r + R_r^*$. We define $(W_1, q_1) = W_0 + \langle e_{n+1}, e_{n+1}^*\rangle$. Let $G_i = \mathrm{Sp}(W_i)$ and $\widetilde{G_i} = \mathrm{Mp}(W_i)$.

Let $0 \le i \le n$ be an integer. Let P_i be the parabolic subgroup of G_2 stabilizing the flag

$$0 = R_0 \subset R_1 \subset \cdots \subset R_i,$$

with the Levi decomposition $P_i = M_i N_i$. Here and below in this article, the notation $P = MN$ signifies that M is the Levi subgroup and N is the unipotent radical of P. We denote by W^i the orthogonal complement of $R_i + R_i^*$ and $G^i = \mathrm{Sp}(W^i)$. Then $M_i = G^i \times \mathrm{GL}_1^i$. Let ψ_m be the character of N_m defined by

$$\psi_m(n) = \psi\left(\sum_{j=1}^{m-1} q_2(ne_{m-j+1}^*, e_{m-j}) + q_2(ne_1^*, e_1^*)\right).$$

Let ψ_i be the restriction of ψ_m to N_i.

Let $H = H(W_0)$ be the Heisenberg group attached to the symplectic space W_0. By definition, $H = W_0 \ltimes F$ and the group law is given by

$$(w_1, t_1)(w_2, t_2) = (w_1 + w_2, t_1 + t_2 + q_0(w_1, w_2)).$$

The group H embeds in G_2 as a subgroup of G_1 and $H = G_1 \cap N_r$, $N_r = N_{r-1}H$. Let $L = \langle e_1, \cdots, e_n\rangle$ and $L^* = \langle e_n^*, \cdots, e_1^*\rangle$. Then $W_0 = L + L^*$ is a complete polarization. We sometimes write an element $h \in H$ as $h(l + l^*, t)$ where $l \in L$, $l^* \in L^*$ and $t \in F$. Let v be a place of F and ω_{ψ_v} be the Weil representation of $H(F_v)$ which is realized on $\mathcal{S}(L^*(F_v))$. It is defined by

$$\omega_{\psi_v}(h(y + x, t))f(l^*) = \psi(t + q_2(2x + l^*, y))f(l^* + x),\ f \in \mathcal{S}(L^*(F_v)),\ l^*, x \in L^*(F_v), y \in L(F_v).$$

This is the unique irreducible infinite dimensional representation of $H(F_v)$ whose central character is ψ_v. It induces an action of $\widetilde{G_0}(F_v)$ on $\mathcal{S}(L^*(F_v))$. We denote the joint action of $H(F_v) \rtimes \widetilde{G_0}(F_v)$ on $\mathcal{S}(L^*(F_v))$ again by ω_{ψ_v}. We take the convention that if $W_0 = \{0\}$, then $\omega_{\psi_v} = \psi_v$.

Taking restricted tensor product of the Weil representations ω_{ψ_v}, we obtain a global Weil representation ω_ψ of $H(\mathbb{A}_F) \rtimes \widetilde{G_0}(\mathbb{A}_F)$ which is realized on $\mathcal{S}(L^*(\mathbb{A}_F))$. We define the theta series

$$\theta_\psi(hg_0, \phi) = \sum_{l^* \in L^*(F)} \omega_\psi(hg_0)\phi(l^*), \quad \phi \in \mathcal{S}(L^*(\mathbb{A}_F)), h \in H(\mathbb{A}_F), g_0 \in \widetilde{G_0}(\mathbb{A}_F).$$

We now talk about automorphic representations. There are two cases.

Case Mp: Let $\pi_2 = \otimes \pi_{2,v}$ be an irreducible cuspidal genuine automorphic representation of $\widetilde{G_2}(\mathbb{A}_F)$ and $\pi_0 = \otimes \pi_{0,v}$ be an irreducible cuspidal automorphic representation of $G_0(\mathbb{A}_F)$.

Case Sp: Let $\pi_2 = \otimes \pi_{2,v}$ be an irreducible cuspidal automorphic representation of $G_2(\mathbb{A}_F)$ and $\pi_0 = \otimes \pi_{0,v}$ be an irreducible cuspidal genuine automorphic representation of $\widetilde{G_0}(\mathbb{A}_F)$.

Let S be a sufficiently large finite set of places of F containing all archimedean places and finite places whose residue characteristic is two, such that $\pi_{2,v}$ and $\pi_{0,v}$ are both unramified and the conductor of ψ_v is $\mathfrak{o}_{F,v}$ if $v \notin S$. Let $(\alpha_{1,v}, \cdots, \alpha_{m,v})$ and $(\beta_{1,v}, \cdots, \beta_{n,v})$ be the Satake parameters of $\pi_{2,v}$ and $\pi_{0,v}$ respectively. Put

$$A_2 = \begin{cases} \mathrm{diag}[\alpha_{1,v}, \cdots, \alpha_{m,v}, \alpha_{m,v}^{-1}, \cdots, \alpha_{1,v}^{-1}] & \text{Case Mp}, \\ \mathrm{diag}[\alpha_{1,v}, \cdots, \alpha_{m,v}, 1, \alpha_{m,v}^{-1}, \cdots, \alpha_{1,v}^{-1}] & \text{Case Sp}, \end{cases}$$

and

$$A_0 = \begin{cases} \mathrm{diag}[\beta_{1,v}, \cdots, \beta_{n,v}, 1, \beta_{n,v}^{-1}, \cdots, \beta_{1,v}^{-1}] & \text{Case Mp}, \\ \mathrm{diag}[\beta_{1,v}, \cdots, \beta_{n,v}, \beta_{n,v}^{-1}, \cdots, \beta_{1,v}^{-1}] & \text{Case Sp}. \end{cases}$$

We then define the tensor product L-function

$$L_{\psi_v}(s, \pi_{2,v} \times \pi_{0,v}) = \det(1 - A_2 \otimes A_0 \cdot q_v^{-s})^{-1}, \quad L_\psi^S(s, \pi_2 \times \pi_0) = \prod_{v \notin S} L_{\psi_v}(s, \pi_{2,v} \times \pi_{0,v}).$$

The partial L-function is convergent for $\Re s \gg 0$. We denote by $L_{\psi_v}(s, \pi_{i,v}, \mathrm{Ad})$ and $L_\psi^S(s, \pi_i, \mathrm{Ad}) = \prod_{v \notin S} L_{\psi_v}(s, \pi_{i,v}, \mathrm{Ad})$ the (local and partial) adjoint L-functions of π_i. If π_i is an automorphic representation of the metaplectic group (resp. symplectic group), then they depend (resp. do not depend) on ψ. We include the subscript ψ in both cases to unify notation. We assume that these L-functions can be meromorphically continued to the whole complex plane.

Let $\varphi_2 \in \pi_2$, $\varphi_0 \in \pi_0$ and $\phi \in \mathcal{S}(L^*(\mathbb{A}_F))$. Define

$$\begin{aligned} &\mathcal{FJ}_\psi(\varphi_2, \varphi_0, \phi) \\ &= \int_{G_0(F)\backslash G_0(\mathbb{A}_F)} \int_{H(F)\backslash H(\mathbb{A}_F)} \int_{N_{r-1}(F)\backslash N_{r-1}(\mathbb{A}_F)} \varphi_2(uhg_0)\varphi_0(g_0)\overline{\psi_{r-1}(u)\theta_\psi(hg_0, \phi)}\mathrm{d}u\mathrm{d}h\mathrm{d}g_0. \end{aligned}$$

The measures $\mathrm{d}u$ and $\mathrm{d}h$ are the self-dual measures on $N_{r-1}(\mathbb{A}_F)$ and $H(\mathbb{A}_F)$ respectively. The measure $\mathrm{d}g_0$ is the Tamagawa measures on $G_0(\mathbb{A}_F)$.

2.2. **Local Fourier–Jacobi periods.** We fix a Haar measure $\mathrm{d}g_{0,v}$ on $G_0(F_v)$ for each v such that the volume of $G_0(\mathfrak{o}_v)$ equals one for almost all v. Then there is a constant C_0 such that $\mathrm{d}g_0 = C_0 \prod_v \mathrm{d}g_{0,v}$. Following [24], we call C_0 the measure constant.

Let $\mathcal{B}_{\pi_i}$ $(i = 0, 2)$ be the canonical bilinear pairing between π_i and $\pi_i^\vee$ defined by

$$\mathcal{B}_{\pi_2}(\varphi, \varphi^\vee) = \int_{G_2(F)\backslash G_2(\mathbb{A}_F)} \varphi(g)\varphi^\vee(g)\mathrm{d}g, \quad \varphi \in \pi_i,\ \varphi^\vee \in \pi_i^\vee.$$

We fix a bilinear pairing $\mathcal{B}_{\pi_{i,v}}$ between $\pi_{i,v}$ and $\pi_{i,v}^\vee$ for each place v such that $\mathcal{B}_{\pi_i} = \prod_v \mathcal{B}_{\pi_{i,v}}$. Put $\Phi_{\varphi_{i,v},\varphi_{i,v}^\vee}(g) = \mathcal{B}_{\pi_{i,v}}(\pi_{i,v}(g)\varphi_{i,v}, \varphi_{i,v}^\vee)$ if $\varphi_{i,v} \in \pi_{i,v}$ and $\varphi_{i,v}^\vee \in \pi_{i,v}^\vee$.

The contragredient representation of ω_ψ is $\omega_{\psi^{-1}}$ (again realized on $\mathcal{S}(L^*(\mathbb{A}_F))$) and there is a canonical pairing between ω_ψ and $\omega_{\psi^{-1}}$ given by

$$\mathcal{B}_{\omega_\psi}(\phi, \phi^\vee) = \int_{L^*(\mathbb{A}_F)} \phi(l^*)\phi^\vee(l^*)\mathrm{d}l^*, \quad \phi, \phi^\vee \in \mathcal{S}(L^*(\mathbb{A}_F)),$$

where the measure $\mathrm{d}l^*$ is the self-dual measure on $L^*(\mathbb{A}_F)$. Similarly, for any place v, there is a canonical pairing between ω_{ψ_v} and $\omega_{\psi_v^{-1}}$ given by

$$\mathcal{B}_{\omega_{\psi_v}}(\phi_v, \phi_v^\vee) = \int_{L^*(F_v)} \phi_v(l^*)\phi_v^\vee(l^*)\mathrm{d}l^*, \quad \phi_v, \phi_v^\vee \in \mathcal{S}(L^*(F_v)),$$

where the measure $\mathrm{d}l^*$ is the self-dual measure on $L^*(F_v)$. Then $\mathcal{B}_{\omega_\psi} = \prod_v \mathcal{B}_{\omega_{\psi_v}}$. Put $\Phi_{\phi_v,\phi_v^\vee}(g) = \mathcal{B}_{\omega_{\psi_v}}(\omega_{\psi_v}(g)\phi_v, \phi_v^\vee)$.

We now fix a place v of F. Recall that the group P_m of G_2 is a minimal parabolic subgroup which is contained in P_{r-1}. For any real number γ or $\gamma = -\infty$, define

$$N_{m,\gamma} = \{u \in N_m(F_v) \mid |q_2(ue_1^*, e_1^*)| \le e^\gamma, |q_2(ue_{i+1}^*, e_i)| \le e^\gamma,\ 1 \le i \le m-1\}.$$

For any $\gamma \ge -\infty$, we define $N_{i,\gamma} = N_i(F_v) \cap N_{m,\gamma}$. Define

$$\mathcal{F}_{\psi_v}\Phi_{\varphi_{2,v},\varphi_{2,v}^\vee}(hg_0) = \lim_{\gamma\to\infty} \int_{N_{r-1,\gamma}(F_v)} \Phi_{\varphi_{2,v},\varphi_{2,v}^\vee}(hg_0u)\overline{\psi_{r-1,v}(u)}\mathrm{d}u, \quad \varphi_{2,v} \in \pi_{2,v}, \varphi_{2,v}^\vee \in \pi_{2,v}^\vee$$

where $h \in H(F_v)$ and $g_0 \in G_0(F_v)$ in the case Sp (resp. $g_0 \in \widetilde{G_0}(F_v)$ in the case Mp). Define

$$\alpha_v(\varphi_{2,v}, \varphi_{2,v}^\vee, \varphi_{0,v}, \varphi_{0,v}^\vee, \phi_v, \phi_v^\vee) = \int_{G_0(F_v)} \int_{H(F_v)} \mathcal{F}_\psi\Phi_{\varphi_{2,v},\varphi_{2,v}^\vee}(hg_0)\Phi_{\varphi_{0,v},\varphi_{0,v}^\vee}(g_0)\Phi_{\phi_v,\phi_v^\vee}(hg_0)\mathrm{d}h\mathrm{d}g_0,$$

for $\varphi_{i,v} \in \pi_{i,v}, \varphi_{i,v}^\vee \in \pi_{i,v}^\vee,\ \phi_v, \phi_v^\vee \in \mathcal{S}(L^*(F_v))$. If $r \le 1$, then it is to be understood that $\mathcal{F}_\psi\Phi_{\varphi_{2,v},\varphi_{2,v}^\vee} = \Phi_{\varphi_{2,v},\varphi_{2,v}^\vee}$. Moreover, if $r = 0$, then it is to be understood that the integral over $H(F_v)$ is void.

Proposition 2.2.1. *Assume that $\pi_{2,v}$ and $\pi_{0,v}$ are both tempered. Then the limit in the definition of $\mathcal{F}_{\psi_v}\Phi_{\varphi_{2,v},\varphi_{2,v}^\vee}$ exists. Moreover, the defining integral of α_v is absolutely convergent.*

If $\pi_{i,v}$ is unitary, then we may identify $\pi_{i,v}^\vee$ with $\overline{\pi_{i,v}}$. We then define

$$\alpha_v(\varphi_{2,v},\varphi_{0,v},\phi_v)=\alpha_v(\varphi_{2,v},\overline{\varphi_{2,v}},\varphi_{0,v},\overline{\varphi_{0,v}},\phi_v,\overline{\phi_v}).$$

Proposition 2.2.2. *Assume that $\pi_{2,v}$ and $\pi_{0,v}$ are unitary and tempered. Then $\alpha_v(\varphi_{2,v},\varphi_{0,v},\phi_v)\geq 0$ for all smooth vectors $\varphi_{2,v}\in\pi_{2,v}$, $\varphi_{0,v}\in\pi_{0,v}$ and $\phi_v\in\mathcal{S}(L^*(F_v))$.*

These two propositions will be proved in Section 3.

We now consider the unramified situation. Note first that the symplectic spaces W_i's, the isotropic subspaces R_i's and hence the groups G_i's are naturally defined over $\mathfrak{o}_F$. Let S be a sufficiently large finite set of places of F containing all archimedean places and finite places whose residue characteristic is two, such that if $v\notin S$, then the following conditions hold.

(1) The conductor of ψ_v is $\mathfrak{o}_{F,v}$.
(2) $\phi_v=\phi_v^\vee=\mathbf{1}_{L^*(\mathfrak{o}_{F,v})}$
(3) For $i=0,2$, $\varphi_{i,v}$ and $\varphi_{i,v}^\vee$ are fixed by $G_i(\mathfrak{o}_{F,v})$ and satisfy $\mathcal{B}_{\pi_{i,v}}(\varphi_{i,v},\varphi_{i,v}^\vee)=1$. In particular, the representations $\pi_{i,v}$ and $\pi_{i,v}^\vee$ are unramified.
(4) $\int_{G_0(\mathfrak{o}_{F,v})}\mathrm{d}g_{0,v}=1$.

Proposition 2.2.3. *If $v\notin S$ and the defining integral of α_v is convergent, then*

$$\alpha_v(\varphi_{2,v},\varphi_{2,v}^\vee,\varphi_{0,v},\varphi_{0,v}^\vee,\phi_v,\phi_v^\vee)=\Delta_{G_{2,v}}\frac{L_{\psi_v}(\frac12,\pi_{2,v}\times\pi_{0,v})}{L_{\psi_v}(1,\pi_{0,v},\mathrm{Ad})L_{\psi_v}(1,\pi_{2,v},\mathrm{Ad})}.$$

We will prove this proposition in Section 4. Note that in this proposition, we do not assume that the representations $\pi_{2,v}$ and $\pi_{0,v}$ are tempered.

2.3. **Conjectures.** Following [24] and [36], we say that the representations π_2 and π_0 are almost locally generic if for almost all places v of F, the local components $\pi_{2,v}$ and $\pi_{0,v}$ are generic. Suppose that we are in the case of Mp. As explained in [24], the automorphic representations π_2 and π_0 should come from some elliptic Arthur parameters

$$\Psi_2: L_F\times\mathrm{SL}_2(\mathbb{C})\to\widehat{\widetilde{G_2}}=\mathrm{Sp}(2m,\mathbb{C}),\quad \Psi_0: L_F\times\mathrm{SL}_2(\mathbb{C})\to\widehat{G_0}=\mathrm{SO}(2n+1,\mathbb{C})$$

where L_F is the (hypothetical) Langlands group of F. If π_i is tempered, then Ψ_i is trivial on $\mathrm{SL}_2(\mathbb{C})$. It is believed (Ramanujan conjecture) that almost locally generic representations are tempered. We define S_{π_2} (resp. S_{π_0}) to be the centralizer of the image of Ψ_2 in $\widehat{\widetilde{G_2}}$ (resp. $\widehat{G_0}$). They are finite abelian 2-groups. In the case Sp, we have the same discussion, except that we replace $\widetilde{G_2}$ by G_2 and replace G_0 by $\widetilde{G_0}$.

Conjecture 2.3.1. *Assume that π_2 and π_0 are irreducible cuspidal automorphic representations that are almost locally generic. Then the following statements hold.*

(1) *The defining integral of* $\alpha_v(\varphi_{2,v}, \varphi_{2,v}^\vee, \varphi_{0,v}, \varphi_{0,v}^\vee, \phi_v, \phi_v^\vee)$ *is convergent for any* K_i*-finite vectors* $\varphi_{i,v}, \varphi_{i,v}^\vee$ *and* K_0*-finite Schwartz functions* $\phi_v, \phi_v^\vee$*, where* K_i *is a maximal compact subgroup of* $G_i(F_v)$*,* $i = 0, 2$*;*

(2) $\alpha_v(\varphi_{2,v}, \varphi_{0,v}, \phi_v) \geq 0$ *for any* K_i*-finite vectors* $\varphi_{i,v}$ *and* K_0*-finite Schwartz function* ϕ_v*. Moreover,* $\alpha_v(\varphi_{2,v}, \varphi_{0,v}, \phi_v) = 0$ *for all* K_i*-finite* $\varphi_{i,v}$ *and* K_0*-finite* ϕ_v *precisely when*

$$\mathrm{Hom}_{N_{r-1}(F_v)\rtimes(H(F_v)\rtimes G_0(F_v))}(\pi_{2,v} \otimes \pi_{0,v} \otimes \overline{\psi_{r-1,v} \otimes \omega_{\psi_v}}, \mathbb{C}) = 0;$$

(3) *Assume that* $\varphi_i = \otimes_v \varphi_{i,v} \in \pi_i$ $(i = 0, 2)$ *and* $\phi = \otimes_v \phi_v \in \mathcal{S}(L^*(\mathbb{A}_F))$ *are factorizable, then*

(2.3.1)

$$|\mathcal{FJ}_\psi(\varphi_2, \varphi_0, \phi)|^2 = \frac{C_0 \Delta_{G_2}^S}{|S_{\pi_2}||S_{\pi_0}|} \frac{L_\psi^S(s, \pi_2 \times \pi_0)}{L_\psi^S(s + \frac{1}{2}, \pi_2, \mathrm{Ad}) L_\psi^S(s + \frac{1}{2}, \pi_0, \mathrm{Ad})}\Big|_{s=\frac{1}{2}} \times \prod_{v \in S} \alpha_v(\varphi_{2,v}, \varphi_{0,v}, \phi_v).$$

Remark 2.3.2. It follows from the Proposition 2.2.3 that the right hand side of (2.3.1) does not depend on the finite set S.

Remark 2.3.3. Assume that π_2 and π_0 are both tempered. It is then believed that $L_\psi^S(s, \pi_2 \times \pi_0)$ and $L_\psi^S(s, \pi_i, \mathrm{Ad})$ should be holomorphic for $\Re s > 0$. Moreover, $L_\psi^S(1, \pi_i, \mathrm{Ad}) \neq 0$.

Remark 2.3.4. Without the assumption of almost local genericity of π_2 and π_0, we expect that local linear forms α_v can be "analytically continued" in some way so that it is defined for all representations $\pi_{2,v}$ and $\pi_{0,v}$. This is indeed the case if $v \notin S$. Thus α_v is well-defined for all v if π_2 and π_0 satisfy the property that $\pi_{2,v}$ and $\pi_{0,v}$ are both tempered if $v \in S$. Moreover, we expect that the identity 2.3.1 holds with the quantity $|S_{\pi_2}||S_{\pi_0}|$ replaced by some $2^{-\beta}$ where β is an integer. The nature of β, however, remains mysterious at this moment.

We end this section by writing Conjecture 2.3.1 (3) in an equivalent form which does not involve the finite set S. We may define the completed L-functions

$$L_\psi(s, \pi_2 \times \pi_0), \quad L_\psi(s, \pi_i, \mathrm{Ad}),\ i = 0, 2.$$

The actual definition of the local Euler factor of these L-functions is not essential to us since Conjecture 2.3.1 does not depend on the definition of these Euler factors. Put

$$\mathcal{L} = \Delta_{G_2} \frac{L_\psi(s, \pi_2 \times \pi_0)}{L_\psi(s + \frac{1}{2}, \pi_2, \mathrm{Ad}) L_\psi(s + \frac{1}{2}, \pi_0, \mathrm{Ad})}\Big|_{s=\frac{1}{2}}$$

and let $\mathcal{L}_v$ be its local Euler factor evaluated at $s = \frac{1}{2}$ at the place v. Define

$$\alpha_v^\natural = \mathcal{L}_v^{-1} \alpha_v.$$

Then the identity (2.3.1) can be rewritten as

(2.3.2)
$$\mathcal{FJ}_\psi \cdot \overline{\mathcal{FJ}_\psi} = \frac{C_0}{|S_{\pi_2}||S_{\pi_0}|} \mathcal{L} \cdot \prod_v \alpha_v^\natural.$$

The product is convergent since there are only finitely many terms which do not equal to 1. This is an equality of elements in

$$\mathrm{Hom}(\pi_2 \otimes \pi_0 \otimes \overline{\psi_{r-1} \otimes \omega_\psi}, \mathbb{C}) \otimes \overline{\mathrm{Hom}(\pi_2 \otimes \pi_0 \otimes \overline{\psi_{r-1} \otimes \omega_\psi}, \mathbb{C})}.$$

Note that by [37, 45, 46], this space is at most one dimensional. So we know *a priori* that there is a constant C such that

$$\mathcal{FJ}_\psi \cdot \overline{\mathcal{FJ}_\psi} = C \cdot \prod_v \alpha_v^\natural.$$

The point of Conjecture 2.3.1 is thus to compute the constant C.

3. Convergence and positivity

For the rest of Part I of this paper, we fix a place v of F and suppress it from all notation. Thus F is a local field of characterstic zero. To shorten notation, for any algebraic group G or $G = \mathrm{Mp}(2n)$ over F, we denote by G instead of $G(F)$ for its group of F-points. We have fixed a basis $\{e_m^*, \cdots, e_1^*, e_1, \cdots, e_m\}$ of W_2. We thus realize the group G_2 and its various subgroups as groups of matrices. We also identify W_i, L, L^* as spaces of row vectors. We put $K_i = G_i(\mathfrak{o}_F)$. This is a maximal compact subgroup of G_i. The group P_m consists of upper triangular matrices. The group $P_m \cap G_i$ is a minimal parabolic subgroup of G_i.

Suppose $a = (a_1, \cdots, a_n) \in (F^\times)^n$. Then we let $d(a) \in G_0$ so that $d(a)e_i^* = a_i e_i^*$ for any $1 \leq i \leq n$. We also put $\underline{a} = \mathrm{diag}[a_n, \cdots, a_1] \in \mathrm{GL}_n$.

3.1. Preliminaries.

We recall some basic estimates in this subsection. We follow [24, Section 4] rather closely.

Let G be a reductive group over F. Let A_G be a maximal split subtorus of G, M_0 the centralizer of A_G in G. We fix a minimal parabolic subgroup P_0 of G with the Levi decomposition $P_0 = M_0 N_0$. Let Δ be the set of simple roots of (P_0, A_G). Let δ_{P_0} be the modulus character of P_0. Let

$$A_G^+ = \{a \in A_0 \mid |\alpha(a)| \leq 1 \text{ for all } \alpha \in \Delta\}.$$

We fix a special maximal compact subgroup K of G. Then we have a Cartan decomposition $G = K A_G^+ K$. We also have the Iwasawa decomposition

$$G = M_0 N_0 K, \quad g = m_0(g) n_0(g) k_0(g).$$

Let f and f' be two nonnegative functions on G. We say that $f \ll f'$ if there is a constant C such that $f(g) \leq C f'(g)$ for all $g \in G$. We say that $f \sim f'$ if $f \ll f'$ and $f' \ll f$. In this case we say that f and f' are equivalent.

For any function $f \in L^1(G)$,

$$\int_G f(g) \mathrm{d}g = \int_{A_G^+} \nu(m) \iint_{K \times K} f(k_1 m k_2) \mathrm{d}k_1 \mathrm{d}k_2 \mathrm{d}m, \tag{3.1.1}$$

where $\nu(m)$ is a positive function on A_G^+ such that

$$\nu(m) \sim \delta_{P_0}(m)^{-1}. \tag{3.1.2}$$

Let $\mathbf{1}$ be the trivial representation of M_0 and let $e(g) = \delta_{P_0}(m_0(g))^{\frac{1}{2}}$ be an element in $\operatorname{Ind}_{P_0}^G \mathbf{1}$. Let $\mathrm{d}k$ be the measure on K such that $\operatorname{vol} K = 1$. We define the Harish–Chandra function

$$\Xi(g) = \int_K e(kg)\mathrm{d}k = \int_K \delta_{P_0}(m_0(kg))^{\frac{1}{2}}\mathrm{d}k.$$

This function is bi-K-invariant. This function depends on the choice of K. However, different choices of K give equivalent functions on G. So this choice will not affect our estimates.

We define a height function on G. We fix an embedding $\tau : G \to \mathrm{GL}_n$. Write $\tau(g) = (a_{ij})$ and $\tau(g^{-1}) = (b_{ij})$. Define

$$\varsigma(g) = \sup\{1, \log|a_{ij}|, \log|b_{ij}| \mid 1 \le i, j \le n\}. \tag{3.1.3}$$

There is a positive real number d such that

$$\delta_0(a)^{\frac{1}{2}} \ll \Xi(a) \ll \delta_0(a)^{\frac{1}{2}}\varsigma(a)^d, \quad a \in A_0^+. \tag{3.1.4}$$

Now let π be an irreducible admissible tempered representation of G. Let Φ be a smooth matrix coefficient of G. Then there is a constant B such that

$$|\Phi(g)| \ll \Xi(g)\varsigma(g)^B. \tag{3.1.5}$$

This is classical and is called the weak inequality when Φ is K-finite and due to [44] when Φ is smooth.

We finally assume that $G = \mathrm{Mp}(2n)$. This is not an algebraic group, but it behaves in many ways like an algebraic group. In particular, we have a Cartan decomposition for G, i.e. $G = KA_G^+K$ where K is the inverse image of a special maximal compact subgroup of $\mathrm{Sp}(2n)$ (e.g. $\mathrm{Sp}(2n)(\mathfrak{o}_F)$ if F is nonarchimedean and $\mathrm{U}(n)$ is F is archimedean) and A_G^+ is the inverse image of $A^+_{\mathrm{Sp}(2n)}$ in G. We define $\Xi_G = \Xi_{\mathrm{Sp}(2n)} \circ p$ where $p : G \to \mathrm{Sp}(2n)$ is the canonical projection. Then the weak inequality holds for tempered representations of G.

3.2. Some estimates.

Lemma 3.2.1. *There is a $d > 0$, such that*

$$\int_{N_{i+1}\cap G^i} \Xi_{G^i}(um)\varsigma(u)^{-d}\mathrm{d}u$$

is absolutely convergent for all $m \in G_0$. Moreover, in this case, there is an $\beta > 0$ so that

$$\int_{N_{i+1}\cap G^i} \Xi_{G^i}(um)\varsigma(u)^{-d}\mathrm{d}u \ll \Xi_{G^{i+1}}(m)\varsigma(m)^{\beta}, \quad m \in G_0.$$

Proof. In the archimedean case, this is [17, § 10, Lemma 2]. In the non-archimedean case, this is [43, Theorem 4.3.20]. □

Lemma 3.2.2. *There is some constant $c > 0$ so that*

$$\Xi_{G^i}(gg') \ll \Xi_{G^i}(g)e^{c\varsigma(g')}.$$

In particular, if $g = 1$, then we have

$$\Xi(g') \gg e^{-c\varsigma(g')}.$$

Proof. This can be proved by mimicking the argument in [48, Section 3.3] and [36, Lemma 3.11]. □

Lemma 3.2.3. *Fix a real number D. Then there exists some $\beta > 0$, such that*

$$\int_{N_{i+1,\gamma}\cap G^i} \Xi_{G^i}(um)\varsigma(u)^D \mathrm{d}u \ll \gamma^\beta \varsigma(m)^\beta \Xi_{G^{i+1}}(m), \quad m \in G^{i+1}.$$

Proof. We fix some real number b to be determined later. We denote the left hand side of the inequality by I. Then, $I = I_{<b} + I_{\geq b}$ with

$$\begin{aligned} I_{<b} &= \int_{N_{i+1,\gamma}\cap G^i} \mathbf{1}_{\varsigma<b}(u)\Xi_{G^i}(um)\varsigma(u)^D \mathrm{d}u \\ I_{\geq b} &= \int_{N_{i+1,\gamma}\cap G^i} \mathbf{1}_{\varsigma\geq b}(u)\Xi_{G^i}(um)\varsigma(u)^D \mathrm{d}u, \end{aligned}$$

where $\mathbf{1}_{\varsigma<b}$ is the characteristic function of $\{u \in N_{i+1} \cap G^i \mid \varsigma(u) < b\}$ and $\mathbf{1}_{\varsigma\geq b}$ is the characteristic function of $\{u \in N_{i+1} \cap G^i \mid \varsigma(u) \geq b\}$.

By Lemma 3.2.1, we have

$$\begin{aligned} I_{<b} &\ll b^d \int_{N_{i+1,\gamma}\cap G^i} \mathbf{1}_{\varsigma<b}(u)\Xi_{G^i}(um)\varsigma(u)^{D-d} \mathrm{d}u \\ &\ll b^d \varsigma(m)^{\beta_1} \Xi_{G^{i+1}}(m), \end{aligned}$$

where β_1 is a positive real number and d is a positive real number so that the integral

$$\int_{N_{i+1}\cap G^i} \Xi_{G^i}(um)\varsigma(u)^{D-d} \mathrm{d}u$$

is convergent.

Let $\lambda : N_{i+1} \cap G^i \to F$ be a character defined by $\lambda(n) = q_2(ne^*_{m-i}, e_{m-i-1})$. Then by [5, Corollary B.3.1], there is an $\epsilon > 0$, such that the integral

$$\int_{N_{i+1}\cap G^i} \Xi_{G^i}(u)e^{\epsilon\varsigma(u)}\varsigma(u)^D(1 + |\lambda(u)|)^{-1} \mathrm{d}u$$

is convergent. We have $\Xi_{G^i}(um) \ll e^{\alpha\varsigma(m)}\Xi_{G^i}(u)$ for some $\alpha > 0$, c.f. Lemma 3.2.2. It follows that

$$
\begin{aligned}
I_{\geq b} &\ll e^{\alpha\varsigma(m)} \int_{N_{i+1,\gamma}\cap G^i} \mathbf{1}_{\varsigma\geq b}(u)\Xi_{G^i}(u)\varsigma(u)^D e^{\epsilon\varsigma(u)}(1+|\lambda(u)|)^{-1}e^{-\epsilon\varsigma(u)}(1+|\lambda(u)|)\mathrm{d}u \\
&\ll e^{\alpha\varsigma(m)-\epsilon b}(1+e^\gamma)\int_{N_{i+1,\gamma}\cap G^i} \mathbf{1}_{\varsigma\geq b}(u)\Xi_{G^i}(u)\varsigma(u)^D e^{\epsilon\varsigma(u)}(1+|\lambda(u)|)^{-1}\mathrm{d}u \\
&\ll e^{\alpha\varsigma(m)-\epsilon b}(1+e^\gamma)\int_{N_{i+1}\cap G^i} \Xi_{G^i}(u)\varsigma(u)^D e^{\epsilon\varsigma(u)}(1+|\lambda(u)|)^{-1}\mathrm{d}u \\
&\ll e^{\alpha\varsigma(m)-\epsilon b}(1+e^\gamma).
\end{aligned}
$$

There is a constant $c > 0$, such that $\Xi_{G^{i+1}}(m) \gg e^{-c\varsigma(m)}$, then we have

$$
I \ll b^d\Xi_{G^{i+1}}(m)\varsigma(m)^{\beta_1} + e^{(\alpha+c)\varsigma(m)-\epsilon b}(1+e^\gamma)\Xi_{G^{i+1}}(m).
$$

We may thus choose $b = \epsilon^{-1}(\log(1+e^\gamma) + (\alpha+c)\varsigma(m))$ and get

$$
I \ll (\epsilon^{-d}\varsigma(m)^{\beta_1}(\gamma + (\alpha+c)\varsigma(m))^d + 1)\Xi_{G^{i+1}}(m).
$$

Note that α, β_1, d and c are constants which are independent of γ or m. We therefore conclude that there is some $\beta > 0$, such that

$$
I \ll \gamma^\beta\varsigma(m)^\beta\Xi_{G^{i+1}}(m).
$$

This proves the lemma. ☐

Lemma 3.2.4. *Fix a real number D. Then there is some $\beta > 0$ such that*

$$
\int_{N_{i+1,-\infty}\cap G^i} \Xi_{G^i}(um)\varsigma(u)^D\mathrm{d}u \ll \varsigma(m)^\beta\Xi_{G^{i+1}}(m), \quad m \in G^{i+1}.
$$

Proof. Choose a subgroup $N^\dagger$ of $N_{i+1}\cap G^i$ so that the multiplication map $N^\dagger \times (N_{i+1,-\infty}\cap G^i) \to N_{i+1}\cap G^i$ is an isomorphism. Recall that Ξ_{G^i} is itself a matrix coefficient of a (unitary) tempered representation which we temporarily denote by e. Thus $\Xi_{G^i}(g) = \langle e(g)v, v^\vee\rangle$ where $\langle -,-\rangle$ is the inner product on e and $v, v^\vee \in e$. It follows from the Dixmier–Milliavin theorem [7] that $v^\vee$ is a finite linear combination of the elements of the form

$$
\int_{N_\dagger} f(n)e(n^{-1})v'^\vee\mathrm{d}n,
$$

where $f \in \mathcal{C}_c^\infty(N_\dagger)$. Thus Ξ_{G^i} is a finite linear combination of the functions of the form

$$
g \mapsto \int_{N^\dagger} f(n)\Phi(ng)\mathrm{d}n,
$$

where $f(n)$ is a compactly supported function on $N^\dagger$ and Φ is a smooth matrix coefficient of a tempered representation of G^i, namely e. The lemma then follows from Lemma 3.2.3. ☐

Lemma 3.2.5. *Let f be a nonnegative function on L^* such that $P(x)f(x)$ is bounded for any polynomial function $P(x)$ on L^* (e.g. f is compactly supported). Let $p : H \rtimes G_0 \to L^*$ be the projection given by*

$$hg_0 \mapsto \sum_{i=1}^{n} q_2(hg_0 e_{n+1}^*, e_i) e_i^*.$$

Then there is a real number B such that

$$\int_H \Xi_{G_1}(hg_0) f(p(hg_0)) \mathrm{d}h \ll \Xi_{G_0}(g_0) \varsigma(g_0)^B, \quad g_0 \in G_0.$$

Proof. By the Cartan decomposition of G_0, we may assume that $g = d(a)$ where $a = (a_1, \cdots, a_n) \in (F^\times)^n$, $|a_n| \le \cdots \le |a_1| \le 1$. Then $p(h(l + l^*, t)d(a)) = l^* \underline{a}$ where $\underline{a} = \mathrm{diag}[a_n, \cdots, a_1]$.

We fix some γ which will be detemined later. Let $H_\gamma = H \cap N_{m,\gamma}$ and H^γ be the complement of H_γ in H. Then

$$\int_H \Xi_{G_1}(hd(a)) f(l^* \underline{a}) \mathrm{d}h = \int_{H_\gamma} \Xi_{G_1}(hd(a)) f(l^* \underline{a}) \mathrm{d}h + \int_{H^\gamma} \Xi_{G_1}(hd(a)) f(l^* \underline{a}) \mathrm{d}h.$$

By Lemma 3.2.3, the first integral is bounded by

$$\gamma^\beta \Xi_{G_0}(d(a)) \varsigma(d(a))^B.$$

Write $l^* = (l_1^*, \cdots, l_n^*)$ and $l'^* = (0, l_2^*, \cdots, l_n^*) \in F^n$. Then

$$\int_{H^\gamma} \Xi_{G_1}(hg_0) f(l^* \underline{a}) \mathrm{d}h = \int_{H^\gamma} \Xi_{G_1}(h(l + l'^*, t) d(a) h(l_1^* a_n, 0, \cdots, 0)) f(l^* \underline{a}) \mathrm{d}h.$$

There is some positive constant α such that

$$\Xi_{G_1}(h(l + l'^*, t) d(a) h(l_1^* a_n, 0)) \ll \Xi_{G_1}(h(l + l'^*, t) d(a)) e^{\alpha \log \max\{|l_1^* a_n|, 1\}}.$$

Therefore

$$\int_{H^\gamma} \Xi_{G_1}(hd(a)) f(l^* \underline{a}) \mathrm{d}h \ll \int_{H_{-\infty}} \Xi_{G_1}(hd(a)) \mathrm{d}h \times \int_{|l_1^* a_n| \ge e^\gamma} e^{\alpha \log \max\{|l_1^* a_n|, 1\}} f_1(l_1^* a_n) \mathrm{d}l_1^*,$$

where f_1 is a function on F such that $f_1(x)P(x)$ is bounded for any polynomial function P on F. It follows from Lemma 3.2.4 that there is a positive real number D such that

$$\int_{H_{-\infty}} \Xi_{G_1}(hd(a)) \mathrm{d}h \ll \Xi_{G_0}(d(a)) \varsigma(d(a))^D.$$

Since $f_1(x)P(x)$ is bounded for any polynomial function P on F, we have

$$\int_{|l_1^* a_n| \ge e^\gamma} e^{\alpha \log \max\{|l_1^* a_n|, 1\}} f_1(l_1^* a_n) \mathrm{d}l_1^* \ll |a_n|^{-1} e^{-\gamma},$$

where the implicit constant in $\ll$ does not depend on a_n or γ. We may choose γ with $\gamma > -\log|a_n|$. Then

$$\int_{|l_1^* a_n| \ge e^\gamma} e^{\alpha \log \max\{|l_1^* a_n|, 1\}} f_1(l_1^* a_n) \mathrm{d}l_1^* \ll 1.$$

Therefore

$$\int_{H^\gamma} \Xi_{G_1}(hd(a))f(l\underline{a})\mathrm{d}h \ll \Xi_{G_0}(d(a))\varsigma(d(a))^D.$$

The desired estimate then follows. □

Lemma 3.2.6. *Let Φ be a smooth matrix coefficient of a tempered representation π of G_2. Then the limit*

$$\lim_{\gamma\to\infty}\int_{N_{r-1,\gamma}} \Phi(ng)\overline{\psi_{r-1}(n)}\mathrm{d}n, \quad g\in G_2$$

exists and defines a continuous function in ψ_{r-1} (for a fixed g). If F is non-archimedean, then the integral is in fact a constant for sufficiently large γ. Moreover if $g\in G_1$, then

$$\left|\lim_{\gamma\to\infty}\int_{N_{r-1,\gamma}} \Phi(ng)\overline{\psi_{r-1}(n)}\mathrm{d}n\right| \ll \Xi_{G_1}(g)\varsigma(g)^D.$$

Proof. First recall that N_{r-1} is the unipotent subgroup of some parabolic subgroup P_{r-1} of G_2, the Levi part being isomorphic to $G_1\times \mathrm{GL}_1^{r-1}$. Put $T=\mathrm{GL}_1^{r-1}$ and denote an element in T by $a=(a_1,\cdots,a_{r-1})$ where $a_i\in F^\times$.

If F is non-archimedean, the constancy of the integral when γ is large can be proved in the same way as [48, Lemma 3.5]. In fact, suppose that $\Phi(g)=\langle \pi(g)v,v^\vee\rangle$ where $v\in\pi$, $v^\vee\in\pi^\vee$ and $\langle -,-\rangle$ stands for the pairing between π and its contragradient $\pi^\vee$. Suppose that K' is an open compact subgroup of G_2 such that v and $v^\vee$ are fixed by K'. Let $K''=K'\cap gK'g^{-1}$. This is an open compact subgroup of G_2. Let $c>0$ and T_c be the subgroup of T consisting of elements $a=(a_1,\cdots,a_{r-1})$ so that $|a_i-1|\le e^{-c}$ for all i. The intersection $T\cap K''$ is an open subgroup of T. Moreover $\pi(g)v$ and $v^\vee$ are both fixed by $T\cap K''$. Thus there is some $c(g)>0$ depending on g, and $c(g)\simeq\varsigma(g)$, such that $\pi(g)v$ and $v^\vee$ are fixed by $T_{c(g)}$. We have

$$\begin{aligned}\int_{N_{r-1,\gamma}} \Phi(g)\overline{\psi_{r-1}(n)}\mathrm{d}n &= \int_{N_{r-1,\gamma}}\int_{T_{c(g)}} \langle \pi(a^{-1}nag)v,v^\vee\rangle\overline{\psi_{r-1}(n)}\mathrm{d}a\mathrm{d}n\\ &= \int_{N_{r-1,\gamma}}\langle\pi(ng)v,v^\vee\rangle\left(\int_{T_{c(g)}}\overline{\psi_{r-1}(ana^{-1})}\mathrm{d}a\right)\mathrm{d}n.\end{aligned}$$

There is some $c'(g)$, $c'(g)\simeq\varsigma(g)$, so that if $\gamma>c'(g)$ and $n\in N_{r-1,\gamma}\backslash N_{r-1,c'(g)}$, then the inner integral vanishes. It follows that if $\gamma>c'(g)$, then

$$\int_{N_{r-1,\gamma}} \Phi(g)\overline{\psi_{r-1}(n)}\mathrm{d}n = \int_{N_{r-1,c'(g)}} \Phi(g)\overline{\psi_{r-1}(n)}\mathrm{d}n.$$

It also follows, by Lemma 3.2.3, that if $\gamma>c'(g)$, then there is some $D>0$ so that

$$\left|\int_{N_{r-1,\gamma}} \Phi(g)\overline{\psi_{r-1}(n)}\mathrm{d}n\right| \ll c'(g)^D\Xi_{G_1}(g)\varsigma(g)^D, \quad g\in G_1.$$

As $c'(g)\simeq\varsigma(g)$, we get the desired estimate (possibly for some larger D). This proves the lemma in the non-archimedean case.

From now on we assume that F is archimedean.

To simplify notation, we put

$$I(\gamma, g, \Phi) = \int_{N_{r-1,\gamma}} \Phi(ng)\overline{\psi_{r-1}(n)}\mathrm{d}n, \quad g \in G_1.$$

Note that to prove the limit exists, we may even assume that $g = 1$. By the Dixmier–Malliavin theorem, it is enough to prove the lemma for $\lim_{\gamma\to\infty} I(\gamma, g, f * \Phi)$ where $f \in \mathcal{C}_c^\infty(T)$ and

$$f * \Phi(g) = \int_T f(t)\Phi(t^{-1}gt)\mathrm{d}t$$

is a function on G_2. When there is no confusion, we write $I(\gamma) = I(\gamma, g, f * \Phi)$ for short.

Let $(x_1, \cdots, x_{r-1}) \in F^{r-1}$ and $n(x_1, \cdots, x_{r-1}) \in N_{r-1}$ so that $n(x_1, \cdots, n_{r-1})e^*_{n+i} = e^*_{n+i} + x_{i-1}e^*_{n+i-1}$ for $i = 2, \cdots, r$. Let $N_\dagger = \{n(x_1, \cdots, x_{r-1}) \mid (x_1, \cdots, x_{r-1}) \in F^{r-1}\}$. It is a subgroup of N_{r-1} which is stable under the conjugation by T and the multiplication map $N_\dagger \times N_{r-1,-\infty} \to N_{r-1}$ is an isomorphism. Let $N_{\dagger,\gamma} = N_\dagger \cap N_{r-1,\gamma}$. We denote by $\widehat{N_\dagger}$ the group of additive characters of $N_\dagger$ and by $\widehat{N_\dagger}^{\mathrm{reg}}$ the open subset consisting of generic characters. Then $\psi_{r-1} \in \widehat{N_\dagger}^{\mathrm{reg}}$. Let ψ^t be the character of $N_\dagger$ defined by $\psi^t(n) = \psi_{r-1}(tnt^{-1})$. The map $t \mapsto \psi^t$ defines a homeomorphism from T to $\widehat{N_\dagger}^{\mathrm{reg}}$. A compactly supported function on T is then identified with a compactly supported function on $\widehat{N_\dagger}^{\mathrm{reg}}$. We may thus talk about the Fourier transform of f, which is a Schwartz function on $N_\dagger$. Let $t_1, \cdots, t_{r-1} \in F^\times$ and $t \in T$ so that $tn(x_1, \cdots, x_{r-1})t^{-1} = n(t_1x_1, \cdots, t_{r-1}x_{r-1})$. The measure $|t_1 \cdots t_{r-1}|\mathrm{d}t$ is, up to a positive constant, the restriction of the self-dual measure of $\widehat{N}$ to $\widehat{N}^{\mathrm{reg}}$ under this homeomorphism. We may assume that the constant is one.

We have

$$\begin{aligned} I(\gamma) &= \int_{N_{r-1,\gamma}} \int_T f(t)\Phi(t^{-1}ntg)\overline{\psi_{r-1}(n)}\mathrm{d}t\mathrm{d}n \\ &= \int_{N_{r-1}} \int_T f(t)\, \mathbf{1}_{N_{r-1,\gamma}}(n)\Phi(t^{-1}ntg)\overline{\psi_{r-1}(n)}\mathrm{d}t\mathrm{d}n \\ &= \int_{N_\dagger} \left(\int_T f(t)\, \mathbf{1}_{N_{\dagger,\gamma}}(tnt^{-1})\overline{\psi^t(n)}\mathrm{d}t\right)\left(\int_{N_{r-1,-\infty}} \Phi(nn'g)\mathrm{d}n'\right)\mathrm{d}n, \end{aligned} \tag{3.2.1}$$

where in the last identity, we have made the change of variable $n \mapsto tnt^{-1}$ and split the integral over N_{r-1} as a double integral over $N_\dagger \times N_{r-1,-\infty}$.

We claim that there is a constant C which does not depend on γ so that

$$\left|\int_T f(t)\, \mathbf{1}_{N_{\dagger,\gamma}}(tnt^{-1})\overline{\psi^t(n)}\mathrm{d}t\right| \le C\prod_{i=1}^{r-1} \max\{1, |x_i|\}^{-1}, \tag{3.2.2}$$

where $n = n(x_1, \cdots, x_{r-1}) \in N_\dagger$. In fact, we integrate $t_i \in F^\times$ with $|x_i| \le 1$ via integration by parts. The anti-derivative of $\mathbf{1}_{\{|\cdot|\le e^\gamma\}}(xt)\psi(xt)$ is a function of the form $|x|^{-1}X_\gamma(xt)$ where X_γ

is bounded by a constant independent of γ. It then follows that

$$\int_T f(t)\,\mathbf{1}_{N_{\dagger,\gamma}}(tnt^{-1})\overline{\psi^t(n)}\mathrm{d}t = \int_{F^{r-1}} \prod_{i:|x_i|\leq 1} |x_i|^{-1} X_\gamma(x_i t_i)\partial f_1(t_1,\cdots,t_{r-1})\mathrm{d}t,$$

where $f_1(t_1,\cdots,t_{r-1}) = f(t_1,\cdots,t_{r-1})|t_1\cdots t_{r-1}|^{-1}$ and ∂f_1 is the partial derivative of f_1 with respect to all t_i such that $|x_i|\leq 1$. As f, so f_1, are in $\mathcal{C}_c^\infty(T)$, and X_γ is bounded by a constant independent of γ, the desired estimate (3.2.2) follows.

By [5, Corollary B.3.1], the integral

$$\int_{N_\dagger}\int_{N_{r-1,-\infty}} \prod_{i=1}^{r-1} \max\{1,|x_i|\}^{-1}\Phi(n(x_1,\cdots,x_{r-1})n'g)\mathrm{d}n'\mathrm{d}n$$

is convergent. By the Lebesgue dominated convergence theorem, we have

$$\begin{aligned}\lim_{\gamma\to\infty} I(\gamma) &= \int_{N_\dagger}\left(\int_T \lim_{\gamma\to\infty} f(t)\,\mathbf{1}_{N_{\dagger,\gamma}}(tnt^{-1})\overline{\psi^t(n)}\mathrm{d}t\right)\left(\int_{N_{r-1,-\infty}}\Phi(nn'g)\mathrm{d}n'\right)\mathrm{d}n\\ &= \int_{N_\dagger}\int_{N_{r-1,-\infty}} \widehat{f_1}(n)\Phi(nn'g)\mathrm{d}n'\mathrm{d}n.\end{aligned}$$

The rest of the assertions of the lemma follow easily from this expression. □

3.3. **Proof of Proposition 2.2.1.** The case $r=0$ is rather straight forward. Indeed, in this case $G_0=G_1=G_2$. By the weak inequality, we only need to prove that

$$\int_{G_0} \Xi_{G_0}(g)^2|\Phi_{\phi,\phi^\vee}(g)|\mathrm{d}g$$

is absolutely convergent. By the Cartan decomposition and the estimates (3.1.2) and (3.1.4), the convergence is reduced to the convergence of

$$\int_{|a_n|\leq\cdots\leq|a_1|\leq 1} |a_1\cdots a_n|^{\frac{1}{2}}\left(-\sum_{i=1}^n \log|a_i|\right)^D \mathrm{d}a_1\cdots\mathrm{d}a_n.$$

This is clear. Proposition 2.2.1 is thus proved when $r=0$.

The case $r\geq 2$ follows from the case $r=1$ by Lemma 3.2.6.

We now treat the case $r=1$. In this case $G_2=G_1$. The defining integral of α reduces to

$$\alpha(\varphi_2,\varphi_2^\vee,\varphi_0,\varphi_0^\vee,\phi,\phi^\vee) = \int_{G_0}\int_H \Phi_{\varphi,\varphi^\vee}(hg_0)\Phi_{\varphi_0,\varphi_0^\vee}(g_0)\overline{\Phi_{\phi,\phi^\vee}(hg_0)}\mathrm{d}h\mathrm{d}g_0,$$

Since π_2 and π_0 are both tempered, we need to prove that

$$\int_{G_0}\int_H \Xi_{G_1}(hg_0)\Xi_{G_0}(g_0)|\Phi_{\phi,\phi^\vee}(hg_0)|\mathrm{d}h\mathrm{d}g_0$$

is convergent.

Let $g_0 = k_1 d(a) k_2$ be the Cartan decomposition of g_0 where $a = (a_1, \cdots, a_n) \in (F^\times)^n$ with $|a_s| \leq \cdots \leq |a_1| \leq 1$. We first estimate $|\Phi_{\phi,\phi^\vee}(hd(a))|$. We claim that there is a function f on L^* so that $f(l^*)P(l^*)$ is bounded for any polynomial function P on L^*, such that

$$|\Phi_{\phi,\phi^\vee}(h(l+l^*,t)d(a))| \ll |\det \underline{a}|^{\frac{1}{2}} f(l^* \underline{a}). \tag{3.3.1}$$

Indeed

$$|\Phi_{\phi,\phi^\vee}(h(l+l^*,t)d(a))| \leq |\det \underline{a}|^{\frac{1}{2}} \int_{L^*} |\phi(x\underline{a} + l^*\underline{a})\phi^\vee(x)| \mathrm{d}x.$$

Thus to prove (3.3.1), it is enough to prove that for any polynomial function P on L^*,

$$\sup_{y \in L^*} |P(y)| \int_{L^*} |\phi(x\underline{a} + y)\phi^\vee(x)| \mathrm{d}x < \infty.$$

We have

$$\sup_{y \in L^*} |P(y)| \int_{L^*} |\phi(x\underline{a} + y)\phi^\vee(x)| \mathrm{d}x \leq \int_{L^*} \left(\sup_{y \in L^*} |P(y)\phi(x\underline{a} + y)| \right) |\phi^\vee(x)| \mathrm{d}x.$$

Since P is a polynomial function, we may choose a sufficiently large N, such that

$$\sup_{y \in L^*} |P(y)\phi(x\underline{a} + y)| \ll (1 + |x_1 a_n| + \cdots |x_n a_1|)^N \leq (1 + |x_1| + \cdots |x_n|)^N,$$

where $x = (x_1, \cdots, x_n) \in L^*$. We have the second inequality because $|a_i| \leq 1$ for all i. Then

$$\int_{L^*} (\sup_{y \in L^*} |P(y)\phi(x\underline{a} + y)|)|\phi^\vee(x)| \mathrm{d}x \ll \int_{L^*} (1 + |x_1| + \cdots |x_n|)^N |\phi^\vee(x)| \mathrm{d}x < \infty.$$

We have thus proved (3.3.1).

By (3.1.1), to prove the convergence of the defining integral of α, it is enough to show the convergence of

$$\int_{A^+_{G_0}} \int_H \Xi_{G_1}(h(l+l^*,t)d(a)) \Xi_{G_0}(d(a)) \delta^{-1}_{P_m \cap G_0}(d(a)) |\det \underline{a}|^{\frac{1}{2}} f(l^* \underline{a}) \mathrm{d}h \mathrm{d}a.$$

Then Lemma 3.2.5 reduces the convergence of this integral to the case $r = 0$.

3.4. **Proof of Proposition 2.2.2.** We are going to use the notation in the proof of Lemma 3.2.6, one paragraph before (3.2.1). To simplify notation, we write $\Phi_{\varphi_i} = \Phi_{\varphi_i,\varphi_i}$, $i = 0, 2$ and $\Phi_\phi = \Phi_{\phi,\phi}$.

To facilitate understanding, we divide the proof into several steps.

Step 1. The goal is to reduce the Proposition to the inequality (3.4.1)

In order to prove that $\alpha(\varphi_2, \varphi_0, \phi) \geq 0$, it is enough to show that for any function $f \in \mathcal{C}^\infty_c(T)$, we have

$$\int_T \int_{G_0} \int_H \left(\lim_{\gamma \to \infty} \int_{N_{r-1,\gamma}} \Phi_{\varphi_2}(nhg_0) \overline{\psi_{r-1}(tnt^{-1})} \mathrm{d}n \right) \Phi_{\varphi_0}(g_0) \overline{\Phi_\phi(hg_0)} f(t) \overline{f(t)} \mathrm{d}h \mathrm{d}g_0 \mathrm{d}t \geq 0.$$

We denote this expression by I. Since $f(t)$ is compactly supported, by Fubini's theorem, we have

$$I=\int_{G_0}\int_H\left(\int_T\lim_{\gamma\to\infty}\int_{N_{r-1,\gamma}}\Phi_{\varphi_2}(nhg_0)f(t)\overline{f(t)\psi_{r-1}(tnt^{-1})}\mathrm{d}n\mathrm{d}t\right)\Phi_{\varphi_0}(g_0)\overline{\Phi_\phi(hg_0)}\mathrm{d}h\mathrm{d}g_0.$$

We denote the integral in the parentheses by II. It follows from Lemma 3.2.6 that

$$\lim_{\gamma\to\infty}\int_{N_{r-1,\gamma}}\Phi_{\varphi_2}(nhg_0)\overline{\psi_{r-1}(tnt^{-1})}\mathrm{d}n$$

is bounded by a constant which depends continuously on ψ_{r-1}. Since f is compactly supported on T, we can choose this constant to be independent of t (but depends on hg_0). Then by the Lebesgue dominated convergence theorem, we have

$$II=\lim_{\gamma\to\infty}\int_T\int_{N_{r-1,\gamma}}\Phi_{\varphi_2}(nhg_0)f(t)\overline{f(t)\psi_{r-1}(tnt^{-1})}\mathrm{d}n\mathrm{d}t.$$

Moreover, the double integral on the right hand side is absolutely convergent. We can thus interchange the order of integration. Finally we conclude that

$$II=\lim_{\gamma\to\infty}\int_{N_{r-1,\gamma}}\int_T\Phi_{\varphi_2}(nhg_0)f(t)\overline{f(t)\psi_{r-1}(tnt^{-1})}\mathrm{d}t\mathrm{d}n.$$

Let $f_1(t)=f(t)|t_1\cdots t_{r-1}|^{-\frac{1}{2}}\in\mathcal{C}_c^\infty(T)$. Recall that the map $t\mapsto\psi^t$ identifies T with $\widehat{N_\dagger}^{\mathrm{reg}}$ which is an open subset of $\widehat{N_\dagger}$ consisting of generic characters. The measure $|t_1\cdots t_{r-1}|\mathrm{d}t$ is identified with the self-dual measure on $\widehat{N_\dagger}$ under this map. In this way, f, as well as f_1, are viewed as compactly supported functions on $\widehat{N_\dagger}$ and we may talk about their Fourier transform $\widehat{f}$ and $\widehat{f_1}$ which are functions on $N_\dagger$. The Fourier transform of a product of two functions is the convolution of the Fourier transforms of these two functions. We conclude that

$$\int_T f(t)\overline{f(t)\psi_{r-1}(tnt^{-1})}\mathrm{d}t=\int_{N_\dagger}\widehat{f_1}(n_1n_2)\overline{\widehat{f_1}}(n_2)\mathrm{d}n_2.$$

Therefore

$$\begin{aligned}II&=\lim_{\gamma\to\infty}\int_{N_{\dagger,\gamma}}\int_{N_{r-1,-\infty}}\int_{N_\dagger}\Phi_{\varphi_2}(n_1n'hg_0)\widehat{f_1}(n_1n_2)\overline{\widehat{f_1}}(n_2)\mathrm{d}n_2\mathrm{d}n'\mathrm{d}n_1\\&=\int_{N_\dagger}\int_{N_\dagger}\int_{N_{r-1,-\infty}}\Phi_{\varphi_2}(n_1n_2^{-1}n'hg_0)\widehat{f_1}(n_1)\overline{\widehat{f_1}}(n_2)\mathrm{d}n_2\mathrm{d}n'\mathrm{d}n_1\\&=\int_{N_{r-1,-\infty}}\Phi_{\pi_2(\widehat{f_1})\varphi_2}(n'hg_0)\mathrm{d}n',\end{aligned}$$

where $\pi_2(\widehat{f_1})\varphi_2=\int_{N_\dagger}\widehat{f_1}(n)\pi_2(n)\varphi_2\mathrm{d}n$. This expression makes sense since $\widehat{f_1}$ is a Schwartz function on $N_\dagger$. Thus to show that $I\geq 0$, it remains to show that

$$\int_{G_0}\int_H\int_{N_{r-1,-\infty}}\Phi_{\pi_2(\widehat{f_1})\varphi_2}(n'hg_0)\Phi_{\varphi_0}(g_0)\overline{\Phi_\phi(hg_0)}\mathrm{d}n'\mathrm{d}h\mathrm{d}g_0\geq 0.$$

Actually, we will show that

$$\int_{G_0}\int_H\int_{N_{r-1,-\infty}} \Phi_{\varphi_2}(n'hg_0)\Phi_{\varphi_0}(g_0)\overline{\Phi_\phi(hg_0)}\mathrm{d}n'\mathrm{d}h\mathrm{d}g_0 \geq 0, \tag{3.4.1}$$

for all smooth vectors $\varphi_2 \in \pi_2$ and $\varphi_0 \in \pi_0$. Unlike the proof of [24, Proposition 1.1] and [36, Theorem 2.1(2)], we cannot apply [20, Theorem 2.1] directly, as $G_2 \times HG_0$ is not reductive. However, we are going to mimic the proof of [20, Theorem 2.1] to prove (3.4.1).

Step 2. The goal is to reduce (3.4.1) to the case of K-finite vectors.

We claim that it is enough to prove (3.4.1) for a K_2-finite (resp. K_0-finite) vector $\varphi_2 \in \pi_2$ (resp. $\varphi_0 \in \pi_0$). This is only an issue when F is archimedean. So we assume temporarily that F is archimedean. Since K_2-finite vectors are dense in the space of smooth vectors in π_2, we may choose a sequence of K_2-finite vectors $\varphi_2^{(i)}$ which is convergent to φ_2. It follows that Φ_{φ_2} is approximated pointwisely by $\Phi_{\varphi_2^{(i)}}$. Moreover by [44], there exists an element X in the Lie algebra of G_2, which depends on K_2 only, such that

$$\Phi_{\varphi_2^{(i)}}(g_2) \leq \mathcal{B}_{\pi_2}\left(\pi_2(X)\varphi_2^{(i)}, \pi_2(X)\varphi_2^{(i)}\right)\Xi_{G_2}(g_2) = |\pi_2(X)\varphi_2^{(i)}|^2\Xi_{G_2}(g_2).$$

Since $\varphi_2^{(i)}$ is convergent to φ_2, we see that $|\pi_2(X)\varphi_2^{(i)}|^2$ is convergent to $|\pi_2(X)\varphi_2|^2$. In particular, it is bounded by some constant which is independent of $\varphi_2^{(i)}$. Similarly we choose a sequence $\varphi_0^{(i)}$ of K_0-finite vectors in π_0 which approximate φ_0. Since

$$\int_{G_0}\int_H\int_{N_{r-1,-\infty}} \Xi_{G_2}(n'hg_0)\Xi_{G_0}(g_0)\overline{\Phi_\phi(hg_0)}\mathrm{d}n'\mathrm{d}h\mathrm{d}g_0$$

is absolutely convergent, by the Lebesgue dominated convergence theorem

$$\begin{aligned}&\int_{G_0}\int_H\int_{N_{r-1,-\infty}} \Phi_{\varphi_2}(n'hg_0)\Phi_{\varphi_0}(g_0)\overline{\Phi_\phi(hg_0)}\mathrm{d}n'\mathrm{d}h\mathrm{d}g_0\\ =&\lim_{i\to\infty}\int_{G_0}\int_H\int_{N_{r-1,-\infty}} \Phi_{\varphi_2^{(i)}}(n'hg_0)\Phi_{\varphi_0^{(i)}}(g_0)\overline{\Phi_\phi(hg_0)}\mathrm{d}n'\mathrm{d}h\mathrm{d}g_0.\end{aligned}$$

So the positivity in (3.4.1) for smooth vectors follows from the positivity for K-finite vectors.

From now on, we assume that φ_2 and φ_0 in (3.4.1) are K_2-finite and K_0-finite respectively. We come back to the situation F being an arbitrary local field of characteristic zero.

Step 3. The goal is to reduce (3.4.1) to the inequality (3.4.2).

Since π_2 is tempered, by (the proof of) [20, Theorem 2.1] (which is also valid when F is non-archimedean), one can find a sequence of compactly supported continuous functions $f_{2,j}^{(i)}$ on G_2 and a sequence of positive real numbers $a_j^{(i)}$, $j = 1, \cdots, s_i$, such that $\sum_{j=1}^{s_i} a_j^{(i)} = 1$ and the functions

$$g_2' \mapsto A^{(i)}(g_2') = \sum_{j=1}^{s_i} a_j^{(i)} \int_{G_2} f_{2,j}^{(i)}(g_2g_2')\overline{f_{2,j}^{(i)}(g_2)}\mathrm{d}g_2$$

approximate $\Phi_{\varphi_2}(g_2')$ pointwisely. Moreover, there is a constant C_2, such that

$$|A^{(i)}(g_2')| \leq C_2 \Xi_{G_2}(g_2').$$

Similarly, we can find a sequence of compactly supported continuous functions $f_{0,j}^{(i)}$ on G_0 and a sequence of positive real numbers $b_j^{(i)}$, $j = 1, \cdots, k_i$, such that $\sum_{j=1}^{k_i} b_j^{(i)} = 1$ and the functions

$$g_0' \mapsto B^{(i)}(g_0') = \sum_{j=1}^{k_i} b_j^{(i)} \int_{G_0} f_{0,j}^{(i)}(g_0 g_0') \overline{f_{0,j}^{(i)}(g_0)} \mathrm{d}g_0$$

approximate $\Phi_{\varphi_0}(g_0')$ pointwisely. Moreover, there is a constant C_0, such that

$$|B^{(i)}(g_0')| \leq C_0 \Xi_{G_0}(g_0').$$

Since the integral

$$\int_{G_0} \int_H \int_{N_{r-1,-\infty}} \Xi_{G_2}(n'hg_0) \Xi_{G_0}(g_0) \Phi_\phi(hg_0) \mathrm{d}n' \mathrm{d}h \mathrm{d}g_0$$

is absolutely convergent, by the Lebesgue dominated convergence theorem, to prove (3.4.1), it is enough to prove that for any i, j,

$$\begin{aligned}(3.4.2) \qquad \int_{G_0} \int_H \int_{N_{r-1,-\infty}} &\left(\int_{G_2} f_{2,j}^{(i)}(g_2 n' h g_0') \overline{f_{2,j}^{(i)}(g_2)} \mathrm{d}g_2 \right) \\ &\left(\int_{G_0} f_{0,j}^{(i)}(g_0 g_0') \overline{f_{0,j}^{(i)}(g_0)} \mathrm{d}g_0 \right) \Phi_\phi(hg_0') \mathrm{d}n' \mathrm{d}h \mathrm{d}g_0' \geq 0.\end{aligned}$$

We denote the left hand by Q. Note that this integral is absolutely convergent. To simplify notation, we write $f_2 = f_{2,j}^{(i)}$ and $f_0 = f_{0,j}^{(i)}$.

<u>*Step 4*</u>. Proof of (3.4.2).

We can write the inner product on $\mathcal{S}(L^*)$ as

$$\mathcal{B}_{\omega_\psi}(\phi, \phi') = \int_{L+F\backslash H} \omega_\psi(h')\phi(0) \overline{\omega_\psi(h')\phi'(0)} \mathrm{d}h'.$$

Using this expression of the inner product, we have

$$\begin{aligned} Q = &\int_{G_0} \int_H \int_{N_{r-1,-\infty}} \left(\int_{G_2} f_2(g_2 n' h g_0') \overline{f_2(g_2)} \mathrm{d}g_2 \right) \left(\int_{G_0} f_0(g_0 g_0') \overline{f_0(g_0)} \mathrm{d}g_0 \right) \\ &\left(\int_{L+F\backslash H} \overline{\omega_\psi(h' h g_0')\phi(0)} \omega_\psi(h')\phi(0) \mathrm{d}h' \right) \mathrm{d}n' \mathrm{d}h \mathrm{d}g_0' \\ = &\int_{G_0} \int_H \int_{N_{r-1,-\infty}} \left(\int_{G_2} f_2(g_2 n' h g_0') \overline{f_2(g_2)} \mathrm{d}g_2 \right) \left(\int_{G_0} f_0(g_0 g_0') \overline{f_0(g_0)} \mathrm{d}g_0 \right) \\ &\left(\int_{L+F\backslash H} \overline{\omega_\psi(h' g_0 h g_0')\phi(0)} \omega_\psi(h' g_0)\phi(0) \mathrm{d}h' \right) \mathrm{d}n' \mathrm{d}h \mathrm{d}g_0'. \end{aligned}$$

Note that we have used the fact the the pairing $\mathcal{B}_{\omega_\psi}$ is G_0-invariant.

We make the following change of variables

$$g_0' \mapsto g_0^{-1}g_0', \quad h \mapsto g_0^{-1}h'^{-1}hg_0, \quad n' \mapsto g_0^{-1}h'^{-1}n'h'g_0, \quad g_2 \mapsto g_2h'g_0.$$

Then

$$Q = \int_{G_2}\int_{N_{r-1,-\infty}}\iint_{(L+F)\backslash H\times H}\iint_{G_0\times G_0} f_2(g_2n'hg_0')\overline{f_2(g_2h'g_0)}$$
$$f_0(g_0')\overline{f_0(g_0)\omega_\psi(hg_0')\phi(0)}\omega_\psi(h'g_0)\phi(0)\mathrm{d}g_0\mathrm{d}g_0'\mathrm{d}h\mathrm{d}h'\mathrm{d}n'\mathrm{d}g_2,$$

where $L+F$ embeds in $H\times H$ diagonally.

Finally we decompose the integral over G_2 as

$$\int_{G_2/(N_{r-1,-\infty}\rtimes(L+F))}\int_{N_{r-1,-\infty}}\int_{L+F}.$$

We conclude that

$$Q = \int_{G_2/(N_{r-1,-\infty}\rtimes(L+F))}\left|\int_{N_{r-1,-\infty}}\int_H\int_{G_0} f_2(g_2nhg_0)f_0(g_0)\overline{\omega_{\psi,\mu}(hg_0)\phi(0)}\mathrm{d}g_0\mathrm{d}h\mathrm{d}n\right|^2\mathrm{d}g_2 \geq 0.$$

We have thus proved (3.4.2) and hence Proposition 2.2.2.

3.5. **Regularization via stable unipotent integral.** In this subsection, we give an alternative but equivalent way to define the linear functional α when F is non-archimedean following [29,36]. This definition is better for the unramified computation and is valid for nontempered representations. In this subsection, F is always assumed to be non-archimedean.

Let N be a unipotent group over F and f a smooth function on N. We say that f is compactly supported on average if there are compact subsets U and U' of N, such that $\mathrm{L}(\delta_{U'})\mathrm{R}(\delta_U)f$ is compactly supported. Here δ_U stands for the Dirac measure on U, i.e. $\delta_U = (\mathrm{vol}\, U)^{-1}\,\mathbf{1}_U$, and

$$\mathrm{L}(\delta_{U'})\mathrm{R}(\delta_U)f(n) = \int_N\int_N \delta_{U'}(u')\delta_U(u)f(u'nu)\mathrm{d}u'\mathrm{d}u.$$

If f is compactly supported on average, we then define

$$\int_N^{\mathrm{st}} f(n)\mathrm{d}n := \int_N \mathrm{L}(\delta_{U'})\mathrm{R}(\delta_U)f(n)\mathrm{d}n.$$

This is called the stable integral of f on N. The definition is independent of the choice of U and U'.

We denote temporarily by G a reductive group over F. Let $P_{\min} = M_{\min}N_{\min}$ be a fixed minimal parabolic subgroup of G. Let $P = MN \supset P_{\min}$ be a parabolic subgroup of G. Let Ψ be a *generic* character of N, i.e. the stabilizer of Ψ in $M_{\min}$ is the center of $M_{\min}$. Let π be an irreducible admissible representation of G and Φ a matrix coefficient of π. Then

Proposition 3.5.1 ([36, Proposition 3.3])**.** *The function $\Phi|_{N_P}\cdot\Psi$ is compactly supported on average.*

Now let $G = \mathrm{Mp}(2n)$. Then Proposition 3.5.1 still holds. The same proof as in [36, Proposition 3.3] goes through as it uses only the Bruhat decomposition and Jacquet's subrepresentation theorem, which are valid for G.

Now we retain the notation G_0, G_1, G_2 etc. Let Φ be a matrix coefficient on G_2 (resp. $\widetilde{G_2}$). Define

$$\mathcal{F}_\psi^{\mathrm{st}}\Phi(g) = \int_{N_{r-1}}^{\mathrm{st}} \Phi(gn)\overline{\psi_{r-1}(n)}\mathrm{d}n,$$

which is a function on G_2 (resp. $\widetilde{G_2}$). This definition makes sense because of Proposition 3.5.1.

Lemma 3.5.2. *Assume that Φ is a matrix coefficient of a tempered representation of G_2 (resp. $\widetilde{G_2}$). Then*

$$\mathcal{F}_\psi^{\mathrm{st}}\Phi(hg_0) = \mathcal{F}_\psi\Phi(hg_0), \quad h \in H, g_0 \in G_0, \textit{(resp. } g_0 \in \widetilde{G_0}\textit{)}.$$

Proof. By definition,

$$\mathcal{F}_\psi^{\mathrm{st}}\Phi_2(hg_0) = \int_{N_{r-1}} \left((\mathrm{vol}\, U)^{-1} \int_U \Phi(unhg_0)\overline{\psi_{r-1}(un)}\mathrm{d}u \right) \mathrm{d}n,$$

where U is an open compact set of N_{r-1}. The inner integral, as a function of n, is compactly supported. Therefore we may take a sufficiently large γ, such that $N_{r-1,\gamma}$ contains U and the support of the inner integral (as a function of n) and that

$$\mathcal{F}_\psi\Phi_2(hg_0) = \int_{N_{r-1,\gamma}} \Phi_2(nhg_0)\overline{\psi_{r-1}(n)}\mathrm{d}n.$$

It follows that

$$\begin{aligned}
\mathcal{F}_\psi^{\mathrm{st}}\Phi_2(hg_0) &= \int_{N_{r-1,\gamma}} (\mathrm{vol}\, U)^{-1} \int_U \Phi(unhg_0)\overline{\psi_{r-1}(un)}\mathrm{d}u\mathrm{d}n, \\
&= \int_{N_{r-1,\gamma}} \Phi(nhg_0)\overline{\psi_{r-1}(n)}\mathrm{d}n \times (\mathrm{vol}\, U)^{-1} \int_U \mathrm{d}u \\
&= \mathcal{F}_\psi\Phi_2(hg_0),
\end{aligned}$$

where in the second equality, we have made a change of variable $n \mapsto u^{-1}n$. □

Thanks to Lemma 3.5.2, if F is nonarchimedean, then we may use $\mathcal{F}_\psi^{\mathrm{st}}$ instead of $\mathcal{F}_\psi$ in the definition of the local linear form α. We will not distinguish $\mathcal{F}_\psi^{\mathrm{st}}$ and $\mathcal{F}_\psi$ from now on and write just $\mathcal{F}_\psi$.

4. Unramified computations

In this section, we assume the conditions prior to Proposition 2.2.3. In particular, F is a non-archimedean local field of residue characteristic different from two. The argument is mostly adapted from [36], except that at the end we use a different trick, which avoids the use of the explicit formulae of the Whittkaer–Shintani functions as in [36, Appendix]. Some of the arguments which are identical to [36] are only sketched.

4.1. **Setup.** For $i = 0, 1, 2$, let $B_i = P_m \cap G_i = T_i U_i$ be the upper triangular Borel subgroup of G_i where T_i is the diagonal maximal torus of G_i. We have a hyperspecial subgroup $K_i = \mathrm{Sp}(W_i)(\mathfrak{o}_F)$ of $\mathrm{Sp}(W_i)$. Recall that the two fold cover $\widetilde{G_i} \to G_i$ splits uniquely over K_i. We can thus view K_i as a subgroup of $\widetilde{G_i}$. Let Ξ (resp. ξ) be an unramified character of T_2 (resp. T_0). In the case Sp, we consider the unramified principal series $\pi_2 = I(\Xi)$ of G_2 and $\pi_0 = I(\xi)$ of $\widetilde{G_0}$. In the case Mp, we consider the unramified principal series $\pi_2 = I(\Xi)$ of $\widetilde{G_2}$ and $\pi_0 = I(\xi)$ of G_0. Note that the unramified principal series representation of the metaplectic group depends on the additive character ψ, even though this is not reflected in the notation. We frequently identify Ξ with an element in $\mathbb{C}^m$ which we also denote by $\Xi = (\Xi_1, \cdots, \Xi_m)$, the correspondence being given by

$$\Xi(\mathrm{diag}[a_m, \cdots, a_1, a_1^{-1}, \cdots, a_m^{-1}]) = |a_1|^{\Xi_1} \cdots |a_m|^{\Xi_m}$$

Similarly we identify ξ with an element in $\mathbb{C}^n$. The contragredient of π_2 (resp. π_0) is $I(\Xi^{-1})$ (resp. $I(\xi^{-1})$). Let $f_\Xi \in I(\Xi)$, $f_{\Xi^{-1}} \in I(\Xi^{-1})$ (resp. $f_\xi \in I(\xi)$, $f_{\xi^{-1}} \in I(\xi^{-1})$) be the K_2-fixed (resp. K_0-fixed) elements with $f_\Xi(1) = f_{\Xi^{-1}}(1) = 1$ (resp. $f_\xi(1) = f_{\xi^{-1}}(1) = 1$). Let

$$\Phi_\Xi(g_2) = \int_{K_2} f_\Xi(k_2 g_2) \mathrm{d}k_2, \quad \Phi_\xi(g_0) = \int_{K_0} f_\xi(k_0 g_0) \mathrm{d}k_0, \quad \Phi_\phi(hg_0) = \int_{L(\mathfrak{o}_F)} \omega_\psi(hg_0)\, \mathbf{1}_{L^*(\mathfrak{o}_F)}(x) \mathrm{d}x.$$

and

$$I(g_2, \Xi, \xi, \psi) = \int_{G_0} \int_H \mathcal{F}_\psi \Phi_\Xi(g_2^{-1} h g_0) \Phi_\xi(g_0) \overline{\Phi_\phi(hg_0)} \mathrm{d}h \mathrm{d}g_0.$$

Then $\alpha(f_\Xi, f_{\Xi^{-1}}, f_\xi, f_{\xi^{-1}}, \phi, \phi) = I(1, \Xi, \xi, \psi)$.

Let $J = H \rtimes G_0$ and $\widetilde{J} = H \rtimes \widetilde{G_0}$. We define the Borel subgroup B_J (resp. $B_{\widetilde{J}}$) of J (resp. $\widetilde{J}$) as a subgroup of J (resp. $\widetilde{J}$) consisting of elements of the form hb_0 where $b_0 \in B_0$ (resp. $\widetilde{B_0}$, the inverse image of B_0 in $\widetilde{G_0}$) and $h \in H$ is of the form $h(l, t)$, $l \in L$. We define the unramified principal series representation of J (resp. $\widetilde{J}$) as

$$I^J(\xi, \overline{\psi}) = \{f \in \mathcal{C}^\infty(J) \mid f(h(l,t) b_0 h g_0) = \delta_{B_J}^{\frac{1}{2}}(b_0) \xi(b_0) \overline{\psi}(t) f(hg_0)\},$$

resp.

$$I^{\widetilde{J}}(\xi, \overline{\psi}) = \{f \in \mathcal{C}^\infty(\widetilde{J}) \mid f(h(l,t) b_0 h g_0) = \delta_{B_J}^{\frac{1}{2}}(b_0) \xi\chi_\psi(b_0) \overline{\psi}(t) f(hg_0)\},$$

where $\xi\chi_\psi(b_0) = \xi(\mathrm{diag}[t_n, \cdots, t_1, t_1^{-1}, \cdots, t_n^{-1}])\chi_\psi(t_1 \cdots t_n)$ and $t_n, \cdots, t_1, t_1^{-1}, \cdots, t_n^{-1}$ are diagonal entries of b_0.

The group J (resp. $\widetilde{J}$) acts on $I^J(\xi, \overline{\psi})$ (resp. $I^{\widetilde{J}}(\xi, \overline{\psi})$) via the right translation. Let $K_J = J \cap K_1$. There is a canonical J (resp. $\widetilde{J}$)-invariant pairing given by

$$\mathcal{B}_J(f, f^\vee) = \int_{L^*} \int_{K_0} f(h(l^*, 0) k_0) f^\vee(h(l^*, 0) k_0) \mathrm{d}k_0 \mathrm{d}l^*,$$

where $f \in I^J(\xi, \overline{\psi}), f^\vee \in I^J(\xi^{-1}, \psi)$ (resp. $f \in I^{\widetilde{J}}(\xi, \overline{\psi}), f^\vee \in I^{\widetilde{J}}(\xi^{-1}, \psi)$).

In the case Mp, there is a canonical inner product preserving isomorphism

$$\omega_{\overline{\psi}} \otimes I(\xi) \to I^{\widetilde{J}}(\xi, \overline{\psi}), \quad \phi \otimes f_\xi \to f_{\xi, \overline{\psi}},$$

where $f_{\xi,\overline{\psi}}(hg_0) = \omega_{\overline{\psi}}(hg_0)\phi(0)f_\xi(g_0)$, $h \in H$ and $g_0 \in \widetilde{G_0}$. In the case Sp, there is a canonical inner product preserving isomorphism

$$\omega_{\overline{\psi}} \otimes I(\xi) \to I^J(\xi, \overline{\psi}), \quad \phi \otimes f_\xi \to f_{\xi, \overline{\psi}},$$

where $f_{\xi,\overline{\psi}}(hg_0) = \omega_{\overline{\psi}}(h\iota(g_0))\phi(0)f_\xi(\iota(g_0))$. Analogues isomorphism also holds in the case Mp.

For the ease of the exposition, we slightly modify our notation in the case Mp for the rest of this section. For $i = 0, 1, 2$, we put $G_i = \mathrm{Mp}(W_i)$ and B_i the standard Borel subgroup of G_i. Denote by $J = H \rtimes \mathrm{Mp}(W_0)$, which is a subgroup of G_1, and B_J its Borel subgroup. We denote by $K_i = \mathrm{Sp}(W_i)(\mathfrak{o}_F)$ a hyperspecial maximal subgroup of $\mathrm{Sp}(W_i)$. The metaplectic cover $\mathrm{Mp}(W_i) \to \mathrm{Sp}(W_i)$ splits canonically over K_i, so we view K_i as a compact (but not maximal) subgroup of G_i and an element in K_i is naturally viewed as an element in G_i. Let $K_J = K_1 \cap J$. The subgroup $P_i = M_i N_i$ $(i = 1, \cdots, r-1)$ is a parabolic subgroup of $\mathrm{Sp}(W_2)$ as before. The metaplectic double cover splits canonically over N_i, so we consider N_i as subgroups of G_2. By the Weyl group of $\mathrm{Mp}(W_i)$, we mean the Weyl group of $\mathrm{Sp}(W_i)$. We let

$$w_{2,\mathrm{long}} = \begin{pmatrix} & \mathsf{w}_m \\ -\mathsf{w}_m & \end{pmatrix}, \quad w_{1,\mathrm{long}} = \begin{pmatrix} & \mathsf{w}_{n+1} \\ -\mathsf{w}_{n+1} & \end{pmatrix}, \quad w_{0,\mathrm{long}} = \begin{pmatrix} & \mathsf{w}_n \\ -\mathsf{w}_n & \end{pmatrix}$$

be representatives of the longest elements in the Weyl groups W_{G_2}, W_{G_1} and W_{G_0} respectively. They are viewed as elements in G_2, G_1 and G_0 respectively.

For $(\Xi_1, \cdots, \Xi_m) \in \mathbb{C}^m$ and $(\xi_1, \cdots, \xi_n) \in \mathbb{C}^n$, we denote by Ξ and ξ the genuine character of B_2 and B_0 respectively, defined by

$$\Xi((\mathrm{diag}[t_m, \cdots, t_1, t_1^{-1}, \cdots, t_m^{-1}], \epsilon)) = \epsilon \cdot (\chi_\psi \Xi_1)(t_1) \cdots (\chi_\psi \Xi_m)(t_m),$$
$$\xi((\mathrm{diag}[t_n, \cdots, t_1, t_1^{-1}, \cdots, t_n^{-1}], \epsilon)) = \epsilon \cdot (\chi_\psi \xi_1)(t_1) \cdots (\chi_\psi \xi_n)(t_n).$$

We have the unramified principal series representation $I(\Xi)$ of G_2 and $I(\xi, \overline{\psi})$ of J. We let $f_{\xi,\overline{\psi}}$ be the K_J fixed element in $I(\xi, \overline{\psi})$ such that $f_{\xi,\overline{\psi}}(1) = 1$. We will need to integrate over $\mathrm{Mp}(W_0)$. For this, we pick a measure $\mathrm{d}x$ on $\mathrm{Mp}(W_0)$, such that for any $f \in \mathcal{C}_c^\infty(\mathrm{Sp}(W_0))$, we have $\int_{\mathrm{Sp}(W_0)} f(g)\mathrm{d}g = \int_{\mathrm{Mp}(W_0)} f(x)\mathrm{d}x$. When integrating over K_i's or K_J, we always use the measure so that the volume of the domain of the integration is one.

With this modification of notation, the integral $I(g_2, \Xi, \xi, \psi)$ in both cases Mp and Sp can be written as

$$I(g_2, \Xi, \xi, \psi) = \int_J \int_{K_J} \mathcal{F}_\psi \Phi_\Xi(g_2^{-1} g_J) f_{\xi,\overline{\psi}}(k_J g_J) \mathrm{d}k_J \mathrm{d}g_J.$$

4.2. **Reduction Steps:** $r \geq 1$**.** We distinguish two cases: $r = 0$ and $r \geq 1$. We treat the case $r \geq 1$ first.

Let $\dot{w} = w_{1,\mathrm{long}}^{-1} w_{2,\mathrm{long}}$ be a representative of the longest element in $W_{G_1} \backslash W_{G_2}$.

Lemma 4.2.1. *If $g_2 \in G_2$ and $g_J \in J$, then*

$$\mathcal{F}_\psi \Phi_\Xi(g_2^{-1} g_J) = \mathbf{w}^{-1} \int_{K_1} \int_{N_{r-1}}^{\mathrm{st}} \mathcal{F}_\psi(\pi_2(g_J) f_\Xi)(k_1 \dot{w} n)(\pi_2^\vee(g_2) f_{\Xi^{-1}})(k_1 \dot{w} n) \mathrm{d}n \mathrm{d}k_1,$$

where

$$\mathbf{w} = \int_{N_{r-1}} f_\Xi(\dot{w} n) f_{\Xi^{-1}}(\dot{w} n) \mathrm{d}n = \frac{\Delta_{T_2}}{\Delta_{G_2}} \left(\frac{\Delta_{T_1}}{\Delta_{G_1}} \right)^{-1}.$$

Proof. By definition,

$$\begin{aligned} \mathcal{F}_\psi \Phi_\Xi(g_2^{-1} g_J) &= \int_{N_{r-1}}^{\mathrm{st}} \mathcal{B}_{\pi_2}(\pi_2(g_2^{-1} g_J u) f_\Xi, f_{\Xi^{-1}}) \psi(u)^{-1} \mathrm{d}u \\ &= \int_{N_{r-1}}^{\mathrm{st}} \mathcal{B}_{\pi_2}(\pi_2(g_J u) f_\Xi, \pi_2^\vee(g_2) f_{\Xi^{-1}}) \psi(u)^{-1} \mathrm{d}u. \end{aligned}$$

By [36, Lemma 3.2] (it is valid also for metaplectic groups since the Bruhat decomposition is valid for metaplectic groups), there is an open compact subgroup U of N_{r-1}, such that $(\pi_2^\vee(g_2) f_{\Xi^{-1}})^\circ = \mathrm{R}(\delta_U \psi)(\pi_2^\vee(g_2) f_{\Xi^{-1}})$ and $(\pi_2(g_2) f_\Xi)^\circ = \mathrm{R}(\delta_U \psi)(\pi_2(g_2) f_\Xi)$ are supported in $B_2 \dot{w} P_{r-1}$. Then

$$\mathcal{F}_\psi \Phi_\Xi(g_2^{-1} g_J) = \int_{N_{r-1}} \mathcal{B}_{\pi_2}(\pi_2(u)(\pi_2(g_J) f_\Xi)^\circ, (\pi_2^\vee(g_2) f_{\Xi^{-1}})^\circ) \psi(u)^{-1} \mathrm{d}u.$$

We use the following realization of $\mathcal{B}_{\pi_2}$:

$$\mathcal{B}_{\pi_2}(\varphi, \varphi^\vee) = \mathbf{w}^{-1} \int_{K_1} \int_{N_{r-1}}^{\mathrm{st}} \varphi(k_1 \dot{w} n) \varphi^\vee(k_1 \dot{w} n) \mathrm{d}n \mathrm{d}k_1,$$

where

$$\mathbf{w} = \int_{N_{r-1}} f_\Xi(\dot{w} n) f_{\Xi^{-1}}(\dot{w} n) \mathrm{d}n = \frac{\Delta_{T_2}}{\Delta_{G_2}} \left(\frac{\Delta_{T_1}}{\Delta_{G_1}} \right)^{-1}.$$

In fact, the pairing is G_2 invariant since $B_2 K_1 \dot{w} N_{r-1}$ is an open subset of G_2. The evaluation of $\mathbf{w}$ is as follows. Denote temporarily by f_i $(i = 1, 2)$ the function on $\mathrm{Sp}(W_i)$ which satisfies $f_i|_{K_i} = 1$, $f_i(bg) = \delta_i(b) f_i(g)$ for all $b \in B_i$ where B_i is the Borel subgroup of $\mathrm{Sp}(W_i)$ and δ_i is the modulus character of B_i. Define a function f_1' on $\mathrm{Sp}(W_1)$ by

$$f_1'(g) = \int_{N_{r-1}} f_2(\dot{w} n g) \mathrm{d}n.$$

Then $\mathbf{w} = f_1'(1)$. Since $f_1'(bg) = \delta_1(b) f_1'(g)$ and $f_1'|_{K_1}$ is a constant, it follows that $f_1' = \mathbf{w} f_1$. Therefore

$$\int_{N_m \cap \mathrm{Sp}(W_1)} f_1'(w_{1,\mathrm{long}} n) \mathrm{d}n = \mathbf{w} \int_{N_m \cap \mathrm{Sp}(W_1)} f_1(w_{1,\mathrm{long}} n) \mathrm{d}n.$$

The left hand side equals

$$\int_{N_m} f_2(w_{2,\mathrm{long}}n)\mathrm{d}n$$

by the definition of f_1'. It follows from [16, Proposition 4.7] that

$$\int_{N_m\cap \mathrm{Sp}(W_i)} f_i(w_{i,\mathrm{long}}n)\mathrm{d}n = \frac{\Delta_{T_i}}{\Delta_{G_i}}.$$

We then conclude that

$$\mathbf{w} = \frac{\Delta_{T_2}}{\Delta_{G_2}}\left(\frac{\Delta_{T_1}}{\Delta_{G_1}}\right)^{-1}.$$

We continue the computation of $\mathcal{F}_\psi\Phi_\Xi(g_2^{-1}g_J)$. We have

$$\mathcal{F}_\psi\Phi_\Xi(g_2^{-1}g_J) = \mathbf{w}^{-1}\int_{N_{r-1}}\int_{K_1}\int_{N_{r-1}} (\pi_2(g_J)f_\Xi)^\circ(k_1\dot{w}nu)(\pi_2^\vee(g_2)f_{\Xi^{-1}})^\circ(k_1\dot{w}n)\psi(u)^{-1}\mathrm{d}n\mathrm{d}k_1\mathrm{d}u,$$

where the integrand is compactly supported. It equals

$$\begin{aligned}&\mathbf{w}^{-1}\int_{K_1}\int_{N_{r-1}} \mathcal{F}_\psi(\pi_2(g_J)f_\Xi)(k_1\dot{w}n)(\pi_2^\vee(g_2)f_{\Xi^{-1}})^\circ(k_1\dot{w}n)\mathrm{d}n\mathrm{d}k_1\\ =\ &\mathbf{w}^{-1}\int_{K_1}\int_{N_{r-1}}^{\mathrm{st}} \mathcal{F}_\psi(\pi_2(g_J)f_\Xi)(k_1\dot{w}n)(\pi_2^\vee(g_2)f_{\Xi^{-1}})(k_1\dot{w}n)\mathrm{d}n\mathrm{d}k_1.\end{aligned}$$

□

By Lemma 4.2.1, we have

$$I(g_2,\Xi,\xi,\psi) = \mathbf{w}^{-1}\int_J\int_{K_1}\int_{N_{r-1}}^{\mathrm{st}}\int_{K_J} \mathcal{F}_\psi f_\Xi(k_1\dot{w}ng_J)\pi_2^\vee(g_2)f_{\Xi^{-1}}(k_1\dot{w}n)f_{\xi,\overline{\psi}}(k_Jg_J)\mathrm{d}k_J\mathrm{d}n\mathrm{d}k_1\mathrm{d}g_J.$$

Let

$$l_0^* = (1,\cdots,1)\in L^*,\quad \eta_1 = w_{1,\mathrm{long}}h(l_0^*,0)\in G_1,\quad \eta=\dot{w}\eta_1\in G_2.$$

Lemma 4.2.2. *The double coset $B_2\eta(N_{r-1}\rtimes B_J)$ is open dense in G_2.*

Proof. This is straightforward to check. □

Thanks to this lemma, we can define a function $Y_{\Xi,\xi,\psi}$ on G_2 with the following properties.

(1) $Y_{\Xi,\xi,\psi}(b_2g_2h(l,t)b_0u) = (\Xi^{-1}\delta_{B_2}^{1/2})(b_2)(\xi\delta_{B_J}^{-1/2})(b_0)\overline{\psi(t)\psi_{r-1}(u)}Y_{\Xi,\xi,\psi}(g_2)$ for any $b_2\in B_2$, $b_0\in B_0$, $l\in L$ and $u\in N_{r-1}$.

(2) The support of $Y_{\Xi,\xi,\psi}$ is $B_2\eta(N_{r-1}\rtimes B_J)$.

(3) $Y_{\Xi,\xi,\psi}(\eta)=1$.

The space of functions that satisfy the first two conditions is one dimensional by Lemma 4.2.2. We have

$$Y_{\Xi,\xi,\psi}(b_2\eta h(l,t)b_0u) = (\Xi^{-1}\delta_2^{1/2})(b_2)(\xi\delta_{B_J}^{-1/2})(b_0)\overline{\psi(t)\psi_{r-1}(u)},$$

for $b_2 \in B_2$, $b_0 \in B_0$, $l \in L$ and $u \in N_{r-1}$. We define a function $T_{\Xi,\xi,\psi}$ on G_2 as

$$T_{\Xi,\xi,\psi}(g_2) = \begin{cases} \displaystyle\int_J \mathcal{F}_\psi f_\Xi(g_2 g_J) f_{\xi,\overline{\psi}}(g_J) \mathrm{d}g_J, & g_2 \in B_2\eta(N_{r-1} \rtimes B_J), \\ 0, & \text{otherwise.} \end{cases}$$

If the defining integral of $T_{\Xi,\xi,\psi}$ is convergent, then we have

$$T_{\Xi,\xi,\psi}(g_2) = T_{\Xi,\xi,\psi}(\eta) Y_{\Xi^{-1},\xi^{-1},\psi^{-1}}(g_2), \quad g_2 \in G_2.$$

We assume that the defining integral of $T_{\Xi,\xi,\psi}$ is convergent for the moment. This will be proved later. It follows that

$$\begin{aligned} & I(g_2, \Xi, \xi, \psi) \\ =& \mathbf{w}^{-1} \int_{K_1} \int_{N_{r-1}}^{\mathrm{st}} \int_{K_J} T_{\Xi,\xi,\psi}(k_1 \dot{w} n k_J) \pi_2^\vee(g_2) f_{\Xi^{-1}}(k_1 \dot{w} n) \mathrm{d}k_J \mathrm{d}n \mathrm{d}k_1 \\ =& \mathbf{w}^{-1} T_{\Xi,\xi,\psi}(\eta) \int_{K_1} \int_{N_{r-1}}^{\mathrm{st}} \int_{K_J} Y_{\Xi^{-1},\xi^{-1},\psi^{-1}}(k_1 \dot{w} n k_J) \pi_2^\vee(g_2) f_{\Xi^{-1}}(k_1 \dot{w} n) \mathrm{d}k_J \mathrm{d}n \mathrm{d}k_1. \end{aligned}$$

Define

(4.2.1)

$$S'_{\Xi^{-1},\xi^{-1},\psi^{-1}}(g_2) = \mathbf{w}^{-1} \int_{K_1} \int_{N_{r-1}}^{\mathrm{st}} \int_{K_J} Y_{\Xi^{-1},\xi^{-1},\psi^{-1}}(k_1 \dot{w} n k_J) \pi_2^\vee(g_2) f_{\Xi^{-1}}(k_1 \dot{w} n) \mathrm{d}k_J \mathrm{d}n \mathrm{d}k_1.$$

Then we have

$$I(g_2, \Xi, \xi, \psi) = T_{\Xi,\xi,\psi}(\eta) S'_{\Xi^{-1},\xi^{-1},\psi^{-1}}(g_2).$$

4.3. **Reduction Steps: $r = 0$.** We now treat the case $r = 0$.

The integral we need to compute is

$$I(g_J, \Xi, \xi, \psi) = \int_{G_0} \int_{K_J} \int_{K_0} f_\Xi(k_0 g) f_{\xi,\overline{\psi}}(k_J g_J^{-1} g) \mathrm{d}k_0 \mathrm{d}k_J \mathrm{d}g.$$

We define

$$l_0 = (1, \cdots, 1) \in L, \quad \eta = w_{0,\mathrm{long}} h(l_0, 0) \in J.$$

Similar to Lemma 4.2.2, it is straightforward to prove the following lemma.

Lemma 4.3.1. *The double coset $B_J \eta B_0$ is open dense in J.*

We define a function $Y_{\Xi,\xi,\psi}$ on J which is supported on $B_J \eta B_0$ by

$$Y_{\Xi,\xi,\psi}(h(l,t) b_0' \eta b_0) = (\xi^{-1} \delta_J^{\frac{1}{2}})(b_0')(\Xi \delta_0^{\frac{1}{2}})(b_0) \psi(t), \quad b_0, b_0' \in B_0, l \in L.$$

We define the function $T_{\Xi,\xi,\psi}$ on J by

$$T_{\Xi,\xi,\psi}(g_J) = \begin{cases} \displaystyle\int_{G_0} f_\Xi(g) f_{\xi,\overline{\psi}}(g_J g) \mathrm{d}g, & g_J \in B_J \eta B_0, \\ 0, & \text{otherwise.} \end{cases}$$

and the function $S_{\Xi,\xi,\psi}$ by

$$S_{\Xi,\xi,\psi}(g_J) = \int_{K_J}\int_{K_0} Y_{\Xi,\xi,\psi}(k_J g_J^{-1} k_0)\mathrm{d}k_0\mathrm{d}k_J.$$

It follows that

$$I(g_J,\Xi,\xi,\psi) = T_{\Xi,\xi,\psi}(\eta)S_{\Xi^{-1},\xi^{-1},\psi^{-1}}(g_J).$$

We now prove the convergence of the defining integral of $T_{\Xi,\xi,\psi}$ and $S_{\Xi^{-1},\xi^{-1},\psi^{-1}}$. Assume that $r \geq 0$.

Lemma 4.3.2. *The defining integrals for $T_{\Xi,\xi,\psi}$ and $S_{\Xi^{-1},\xi^{-1},\psi^{-1}}$ are absolutely convergent if Ξ' and ξ are sufficiently close to the unitary axis, where Ξ' is the restriction of Ξ to T_1.*

Proof. If $r = 0$, then it follows from Proposition 2.2.1 (or its proof, applied to $|\Xi|$ and $|\xi|$) that $I(g_J,\Xi,\xi,\psi)$ is convergent if Ξ and ξ are sufficiently close to the unitary axis. It then follows that for a fixed $g_J \in J$, the defining integral of $T_{\Xi,\xi,\psi}(k_J g_J k_0)$ is convergent for almost all $k_J \in K_J$ and $k_0 \in K_0$ such that $k_J g_J^{-1} k_0 \in B_J \eta B_0$. By the definition of $T_{\Xi,\xi,\psi}$, its defining integral is convergent for some $g_J \in B_J\eta B_0$ is and only if it is convergent for all $g_J \in B_J\eta B_0$. Therefore the defining integral of $T_{\Xi,\xi,\psi}(\eta)$ is convergent. This then implies that the defining integral of $S_{\Xi^{-1},\xi^{-1},\psi^{-1}}$ is convergent.

The convergence in the case of $r = 1$ can be proved similarly. We only need to change the notation at several places.

Now assume that $r \geq 2$. By [36, Lemma 3.3], there is an open compact subgroup U of N_{r-1}, such that for all $g_J \in J$,

$$\mathcal{F}_\psi f_\Xi(\eta g_J) = \int_U f_\Xi(\eta g_J u)\overline{\psi_{r-1}(u)}\mathrm{d}u.$$

Therefore there is a constant C, such that

$$|\mathcal{F}_\psi f_\Xi(\eta g_J)| \leq C \times f_{|\Xi'|}(\eta_1 g_J).$$

The Lemma in the case $r \geq 2$ then follows from the case $r = 1$. ☐

4.4. **Proof of Proposition 2.2.3.** Assume that $r \geq 1$. Let $\Xi^0 = (\Xi_1, \cdots, \Xi_n) \in \mathbb{C}^n$. Let σ be the unramified principal series representation of G_0 defined by Ξ^0. We let τ be the unramified principal series representation of GL_r defined by the unramified characters $(\Xi_{n+1}, \cdots, \Xi_m)$.

Following the notation of [24] and [36], we shall denote $T_{\Xi,\xi,\psi}(\eta)$ by $\zeta(\Xi,\xi,\psi)$.

Lemma 4.4.1. *We have*

$$\zeta(\Xi,\xi,\psi) = \begin{cases} \dfrac{L_\psi(\frac{1}{2},\pi_0\times\tau)}{L(1,\sigma\times\tau)L(1,\tau,\wedge^2)}\displaystyle\prod_{1\leq i<j\leq r}\frac{1}{L(1,\Xi_{n+i}\Xi_{n+j}^{-1})}\zeta(\Xi^0,\xi,\psi), & \textit{Case } \mathrm{Sp},\\ \dfrac{L(\frac{1}{2},\pi_0\times\tau)}{L_\psi(1,\sigma\times\tau)L(1,\tau,\mathrm{Sym}^2)}\displaystyle\prod_{1\leq i<j\leq r}\frac{1}{L(1,\Xi_{n+i}\Xi_{n+j}^{-1})}\zeta(\Xi^0,\xi,\psi), & \textit{Case } \mathrm{Mp}.\end{cases}$$

Proof. Recall that $l_0^* = (1, \cdots, 1) \in L^*$. By definition,

$$\zeta(\Xi, \xi, \psi) = \int_{G_0} \int_H \int_{N_{r-1}} f_\Xi(w_{2,\mathrm{long}} h(l_0^*, 0) u h g_0) f_\xi(g_0) \overline{\psi_{r-1}(u) \omega_\psi(h g_0) \phi(0)} \mathrm{d}u \mathrm{d}h \mathrm{d}g_0. \tag{4.4.1}$$

We combine the integral over H and N_{r-1} to get an integral over N_r and get

$$\zeta(\Xi, \xi, \psi) = \int_{G_0} \int_{N_r} f_\Xi(w_{2,\mathrm{long}} h(l_0^*, 0) v g_0) f_\xi(g_0) \overline{\psi_{r-1}(v) \omega_\psi(\ell(v) g_0) \phi(0)} \mathrm{d}v \mathrm{d}g_0,$$

where $\ell : N_r \to H$ is the natural projection whose kernel is N_{r-1}. We make a change of variable $v \mapsto h(l_0^*, 0)^{-1} v$ and get

$$\begin{aligned}
\zeta(\Xi, \xi, \psi) &= \int_{G_0} \int_{N_r} f_\Xi(w_{2,\mathrm{long}} v g_0) f_\xi(g_0) \overline{\psi_{r-1}(v) \omega_\psi(h(l_0^*, 0)^{-1} \ell(v) g_0) \phi(0)} \mathrm{d}v \mathrm{d}g_0 \\
&= \int_{G_0} \int_{N_r} f_\Xi(w_{2,\mathrm{long}} g_0 v) f_\xi(g_0) \overline{\psi_{r-1}(v) \omega_\psi(h(l_0^*, 0)^{-1} g_0 \ell(v)) \phi(0)} \mathrm{d}v \mathrm{d}g_0,
\end{aligned}$$

where in the second equality we made a chance of variable $v \mapsto g_0 v g_0^{-1}$ and used the fact that $\psi_{r-1}(g_0 v g_0^{-1}) = \psi_{r-1}(v)$.

Let N_R be the unipotent radical of the upper triangular Borel subgroup of GL_r and

$$f_{W_\tau, \Xi^0}(g) = \int_{N_R} f_\Xi(\mathsf{w}_r n g) \overline{\psi_r(n)} \mathrm{d}n, \quad g \in G_2.$$

Then by the Casselman–Shalika formula, we have

$$f_{W_\tau, \Xi^0}(1) = \prod_{1 \leq i < j \leq r} \frac{1}{L(1, \Xi_{n+i} \Xi_{n+j}^{-1})}.$$

We can then write the integral (4.4.1) as

$$\prod_{1 \leq i < j \leq r} \frac{1}{L(1, \Xi_{n+i} \Xi_{n+j}^{-1})} \times \int_{N_R \backslash N_r} \int_{G_0} f_{W_\tau, \Xi^0}(w_{0,\mathrm{long}} g_0 \ddot{w} v) f_\xi(g_0) \overline{\omega(h(l_0^*, 0)^{-1} g_0 \ell(v)) \phi(0)} \mathrm{d}v \mathrm{d}g_0,$$

where $\ddot{w} = \begin{pmatrix} & & 1_r \\ & 1_{2n} & \\ -1_r & & \end{pmatrix}$. We make a change of variable $g \mapsto w_{0,\mathrm{long}}^{-1} g w_{0,\mathrm{long}}$ and $v \mapsto w_{0,\mathrm{long}}^{-1} v w_{0,\mathrm{long}}$. Then since $w_{0,\mathrm{long}} \in K_0$ and f_{W_τ, Ξ^0}, f_ξ, ϕ are all K_0-fixed, we conclude that

$$\begin{aligned}
\zeta(\Xi, \xi, \psi) = &\prod_{1 \leq i < j \leq r} \frac{1}{L(1, \Xi_{n+i} \Xi_{n+j}^{-1})} \\
&\times \int_{N_R \backslash N_r} \int_{G_0} f_{W_\tau, \Xi^0}(g \ddot{w} v) f_\xi(w_{0,\mathrm{long}} g) \overline{\omega(w_{0,\mathrm{long}} h(l_0^*, 0) g \ell(v)) \phi(0)} \mathrm{d}v \mathrm{d}g.
\end{aligned}$$

By definition,

$$\zeta(\Xi^0, \xi, \psi) = \int_{G_0} f_{\Xi^0}(g) f_\xi(w_{0,\mathrm{long}} g) \overline{\omega_\psi(w_{0,\mathrm{long}} h(l_0^*, 0) g) \phi(0)} \mathrm{d}g.$$

We then apply [14, Theorem 4.3] and [14, End of Section 4, (4.7)] to get the lemma. (In the notation of [14], we apply this to the case $r = 0$ and $b_\nu(f_{\Xi^0}, f_\xi, \phi) = \zeta(\Xi^0, \xi, \psi)$). $\square$

We now compute $S'_{\Xi^{-1},\xi^{-1},\psi^{-1}}(1)$. Define the projection $\mathrm{pr}_2 : \mathcal{C}_c^\infty(G_2) \to I(\Xi)$ by

$$\mathrm{pr}_2(F_2)(g_2) = \int_{B_2} F_2(b_2 g_2)(\Xi^{-1}\delta_2^{1/2})(b_2)\mathrm{d}b_2,$$

where the measure $\mathrm{d}b_2$ is the left invariant measure on B_2 so that $\mathrm{pr}_2(\mathbf{1}_{K_2}) = f_\Xi$. Then we define

$$l_{\Xi,\xi,\psi} \in \mathrm{Hom}_{N_{r-1}\rtimes J}(I(\Xi), I^J(\xi^{-1}, \psi) \otimes \psi_{r-1})$$

by

$$l_{\Xi,\xi,\psi}(f_2)(g_J) = \int_{G_2} f_2'(g_2 g_J) Y_{\Xi,\xi,\psi}(g_2)\mathrm{d}g_2,$$

where f_2' is any element in $\mathcal{C}_c^\infty(G_2)$ with $\mathrm{pr}_2(f_2') = f_2$. It is not hard to check that $l_{\Xi,\xi,\psi}$ is independent of the choice of f_2'. We define

$$S_{\Xi,\xi,\psi}(g_2) = \mathcal{B}_{I^J(\xi,\overline{\psi})}(f_{\xi,\overline{\psi}}, l_{\Xi,\xi,\psi}(\pi_2(g_2) f_\Xi)).$$

The defining integral of $l_{\Xi,\xi,\psi}$ is convergent if $Y_{\Xi,\xi,\psi}$ is continuous. By [42, Section 3], whose method is valid for both cases Mp and Sp, $Y_{\Xi,\xi,\psi}$ is continuous if (Ξ, ξ) lie in some (nonempty) open subset of $\mathbb{C}^{r+s} \times \mathbb{C}^s$. We refer the readers to [42, Section 3] for a precise description of this open subset.

Lemma 4.4.2. $S'_{\Xi,\xi,\psi} = S_{\Xi,\xi,\psi}$.

Proof. We check that $S_{\Xi,\xi,\psi}$ and $S'_{\Xi,\xi,\psi}$ agree when $Y_{\Xi,\xi,\psi}$ is continuous. We divide the proof into two steps.

Step 1. The goal is to reduce the Lemma to the identity (4.4.2).

Let $\Xi^1 = (\Xi_1, \cdots, \Xi_{n+1})$ and $I(\Xi^1)$ be the unramified principal series representation of G_1 defined by the character Ξ^1. Let $\mathcal{F}'_\psi(f_2)(g_2) := \mathcal{F}_\psi(f_2)(g_2\dot{w})$. Then $\mathcal{F}'_\psi(f_2)|_{G_1} \in I(\Xi^1)$. Define the projection $\mathrm{pr}_1 : \mathcal{C}_c^\infty(G_1) \to I(\Xi^1)$ by

$$\mathrm{pr}_1(F)(g_1) = \int_{B_1} F(b_1 g_1)((\Xi^1)^{-1}\delta_1^{1/2})(b_1)\mathrm{d}b_1,$$

where the left invariant measure $\mathrm{d}b_1$ is the one so that $\mathrm{pr}_1(\mathbf{1}_{K_1}) = f_{\Xi^1}$. Note that pr_1 is surjective and for any element $f \in I(\Xi^1)$, one can choose F whose support lies in K_1 such that $\mathrm{pr}_1(F) = f$.

Define the intertwining operator $l'_{\Xi,\xi,\psi} \in \mathrm{Hom}_{N_{r-1}\rtimes J}(I(\Xi), I^J(\xi^{-1}, \psi) \otimes \psi_{r-1})$ by

$$l'_{\Xi,\xi,\psi}(f_2)(g_J) = \int_{G_1} f_2''(g_1 g_J) Y_{\Xi,\xi,\psi}(g_1\dot{w})\mathrm{d}g_1,$$

where f_2'' is any element in $\mathcal{C}_c^\infty(G_1)$ with $\mathrm{pr}_1(f_2'') = \mathcal{F}'_\psi(f_2)|_{G_1}$.

Fix $g_2 \in G_2$ and let $f_2'' \in \mathcal{C}_c^\infty(G_1)$ be a smooth function whose support is contained in K_1 and $\mathrm{pr}_1(f_2'') = \mathcal{F}'_\psi(\pi_2(g_2)f_\Xi)|_{G_1}$. Then

$$\begin{aligned}
S'_{\Xi,\xi,\psi}(g_2) &= \mathbf{w}^{-1}\int_{K_1}\int_{K_J} Y_{\Xi,\xi,\psi}(k_1\dot{w}k_J)\mathcal{F}'_\psi(\pi_2(g_2)f_\Xi)(k_1)\mathrm{d}k_J\mathrm{d}k_1 \\
&= \mathbf{w}^{-1}\int_{K_1}\int_{K_J} Y_{\Xi,\xi,\psi}(k_1\dot{w}k_J)f_2''(k_1)\mathrm{d}k_J\mathrm{d}k_1 \\
&= \mathbf{w}^{-1}\int_{K_1}\int_{K_J} Y_{\Xi,\xi,\psi}(k_1\dot{w})f_2''(k_1k_J)\mathrm{d}k_J\mathrm{d}k_1 \\
&= \mathbf{w}^{-1}\mathcal{B}_{I^J(\xi,\overline{\psi})}(f_{\xi,\overline{\psi}}, l'_{\Xi,\xi,\psi}(\pi_2(g_2)f_\Xi)).
\end{aligned}$$

Therefore in order to prove the lemma, we only need to show $\mathbf{w}\cdot l_{\Xi,\xi,\psi} = l'_{\Xi,\xi,\psi}$. We have

$$\dim \mathrm{Hom}_{N_{r-1}\rtimes J}(I(\Xi), I^J(\xi^{-1},\psi)\otimes\psi_{r-1}) = 1.$$

This is proved in [42] in the case Sp, but the proof works equally well in the case Mp as it uses only the decomposition $G_i = B_iK_i$. Therefore we only have to find a function $\varphi \in I(\Xi)$ such that $l_{\Xi,\xi,\psi}(\varphi)(1) \neq 0$ and show that

$$l'_{\Xi,\xi,\psi}(\varphi)(1)/l_{\Xi,\xi,\psi}(\varphi)(1) = \mathbf{w}. \tag{4.4.2}$$

Step 2. Proof of (4.4.2).

Let $K_i^{(1)}$ be the Iwahori subgroup of K_i. Let $T_i^{(0)} = T_i(\mathfrak{o}_F)$ and $T_i^{(1)}$ be the kernel of the reduction map $T_i^{(0)} \to T_i(\mathfrak{o}_F/\varpi)$. Note here that by T_i, we mean the diagonal torus of $\mathrm{Sp}(W_i)$ in both cases Sp and Mp. Let $\overline{B}_i$ be the opposite Borel subgroup of G_i and $\overline{N}_i$ be its unipotent radical. Let $N_i^{(0)} = N_i \cap K_i$, $\overline{N}_i^{(1)} = \overline{N}_i \cap K_i^{(1)}$ and $N_i^{(1)} = w_{i,\mathrm{long}}^{-1}\overline{N}_i^{(1)}w_{i,\mathrm{long}}$. Let $N_{r-1}^{(1)} = N_{r-1}\cap N_2^{(1)}$. Note that in the case Mp, these subgroups of K_i are considered as subgroups of G_i via the splitting $K_i \to G_i$.

Let $\varphi = \mathrm{pr}_2(\mathbf{1}_{K_2^{(1)}\eta}) \in \mathcal{C}_c^\infty(G_2)$. Then

$$l_{\Xi,\xi,\psi}(\mathbf{1}_{K_2^{(1)}\eta})(1) = \int_{K_2^{(1)}} Y_{\Xi,\xi,\psi}(k_2\eta)\mathrm{d}k_2.$$

Recall that $l_0^* = (1,\cdots,1) \in L^*$ and $\eta = w_{2,\mathrm{long}}h(l_0^*,0)$. By the Iwahori decomposition of $K_2^{(1)}$, it is not hard to check that

$$K_2^{(1)}\eta = T_2^{(0)}N_2^{(0)}w_{2,\mathrm{long}}h(l_0^*,0)T_0^{(1)}N_J^{(1)}N_{r-1}^{(1)}. \tag{4.4.3}$$

Therefore $Y_{\Xi,\xi,\psi}(k_2\eta) = Y_{\Xi,\xi,\psi}(\eta) = 1$ for any $k_2 \in K_2^{(1)}$. Thus

$$l_{\Xi,\xi,\psi}(\mathbf{1}_{K_2^{(1)}\eta})(1) = \mathrm{vol}\, K_2^{(1)}.$$

We now compute $l'_{\Xi,\xi,\psi}(\mathrm{pr}_2(\mathbf{1}_{K_2^{(1)}\eta}))(1)$. First

$$\mathcal{F}'_\psi(\mathrm{pr}_2(\mathbf{1}_{K_2^{(1)}\eta}))(g_1) = \int_{N_{r-1}} \int_{B_2} \mathbf{1}_{K_2^{(1)}\eta}(b_2 g_1 \dot{w} u)(\Xi^{-1}\delta_2^{1/2})(b_2)\overline{\psi_{r-1}(u)}\mathrm{d}b_2\mathrm{d}u, \quad g_1 \in G_1.$$

By the decomposition (4.4.3) again, for any $u \in N_{r-1}$, if $b_2 g_1 \dot{w} u \in K_2^{(1)}\eta$, then $u \in N_{r-1}^{(1)}$ and $b_2 g_1 \dot{w} \in K_2^{(1)}\eta$. Therefore

$$\begin{aligned}\mathcal{F}'_\psi(\mathrm{pr}_2(\mathbf{1}_{K_2^{(1)}\eta}))(g_1) &= \mathrm{vol}\, N_{r-1}^{(1)} \cdot \int_{B_2} \mathbf{1}_{K_2^{(1)}\eta}(b_2 g_1 \dot{w})(\Xi^{-1}\delta_2^{1/2})(b_2)\mathrm{d}b_2 \\ &= \mathrm{vol}\, N_{r-1}^{(1)} \cdot \int_{B_1} \mathbf{1}_{K_2^{(1)}\eta}(b_1 g_1 \dot{w})((\Xi^1)^{-1}\delta_2^{1/2})(b_1)\mathrm{d}b_1\end{aligned}$$

Thus

$$\begin{aligned}l'_{\Xi,\xi,\psi}(\mathrm{pr}_2(\mathbf{1}_{K_2^{(1)}\eta}))(1) &= \mathrm{vol}\, N_{r-1}^{(1)} \cdot \int_{G_1} \mathbf{1}_{K_2^{(1)}\eta}(g_1\dot{w}) Y_{\Xi^1,\xi,\psi}(g_1\dot{w})\mathrm{d}g_1 \\ &= \mathrm{vol}\, N_{r-1}^{(1)} \cdot \mathrm{vol}\, K_1^{(1)}.\end{aligned}$$

The lemma then follows since $\mathrm{vol}\, N_{r-1}^{(1)} \cdot \mathrm{vol}\, K_1^{(1)} = \mathbf{w}\,\mathrm{vol}\, K_2^{(1)}$. □

Lemma 4.4.3. *We have*

$$S_{\Xi,\xi,\psi}(1) = \frac{\Delta_{G_2}}{\Delta_{T_2}\Delta_{T_0}}\zeta(\Xi,\xi,\psi), \quad S_{\Xi^0,\xi,\psi}(1) = \frac{\Delta_{G_0}}{\Delta_{T_0}^2}\zeta(\Xi^0,\xi,\psi).$$

Proof. We claim that the restriction of the measure $\mathrm{d}g$ to the open subset $B_2\eta B_J N_{r-1}$ decomposes as

$$\mathrm{d}g|_{B_2\eta B_J N_{r-1}} = \frac{\Delta_{G_2}}{\Delta_{T_2}\Delta_{T_0}}\mathrm{d}b_2\mathrm{d}n_{r-1}\mathrm{d}b_J,$$

where $\mathrm{d}b_J = \mathrm{d}b_0\mathrm{d}l\mathrm{d}t$ if $b_J = b_0 h(l,t)$. In fact, on the one hand,

$$\int_{G_2} \mathbf{1}_{K_2^{(1)}\eta}(g)\mathrm{d}g = [K_2 : K_2^{(1)}]^{-1} = q^{-\dim G_2 + \dim N_2 + \dim T_2}\frac{\Delta_{G_2}}{\Delta_{T_2}}.$$

On the other hand, it follows from (4.4.3) that

$$\int_{B_2}\int_{N_{r-1}}\int_{B_J} \mathbf{1}_{K_2^{(1)}\eta}(b_2\eta b_J n_{r-1})\mathrm{d}b_2\mathrm{d}n_{r-1}\mathrm{d}b_J = q^{-\dim T_0 - \dim N_J - \dim N_{r-1}}\Delta_{T_0}.$$

The claim then follows. Therefore

$$l_{\Xi,\xi,\psi}(f_\Xi)(g_J) = \frac{\Delta_{G_2}}{\Delta_{T_2}\Delta_{T_0}}\int_{B_J}\int_{N_{r-1}} f_\Xi(\eta b_J n_{r-1} g_J)(\xi\delta_J^{-\frac{1}{2}})(b_J)\overline{\psi_{r-1}(n_{r-1})}\mathrm{d}b_J\mathrm{d}n_{r-1}.$$

We have

$$\begin{aligned}S_{\Xi,\xi,\psi}(1) = \frac{\Delta_{G_2}}{\Delta_{T_2}\Delta_{T_0}}\int_{L^*}\int_{K_0}\int_{B_J}\int_{N_{r-1}} & f_\Xi(w_{2,\mathrm{long}}h(l_0^*,0)b_J n_{r-1} h(l^*,0)k) \\ & (\xi^{-1}\delta_J^{\frac{1}{2}})(b_J)\overline{\psi_{r-1}(n_{r-1})} f_{\xi,\overline{\psi}}(h(l^*,0)k)\mathrm{d}n_{r-1}\mathrm{d}b_J\mathrm{d}k\mathrm{d}l^*.\end{aligned}$$

We combine the integration over L, K_0 and B_J as an integral over J and then conclude that

$$\begin{aligned}S_{\Xi,\xi,\psi}(1) =&\frac{\Delta_{G_2}}{\Delta_{T_2}\Delta_{T_0}}\int_J\int_{N_{r-1}} f_\Xi(w_{2,\mathrm{long}}h(l_0^*,0)n_{r-1}g_J)\overline{\psi_{r-1}(n_{r-1})}f_{\xi,\overline{\psi}}(g_J)\mathrm{d}n_{r-1}\mathrm{d}g_J\\ =&\frac{\Delta_{G_2}}{\Delta_{T_2}\Delta_{T_0}}\zeta(\Xi,\xi,\psi).\end{aligned}$$

The equality

$$S_{\Xi^0,\xi,\psi}(1) = \frac{\Delta_{G_0}}{\Delta_{T_0}^2}\zeta(\Xi^0,\xi,\psi)$$

can be proved similarly. In fact,

$$\mathrm{d}g_J|_{B_J\eta B_0} = \frac{\Delta_{G_0}}{\Delta_{T_0}^2}\mathrm{d}b_J\mathrm{d}b_0.$$

Therefore

$$\begin{aligned}S_{\Xi^0,\xi,\psi}(1) &= \int_J\int_{G_0}\mathbf{1}_{K_J}(g_J)\,\mathbf{1}_{K_0}(g_0)Y_{\Xi^0,\xi,\psi}(g_Jg_0^{-1})\mathrm{d}g_J\mathrm{d}g_0\\ &= \frac{\Delta_{G_0}}{\Delta_{T_0}^2}\int_{G_0}\int_{B_J}\int_{B_0}\mathbf{1}_{K_J}(b_J\eta b_0g_0)\,\mathbf{1}_{K_0}(g_0)Y_{\Xi^0,\xi,\psi}(b_J\eta b_0)\mathrm{d}b_J\mathrm{d}b_0\mathrm{d}g_0\\ &= \frac{\Delta_{G_0}}{\Delta_{T_0}^2}\zeta(\Xi^0,\xi,\psi).\end{aligned}$$

□

Proof of Proposition 2.2.3. If $r = 0$, then Proposition 2.2.3 can be proved in exactly the same way as [50, Appendix D.3]. We omit the details. See also Lemma 7.2.2.

Assume that $r \geq 1$. Suppose that we are in the case Sp. It follows from Lemma 4.4.1 and Lemma 4.4.3 that

$$\begin{aligned}I(1,\Xi,\xi,\psi) = \left(\frac{\Delta_{T_2}}{\Delta_{G_2}}\right)^{-1}\left(\frac{\Delta_{T_0}}{\Delta_{G_0}}\right)&\left(\frac{L_\psi(\frac{1}{2},\pi_0\times\tau)}{L(1,\sigma\times\tau)L(1,\tau,\wedge^2)}\prod_{1\leq i<j\leq r}\frac{1}{L(1,\Xi_{n+i}\Xi_{n+j}^{-1})}\right)\\ &\left(\frac{L_\psi(\frac{1}{2},\pi_0^\vee\times\tau^\vee)}{L(1,\sigma^\vee\times\tau^\vee)L(1,\tau^\vee,\wedge^2)}\prod_{1\leq i<j\leq r}\frac{1}{L(1,\Xi_{n+i}^{-1}\Xi_{n+j})}\right)I(1,\xi,\Xi^0,\psi).\end{aligned}$$

Proposition 2.2.3 in the case $r \geq 1$ is then reduced to the case $r = 0$. The case Mp can be proved in the same way. We only need to change notation at all necessary places. □

Part 2. Compatibility with Ichino–Ikeda's conjecture

The notation in this part of the paper is independent from Part I. We keep the notation and convention in the Introduction. Additional notation will be fixed in each section.

5. Some assumptions and remarks

5.1. **Parameters.** We will prove that Conjecture 2.3.1(3) is compatible with Ichino–Ikeda's conjecture [24, Conjecture 2.1]. The most subtle part is the appearance of the size of the centralizer of the global L-parameters in the formula. To address this issue, of course, one has to assume that the Langlands correspondence exists and satisfies some expected properties. We begin by setting down the precise hypotheses that we require. We remark that for orthogonal groups and symplectic groups, they follow from the work of Arthur [2] and the recent work of Atobe–Gan [4]. For metaplectic groups, they should eventually follow from the on-going work of Wen-Wei Li (e.g. [34]).

We first state the hypothesis on the local Langlands correspondences.

Hypothesis LLC. We assume the Hypothesis (LLC), (Local factors), (Plancherel measures) from [9, Appendix C] at all non-archimedean places v of F. Thus [9, Theorem C.5] holds if v is non-archimedean. It also holds if v is archimedean by [38].

We note that if v is an archimedean place, then the Hypothesis (LLC) is established by Langlands [28]. Hypothesis (Local factors) is proved in [33]. Hypothesis (Plancherel measures) is proved by [1]. If v is non-archimedean, then they should follow from [2, Theorem 1.5.1, Theorem 9.4.1, Conjecture 9.4.2].

Thus, if v is a place of F and π_v is an irreducible admissible representation of $G(F_v)$, where $G = \mathrm{SO}(2n+1)$ (resp. $\mathrm{SO}(2n)$, resp. $\mathrm{Sp}(2n)$) gives rise to a $2n$ (resp. $2n$, resp. $2n+1$) dimensional selfdual representation Ψ_{π_v} of the Weil–Deligne group $\mathrm{WD}(F_v)$ of sign -1 (resp. $+1$, resp. $+1$). We call it the local L-parameter of π_v.

Let π_v be an irreducible admissible genuine representation of $\mathrm{Mp}(2n)(F_v)$ and $\Theta_{\psi_v}(\pi_v)$ be the restriction to $\mathrm{SO}(V)(F_v)$ of its theta lift to $\mathrm{O}(V)(F_v)$ where V is a $2n+1$ dimensional orthogonal space over F_v of discriminant 1. By [9, Theorem 1.1], the map $\pi_v \mapsto \Theta_{\psi_v}(\pi_v)$ gives a bijection between the set of irreducible admissible genuine representations of $\mathrm{Mp}(2n)(F_v)$ and the union of the sets of irreducible admissible representations of $\mathrm{SO}(V)(F_v)$ where V ranges over all $2n+1$ dimensional orthogonal spaces over F_v of discriminant 1. This bijection satisfies several expected properties (c.f. [9, Theorem 1.3] for a list). The local L-parameter of π_v is defined to be $\Psi_{\Theta_{\psi_v}(\pi_v)}$. Note that the local L-parameter of π_v depends on ψ_v.

We now turn to the global Langlands correspondences. We shall be concerned only with tempered cuspidal automorphic representations. To avoid mentioning the hypothetical Langlands group L_F, we use the following substitute of the global L-parameters following [2, Section 1.4] and [8, Section 25, p. 103–105].

Let π be an irreducible cuspidal tempered automorphic representation of $G(\mathbb{A}_F)$, where $G = \mathrm{SO}(2n+1)$ (resp. $\mathrm{SO}(2n)$, $\mathrm{Sp}(2n)$, $\mathrm{Mp}(2n)$). By the global L-parameter of π, we mean the following data:

- a partition $N = N_1 + \cdots + N_r$, where $N = 2n$ (resp. $2n$, $2n+1$, $2n$);
- a collection of pairwisely inequivalent selfdual irreducible cuspidal automorphic representations Π_i of $\mathrm{GL}_{N_i}(\mathbb{A}_F)$ of sign -1 (resp. $+1$, $+1$, -1), $i = 1, \cdots, r$,

which satisfy the condition that for all places v of F, $\Psi_{\pi_v} \simeq \oplus_{i=1}^r \Psi_{\Pi_{i,v}}$ as representations of $\mathrm{WD}(F_v)$, where $\Psi_{\Pi_{i,v}}$ is an N_i dimensional representation of $\mathrm{WD}(F_v)$ associated to $\Pi_{i,v}$ by the local Langlands correspondences for GL_{N_i} (which is known due to [19] and [21]). By [26, Theorem 4.4], the global L-parameter of π is unique if it exists. We write formally $\Psi_\pi = \boxplus_{i=1}^r \Pi_i$.

We now state the hypothesis on the global Langlands correspondences.

Hypothesis GLC. The global L-parameter of π exists.

For orthogonal and symplectic groups, a weaker version of this (namely, replacing the requirement "for all places v" by "for almost all places v") follows from [2, Theorem 1.5.2, Theorem 9.5.3]. For metaplectic groups, this should follow from the work of Wen-Wei Li.

With this reformulation of the L-parameter of π, we (re-)define the centralizer

$$S_\pi = S_{\Psi_\pi} = \{(a_i) \in (\mathbb{Z}/2\mathbb{Z})^r \mid a_1^{N_1} \cdots a_r^{N_r} = 1\}.$$

From now on, when we speak of the global L-parameters and their centralizers, we always mean the one defined here.

We end this subsection by some discussions on the automorphic representations on the even orthogonal groups. Suppose that π is an irreducible cuspidal tempered automorphic representation of $\mathrm{O}(2n)(\mathbb{A}_F)$. We are interested in the restriction $\pi|_{\mathrm{SO}(2n)(\mathbb{A}_F)}$. Here by $\pi|_{\mathrm{SO}(2n)(\mathbb{A}_F)}$, we mean the following. Suppose that π is realized on V, which is a subspace of the cuspidal automorphic spectrum of $\mathrm{O}(2n)(\mathbb{A}_F)$. Let $V^0 = \{f|_{\mathrm{SO}(2n)(\mathbb{A}_F)} \mid f \in V\}$. Then $\pi|_{\mathrm{SO}(2n)(\mathbb{A}_F)}$ stands for the natural action of $\mathrm{SO}(2n)(\mathbb{A}_F)$ on V^0. We summarize some recent results of Atobe–Gan [4] as the following Hypothesis O.

Hypothesis O. Each tempered automorphic representation π appears with multiplicity one in the discrete spectrum of $\mathrm{O}(2n)(\mathbb{A}_F)$. The following three cases exhaust all possibilities of $\pi|_{\mathrm{SO}(2n)(\mathbb{A}_F)}$.

(1) $\pi|_{\mathrm{SO}(2n)(\mathbb{A}_F)}$ is irreducible and appears with multiplicity one in the discrete spectrum of $\mathrm{SO}(2n)(\mathbb{A}_F)$.

(2) $\pi|_{\mathrm{SO}(2n)(\mathbb{A}_F)}$ is irreducible and appears with multiplicity two in the discrete spectrum of $\mathrm{SO}(2n)(\mathbb{A}_F)$. In this case, there is an automorphic representation π' of $\mathrm{O}(2n)(\mathbb{A}_F)$ such that $\pi \neq \pi'$ and $\pi|_{\mathrm{SO}(2n)(\mathbb{A}_F)} \oplus \pi'|_{\mathrm{SO}(2n)(\mathbb{A}_F)}$ is the $\pi|_{\mathrm{SO}(2n)(\mathbb{A}_F)}$-isotypic component of the discrete spectrum of $\mathrm{SO}(2n)(\mathbb{A}_F)$. Note that π' is not uniquely determined.

(3) $\pi|_{\mathrm{SO}(2n)(\mathbb{A}_F)} = \pi^+ \oplus \pi^-$ where π^+ and π^- are inequivalent automorphic representations of $\mathrm{SO}(2n)(\mathbb{A}_F)$. Both π^+ and π^- appear with multiplicity one in the discrete spectrum of $\mathrm{SO}(2n)(\mathbb{A}_F)$. Moreover, $\Psi_{\pi^+} = \Psi_{\pi^-}$.

In each case, let π^0 be an irreducible component of $\pi|_{\mathrm{SO}(2n)(\mathbb{A}_F)}$. Then we define the L-parameter Ψ_π of π by $\Psi_\pi = \Psi_{\pi^0}$. Suppose that $\Psi_\pi = \Pi_1 \boxplus \cdots \boxplus \Pi_r$ where Π_i is an irreducible cuspidal automorphic representation of $\mathrm{GL}_{N_i}(\mathbb{A}_F)$. Then in the first (resp. second and third) case (resp. cases), at least one of N_i's is odd (resp. all N_i's are even).

Let $\epsilon \in \mathrm{O}(2n)(F) \setminus \mathrm{SO}(2n)(F)$. Conjugation by ϵ induces an outer automorphism of order two of $\mathrm{SO}(2n)$ which does not depend on the choice of the element ϵ. We denote this outer automorphism also by ϵ. If $n \neq 2$, then this is the unique nontrivial outer automorphism of $\mathrm{SO}(2n)$. For any automorphic representation σ of $\mathrm{SO}(2n)(\mathbb{A}_F)$, we let σ^ϵ be its twist by ϵ. In the first two cases, $(\pi|_{\mathrm{SO}(2n)(\mathbb{A}_F)})^\epsilon = \pi|_{\mathrm{SO}(2n)(\mathbb{A}_F)}$. In the third case, $(\pi^\pm)^\epsilon = \pi^\mp$. Here we use "=" to indicate that not only the automorphic representations are isomorphic, but the spaces on which they realize are the same.

The automorphic representation π appears with multiplicity one in the discrete spectrum of $\mathrm{O}(2n)(\mathbb{A}_F)$, so the space on which it realizes is canonical. Suppose that $\pi|_{\mathrm{SO}(2n)(\mathbb{A}_F)}$ is irreducible and appears with multiplicity two in the discrete spectrum of $\mathrm{SO}(2n)(\mathbb{A}_F)$. The restrictions of π and π' to $\mathrm{SO}(2n)(\mathbb{A}_F)$ are canonical subspaces of the discrete spectrum of $\mathrm{SO}(2n)(\mathbb{A}_F)$ and give a canonical decomposition of the $\pi|_{\mathrm{SO}(2n)(\mathbb{A}_F)}$-isotypic component of the discrete spectrum of $\mathrm{SO}(2n)(\mathbb{A}_F)$ (we are not able to distinguish the restrictions of π and π'). Moreover, these subspaces are characterized by the fact that they are invariant under the outer twist ϵ. In other words, if π^0 (as an abstract representation) is an automorphic representation of $\mathrm{SO}(2n)(\mathbb{A}_F)$ and appears with multiplicity two in the discrete spectrum of $\mathrm{SO}(2n)(\mathbb{A}_F)$, then there are precisely two automorphic realizations V_1 and V_2 of π^0 that are invariant under the outer twist by ϵ. Both V_1 and V_2 can be extended to automorphic representations of of $\mathrm{O}(2n)(\mathbb{A}_F)$. Moreover, V_1 and V_2 are orthogonal in the discrete spectrum of $\mathrm{SO}(2n)(\mathbb{A}_F)$ and $V_1 \oplus V_2$ is the π^0-isotypic component of the discrete spectrum of $\mathrm{SO}(2n)(\mathbb{A}_F)$.

Finally, assume that $\mathrm{SO}(2n)$ is quasi-split and π^0 is an irreducible cuspidal tempered generic automorphic representation of $\mathrm{SO}(2n)(\mathbb{A}_F)$ which appears with multiplicity two in the discrete spectrum. Suppose that $\Psi_{\pi^0} = \Pi_1 \boxplus \cdots \boxplus \Pi_r$. Then (at least conjecturally) the descent construction [15] provides us with an automorphic realization of π^0 which is invariant under the outer twist ϵ. We refer the readers to [30, Section 5] for some further discussions on the descent construction.

Convention. We assume the hypotheses LLC, GLC and O from now on, unless otherwise specified.

5.2. **Theta correspondences.** We are going to use the Rallis inner product formula in the later sections of this paper. We will not recall the precise form of this formula in various cases, but refer the readers to [52, 53] for the formula in the first term range and to [12] for the formula in the second term range.

We now consider the behavior of the L-parameters under theta correspondences.

Lemma 5.2.1. *Let V be a $2n$ dimensional orthogonal space over F and π an irreducible cuspidal tempered automorphic representation of $\mathrm{O}(V)(\mathbb{A}_F)$. Let $\Theta_\psi(\pi)$ be its theta lift to $\mathrm{Sp}(2n)(\mathbb{A}_F)$ with additive character ψ. Suppose that $\Theta_\psi(\pi)$ is nonzero and cuspidal. Let $\Psi_\pi = \boxplus_{i=1}^r \Pi_i$ be the L-parameter of π. Then $\Pi_i \neq \mathbf{1}$ (the trivial character of $\mathbb{A}_F^\times$) for all i.*

Proof. Suppose that $\Pi_i = \mathbf{1}$ for some i. We may assume that $i = 1$. Then by Hypothesis O, $\pi|_{\mathrm{SO}(V)(\mathbb{A}_F)}$ is irreducible. We prove that π has a nonzero theta lift to $\mathrm{Sp}(2n-2)(\mathbb{A}_F)$. The lemma then follows from the tower property of the theta lift [40].

If π has a nonzero theta lift to $\mathrm{Sp}(2n-2r)(\mathbb{A}_F)$ for some $r > 1$, then by the tower property of the theta lift, π has a nonzero theta lift to $\mathrm{Sp}(2n-2)(\mathbb{A}_F)$. Thus we may assume that π does not have a nonzero theta lift to any $\mathrm{Sp}(2n-2r)(\mathbb{A}_F)$ for any $r > 1$.

We fix a sufficiently large finite set S of places of F which contains all the archimedean places, so that if $v \notin S$, then π (hence Π_i) is unramified. By the hypotheses LLC and GLC,

$$L^S(s, \pi) = \prod_{i=1}^r L^S(s, \Pi_i),$$

where the left hand side is the standard L-function of π defined by the doubling method and the right hand side is the standard L-function of Π_i. If $i \neq 1$, then $L^S(s, \Pi_i)$ is holomorphic and does not vanish at $s = 1$ [27] and $L^S(s, \mathbf{1})$ have a simple pole at $s = 1$. Therefore $L^S(s, \pi)$ has a simple pole at $s = 1$.

Let v be a place of F. By assumption, $\pi_v|_{\mathrm{SO}(V)(F_v)}$ is irreducible. By [9, Theorem C.5], there is an irreducible admissible representation σ of $\mathrm{Sp}(2n-2)(F_v)$ such that $\pi_v = \Theta_{\psi_v}(\sigma)$. This means that π_v has a nonzero theta lift to $\mathrm{Sp}(2n-2)(F_v)$.

It then follows from [53, Theorem 10.1] that π has a nonzero theta lift to $\mathrm{Sp}(2n-2)(\mathbb{A}_F)$. This proves the lemma. □

Lemma 5.2.2. *Let V be a $2n+1$ (resp. $2n$) dimensional orthogonal space over F and π be an irreducible cuspidal tempered automorphic representation of $\mathrm{O}(V)(\mathbb{A}_F)$. Let $\Theta_\psi(\pi)$ be its theta lift to $\mathrm{Mp}(2n)(\mathbb{A}_F)$ (resp. $\mathrm{Sp}(2n)(\mathbb{A}_F)$) with additive character ψ. Assume that $\Theta_\psi(\pi)$ is cuspidal and nonzero. Then*

$$\Psi_{\Theta_\psi(\pi)} = \Psi_\pi \otimes \chi_V, \quad \textit{resp. } \Psi_{\Theta_\psi(\pi)} = (\Psi_\pi \boxplus \mathbf{1}) \otimes \chi_V,$$

where $\mathbf{1}$ stands for the trivial character of $\mathbb{A}_F^\times$.

Proof. Let v be a place of F. By [9, Theorem C.5] and [13], we see that

$$\Psi_{\Theta_{\psi_v}(\pi_v)} = \Psi_{\pi_v} \otimes \chi_{V,v}, \quad \text{resp. } \Psi_{\Theta_{\psi_v}(\pi_v)} = (\Psi_{\pi_v} \oplus \mathbf{1}_v) \otimes \chi_{V,v},$$

By the previous lemma, in the case $\dim V = 2n$, Ψ_π does not contain $\mathbf{1}$. The lemma then follows from [26, Theorem 4.4]. □

Lemma 5.2.3. *Let π be an irreducible cuspidal tempered automorphic representation of $\mathrm{O}(V)(\mathbb{A}_F)$ where V is a $2n$ dimensional orthogonal space over F. There is a canonical injective map $S_\pi \to S_{\Theta_\psi(\pi)}$. It is not bijective if and only if $\Psi_\pi = \Pi_1 \boxplus \cdots \boxplus \Pi_r$ where Π_i is an irreducible cuspidal automorphic representation of $\mathrm{GL}_{N_i}(\mathbb{A}_F)$ with N_i being even. In this case, S_π is an index two subgroup of $S_{\Theta_\psi(\pi)}$.*

Proof. Suppose $\Psi_\pi = \boxplus_{i=1}^r \Pi_i$, where Π_i is an irreducible cuspidal automorphic representation of $\mathrm{GL}_{N_i}(\mathbb{A}_F)$ and $\sum_{i=1}^r = 2n$. By Lemma 5.2.2,

$$S_\pi = \{(a_i) \in (\mathbb{Z}/2\mathbb{Z})^r \mid a_1^{N_1} \cdots a_r^{N_r} = 1\}, \quad S_{\Theta_\psi(\pi)} = \{(a_i) \in (\mathbb{Z}/2\mathbb{Z})^{r+1} \mid a_1^{N_1} \cdots a_r^{N_r} a_{r+1} = 1\}.$$

The map $(a_1, \cdots, a_r) \mapsto (a_1, \cdots, a_r, 1)$ is clearly injective. It is not bijective if and only if there are elements $(a_1, \cdots, a_r) \in (\mathbb{Z}/2\mathbb{Z})^r$ so that $a_1^{N_1} \cdots a_r^{N_r} = -1$. This is equivalent to that at least one of N_i's is odd. □

6. Ichino–Ikeda's conjecture for the full orthogonal group

We review in this section the conjecture of Ichino–Ikeda [24] and extend it to the full orthogonal group. There are minor inaccuracies in the formulation of the conjecture in [24] when the automorphic representation on the even orthogonal group appears with multiplicity two in the discrete automorphic spectrum. We will take care of this issue in Subsection 6.2. The Ichino–Ikeda's conjecture for the full orthogonal groups is stated in Subsection 6.3. We will show that it follows from the Ichino–Ikeda's conjecture for the special orthogonal groups. The argument is close to [10, § 2, 3] at various points. We give details on the new difficulties that arise in our situation (mainly due to the failure of multiplicity one in the discrete automorphic spectrum) and only state the result when its proof is identical to that in [10].

6.1. **Inner products.** Let F be a number field and (U, q_U) be an n-dimensional orthogonal group over F. Let $H = \mathrm{O}(U)$ and $H^0 = \mathrm{SO}(U)$. Recall that there is an exact sequence

$$1 \to H^0 \to H \to \mu_2 \to 1.$$

We view μ_2 as an algebraic group over F. We write t for the nonidentity element in $\mu_2(F)$ and t_v its image in $\mu_2(F_v)$ for each place v of F. Note that if n is odd, then we may take $t = -1$. The sequence splits canonically and gives an isomorphism $H \simeq H^0 \times \mu_2$.

Let $\mathrm{d}\epsilon_v$ be the measure on $\mu_2(F_v)$ so that $\mathrm{vol}\, \mu_2(F_v) = 1$. Then $\mathrm{d}\epsilon = \prod_v \mathrm{d}\epsilon_v$ is the Tamagawa measure of $\mu_2(\mathbb{A}_F)$. Let Z be the center of H^0. Note that the group Z is trivial unless $n = 2$.

Let $\mathrm{d}h$ and $\mathrm{d}h^0$ be the Tamagawa measure of $Z(\mathbb{A}_F)\backslash H(\mathbb{A}_F)$ and $Z(\mathbb{A}_F)\backslash H^0(\mathbb{A}_F)$ respectively. Then we have

$$\int_{Z(\mathbb{A}_F)H(F)\backslash H(\mathbb{A}_F)} f(h)\mathrm{d}h = \int_{\mu_2(F)\backslash\mu_2(\mathbb{A}_F)} \int_{Z(\mathbb{A}_F)H^0(F)\backslash H^0(\mathbb{A}_F)} f(h^0\epsilon)\mathrm{d}h^0\mathrm{d}\epsilon,$$

for all $f \in L^1(Z(\mathbb{A}_F)H(F)\backslash H(\mathbb{A}_F))$.

We fix a decomposition $\mathrm{d}h = \prod_v \mathrm{d}h_v$ where $\mathrm{d}h_v$ is a measure on $H(F_v)$. Let $\mathrm{d}h_v^0 = 2\mathrm{d}h_v|_{H^0(F_v)}$ be a measure on $H^0(F_v)$. Then $\mathrm{d}h^0 = \prod_v \mathrm{d}h_v^0$.

Let π be an irreducible cuspidal automorphic representation of $H(\mathbb{A}_F)$. We denote by V the space of automorphic functions on which π is realized. Let $\pi^0 = \pi|_{H^0(\mathbb{A}_F)}$ and $V^0 = \{f|_{H^0(\mathbb{A}_F)} \mid f \in V\}$. Let $\mathfrak{S}$ be the set of places v of F such that $\pi_v|_{H^0(F_v)}$ is reducible. This is also the set of places v of F so that $\pi_v \otimes \det_v \simeq \pi_v$. Let $\mathcal{B}_\pi$ be the Petersson inner product on V given by

$$\mathcal{B}_\pi(f, f') = \int_{Z(\mathbb{A}_F)H(F)\backslash H(\mathbb{A}_F)} f(h)\overline{f'(h)}\mathrm{d}h, \quad f, f' \in V,$$

We fix a decomposition $\mathcal{B}_\pi = \prod_v \mathcal{B}_{\pi_v}$ where $\mathcal{B}_{\pi_v}$ is an inner product on π_v.

We distinguish two cases.

Case I: $\mathfrak{S} = \varnothing$.

In this case, π^0 is irreducible and the restriction to $H^0(\mathbb{A}_F)$ as functions induces an isomorphism $V \simeq V^0$ as representations of $H^0(\mathbb{A}_F)$. Let $\mathcal{B}_{\pi^0}$ be the Petersson inner product on V^0 (defined using the Tamagawa measure on $H^0(\mathbb{A}_F)$).

Lemma 6.1.1. *For any $f, f' \in V$, we have*

$$\mathcal{B}_{\pi^0}(f|_{H^0(\mathbb{A}_F)}, f'|_{H^0(\mathbb{A}_F)}) = 2\mathcal{B}_\pi(f, f').$$

Proof. This can be proved in the same way as [10, Lemma 2.1]. □

Case II: $\mathfrak{S} \neq \varnothing$.

We fix an isomorphism

$$V \simeq \varinjlim_S \left(\bigotimes_{v \in S} V_v \right) \otimes \left(\bigotimes_{v \notin S} \phi_v \right),$$

where V_v is the space on which π_v is realized and ϕ_v is an $H(\mathfrak{o}_{F,v})$-invariant vector in V_v for $v \notin S$.

If $v \in \mathfrak{S}$, then $\pi_v \otimes \det_v \not\simeq \pi_v$ and $\pi_v^0 \simeq \pi_v^+ \oplus \pi_v^-$ where $\pi_v^\pm$ are irreducible admissible representations of $H^0(F_v)$. We have $V_v^0 \simeq V_v^+ \oplus V_v^-$ where V_v^* is the space on which π_v^* are realized and $* = \pm$ or 0. Note that $V_v^- \simeq \pi_v(t)V_v^+$. For almost all places $v \in \mathfrak{S}$, we have $\phi_v = \phi_v^+ + \phi_v^-$ where $\phi_v^\pm$ is an $H^0(\mathfrak{o}_{F,v})$-invariant element in $V_v^\pm$ and $\phi_v^- = \pi_v(t_v)\phi_v^+$. If $v \notin \mathfrak{S}$, then π_v^0 is an irreducible admissible representation on the space V_v.

In this case, by the Hypothesis O, there are two irreducible cuspidal automorphic representations π^+ and π^- so that $\pi^0 \simeq \pi^+ \oplus \pi^-$, $\pi^- \simeq \pi^+ \circ \mathrm{Ad}\, t$, $V^0 = V^+ \oplus V^-$ where $V^\pm$ are the spaces on which $\pi^\pm$ are realized. We may label the two irreducible components of π_v^0 for $v \in \mathfrak{S}$ so that

$$\pi^\pm \simeq \left(\bigotimes_{v\in\mathfrak{S}} \pi_v^\pm\right) \otimes \left(\bigotimes_{v\notin\mathfrak{S}} \pi_v^0\right),$$

$$V^\pm = \varinjlim_S \left(\bigotimes_{\substack{v\in S\\ v\in\mathfrak{S}}} V_v^\pm\right) \otimes \left(\bigotimes_{\substack{v\in S\\ v\notin\mathfrak{S}}} V_v\right) \otimes \left(\bigotimes_{\substack{v\notin S\\ v\in\mathfrak{S}}} \phi_v^\pm\right) \otimes \left(\bigotimes_{\substack{v\notin S\\ v\notin\mathfrak{S}}} \phi_v\right)$$

Let $\mathcal{B}_{\pi^+}$ be the Petersson inner product on V^+ with a fixed decomposition

$$\mathcal{B}_{\pi^+} = \prod_{v\in\mathfrak{S}} \mathcal{B}_{\pi_v^+} \prod_{v\notin\mathfrak{S}} \mathcal{B}_{\pi_v},$$

where

- $\mathcal{B}_{\pi_v^+}$ is an $H^0(F_v)$ invariant pairing on V_v^+ if $v \in \mathfrak{S}$ and $\mathcal{B}_{\pi_v}$ is an $H(F_v)$ invariant pairing on V_v if $v \notin \mathfrak{S}$.
- $\mathcal{B}_{\pi_v^+}(\phi_v^+, \phi_v^+) = \mathcal{B}_v(\phi, \phi) = 1$ for almost all v.

If $v \in \mathfrak{S}$, we define an $H^0(F_v)$ invariant pairing on V_v^- by $\mathcal{B}_v^-(\phi_v, \phi_v) = \mathcal{B}_v^+(\pi_v(t_v)\phi_v, \pi_v(t_v)\phi_v)$. Then for almost all v, we have $\mathcal{B}_v^-(\phi_v^-, \phi_v^-) = 1$. We then define an $H(F_v)$ invariant pairing on V_v by

$$\mathcal{B}_v^\natural(\phi_v, \phi_v) = \begin{cases} \frac{1}{2}(\mathcal{B}_v^+(\phi_v^+, \phi_v^+) + \mathcal{B}_v^-(\phi_v^-, \phi_v^-)), & \text{if } v \in \mathfrak{S} \\ \mathcal{B}_v(\phi_v, \phi_v), & \text{if } v \notin \mathfrak{S}. \end{cases}$$

Then for almost all v, $\mathcal{B}_v^\natural(\phi_v, \phi_v) = 1$.

Lemma 6.1.2.

$$\mathcal{B}_\pi = \prod_v \mathcal{B}_v^\natural.$$

Proof. This can be proved in the same way as [10, Lemma 2.3]. □

6.2. **Ichino–Ikeda's conjecture for special orthogonal groups.** We review Ichino–Ikeda's conjecture [24, Conjecture 2.1] in this subsection. There is a slight inaccuracy in its orginal formulation in [24] when the multiplicity of the automorphic representation on the even orthogonal group in the discrete automorphic spectrum is two. We will make some modifications to the conjecture in this case.

Let $n \geq 2$ and U_{n+1} and U_n be orthogonal spaces of dimension $n+1$ and n with an embedding $U_n \subset U_{n+1}$. Let $H_i^0 = \mathrm{SO}(U_i)$ $(i = n, n+1)$. Let $\mathrm{d}h$ be the Tamagawa measure on $H_n^0(\mathbb{A}_F)$ and we fix a decomposition $\mathrm{d}h = \prod_v \mathrm{d}h_v$ where $\mathrm{d}h_v$ is a Haar measure on $H_n^0(F_v)$ and $\mathrm{vol}\, H_n^0(\mathfrak{o}_{F,v}) = 1$ for almost all v.

Let $\pi_{n+1} = \otimes_v \pi_{n+1,v}$ and $\pi_n = \otimes_v \pi_{n,v}$ be irreducible cuspidal tempered automorphic representations of $H_{n+1}^0(\mathbb{A}_F)$ and $H_n^0(\mathbb{A}_F)$ respectively. Let $V_{n+1} = \otimes_v V_{n+1,v}$ and $V_n = \otimes_v V_{n,v}$ be the space on which π_{n+1} and π_n are realized respectively. Let $\mathcal{B}_{\pi_{n+1}}$ and $\mathcal{B}_{\pi_n}$ be the Petersson inner products on V_{n+1} (resp. V_n) respectively. We fix a decomposition

$$\mathcal{B}_{\pi_{n+1}} = \prod_v \mathcal{B}_{\pi_{n+1,v}}, \quad \mathcal{B}_{\pi_n} = \prod_v \mathcal{B}_{\pi_{n,v}}$$

where $\mathcal{B}_{\pi_{n+1,v}}$ and $\mathcal{B}_{\pi_{n,v}}$ are inner products on $V_{n+1,v}$ and $V_{n,v}$ respectively.

Let $f_{n+1} = \otimes f_{n+1,v}, f'_{n+1} = \otimes f'_{n+1,v} \in V_{n+1}$ and $f_n = \otimes f_{n,v}, f'_n = \otimes f'_{n,v} \in V_n$. Define

$$\mathcal{J}(f_{n+1}, f'_{n+1}, f_n, f'_n) = \int_{H_n^0(F)\backslash H_n^0(\mathbb{A}_F)} f_{n+1}(h) f_n(h) \mathrm{d}h \cdot \overline{\int_{H_n^0(F)\backslash H_n^0(\mathbb{A}_F)} f'_{n+1}(h) f'_n(h) \mathrm{d}h}.$$

For each place v, we define

$$\mathcal{J}_v(f_{n+1,v}, f'_{n+1,v}, f_{n,v}, f'_{n,v}) = \int_{H_n^0(F_v)} \mathcal{B}_{n+1,v}(\pi_{n+1,v}(h_v) f_{n+1,v}, f'_{n+1,v}) \mathcal{B}_{n,v}(\pi_{n,v}(h_v) f_{n,v}, f'_{n,v}) \mathrm{d}h_v.$$

Let S be a sufficiently large finite set of places of F containing all archimedean places so that if $v \notin S$, then $f_{n+1,v}, f'_{n+1,v}$ (resp. $f_{n,v}$, $f'_{n,v}$) are $H_{n+1}^0(\mathfrak{o}_{F,v})$ (resp. $H_n^0(\mathfrak{o}_{F,v})$) fixed and $\mathcal{B}_{\pi_{n+1,v}}(f_{n+1,v}, f'_{n+1,v}) = \mathcal{B}_{\pi_{n,v}}(f_{n,v}, f'_{n,v}) = 1$. In particular, $\pi_{n+1,v}$ and $\pi_{n,v}$ are both unramified if $v \notin S$. Let $\{\alpha_{1,v}, \cdots, \alpha_{[\frac{n+1}{2}],v}\}$ and $\{\beta_{1,v}, \cdots, \beta_{[\frac{n}{2}],v}\}$ be the Satake parameters of $\pi_{n+1,v}$ and $\pi_{n,v}$ respectively. Let

$$A_{n+1,v} = \mathrm{diag}[\alpha_{1,v}, \cdots, \alpha_{[\frac{n+1}{2}],v}, \alpha^{-1}_{[\frac{n+1}{2}],v}, \cdots, \alpha^{-1}_{1,v}]$$
$$A_{n,v} = \mathrm{diag}[\beta_{1,v}, \cdots, \beta_{[\frac{n}{2}],v}, \beta^{-1}_{[\frac{n}{2}],v}, \cdots, \beta^{-1}_{1,v}].$$

Let

$$L^S(s, \pi_{n+1} \times \pi_n) = \prod_{v \notin S} \det(1 - A_{n+1,v} \otimes A_{n,v} \cdot q_v^{-s})^{-1}$$

be the tensor product L-function and $L^S(s, \pi_{n+1}, \mathrm{Ad})$ and $L^S(s, \pi_n, \mathrm{Ad})$ be the adjoint L-functions.

Conjecture 6.2.1 (Ichino–Ikeda [24, Conjecture 2.1])**.** (1) *Suppose that π_{n+1} and π_n appear with multiplicity one in the discrete spectrum. Then the automorphic realization V_{n+1} (resp. V_n) of π_{n+1} (resp. π_n) is canonical. We have*

$$\mathcal{J}(f_{n+1}, f'_{n+1}, f_n, f'_n) = \frac{1}{|S_{\pi_{n+1}}||S_{\pi_n}|} \Delta^S_{H^0_{n+1}} \frac{L^S(\frac{1}{2}, \pi_{n+1} \times \pi_n)}{L^S(1, \pi_{n+1}, \mathrm{Ad}) L^S(1, \pi_n, \mathrm{Ad})} \prod_{v \in S} \mathcal{J}_v(f_{n+1,v}, f'_{n+1,v}, f_{n,v}, f'_{n,v}).$$

(2) *Suppose n is odd and π_{n+1} appears with multiplicity two in the discrete spectrum of $H^0_{n+1}(\mathbb{A}_F)$. Then the automorphic realization V_n of π_n is canonical. Let $L^2_{\pi_{n+1}}$ be the isotypic component of π_{n+1} in the discrete automorphic spectrum of $H^0_{n+1}(\mathbb{A}_F)$. Then there are two possibilities.*

  (a) *The linear form $\mathcal{J}$ is identically zero on $L^2_{\pi_{n+1}} \times L^2_{\pi_{n+1}} \times V_n \times V_n$. This is equivalent to that either $\mathrm{Hom}_{H^0_n(\mathbb{A}_F)}(\pi_{n+1} \otimes \pi_n, \mathbb{C}) = 0$ or $L^S(\frac{1}{2}, \pi_{n+1} \times \pi_n) = 0$.*

  (b) *There is a unique irreducible subrepresentation V_{n+1} of $L^2_{\pi_{n+1}}$ such that it is invariant under the outer automorphism of H^0_{n+1} and $\mathcal{J}$ is not identically zero on $V_{n+1} \times V_{n+1} \times V_n \times V_n$. We have*

$$\mathcal{J}(f_{n+1}, f'_{n+1}, f_n, f'_n) = \frac{2}{|S_{\pi_{n+1}}||S_{\pi_n}|} \Delta^S_{H^0_{n+1}} \frac{L^S(\frac{1}{2}, \pi_{n+1} \times \pi_n)}{L^S(1, \pi_{n+1}, \mathrm{Ad}) L^S(1, \pi_n, \mathrm{Ad})} \prod_{v \in S} \mathcal{I}_v(f_{n+1,v}, f'_{n+1,v}, f_{n,v}, f'_{n,v}),$$

  if $f_{n+1}, f'_{n+1} \in V_{n+1}$, $f_n, f'_n \in V_n$. Let V'_{n+1} ($\neq V_{n+1}$) be the other irreducible subrepresentation of $L^2_{\pi_{n+1}}$ that is invariant under the outer automorphism of H^0_{n+1}. Then $\mathcal{J}$ is identically zero on $V'_{n+1} \times V'_{n+1} \times V_n \times V_n$.

If n is even, then we have a similar statement, with the role of π_{n+1} and π_n being switched.

Remark 6.2.2. The same inaccuracy also occurs in [36]. One also needs to modify [36, Conjecture 2.5] in a similar way when the automorphic representation on the even orthogonal group has multiplicity two. In this case, the automorphic realization is required to be invariant under the outer twist and (in the notation of [36]) $1/|S_{\Psi(\pi_2)}||S_{\Psi(\pi_0)}|$ needs to be replaced by $2/|S_{\Psi(\pi_2)}||S_{\Psi(\pi_0)}|$.

6.3. **Ichino–Ikeda's conjecture for full orthogonal groups.** Let U_{n+1} and U_n be orthogonal spaces of dimension $n+1$ and n with an embedding $U_n \subset U_{n+1}$. Let $H_i = \mathrm{O}(U_i)$ and $H^0_i = \mathrm{SO}(U_i)$ ($i = n, n+1$). Let $\mathrm{d}h$ be the Tamagawa measure on $H_n(\mathbb{A}_F)$ and we fix a decomposition $\mathrm{d}h = \prod_v \mathrm{d}h_v$ where $\mathrm{d}h_v$ is a Haar measure on $H_n(F_v)$ and $\mathrm{vol}\, H_n(\mathfrak{o}_{F,v}) = 1$ for almost all v.

Let $\pi_{n+1} = \otimes_v \pi_{n+1,v}$ and $\pi_n = \otimes_v \pi_{n,v}$ be irreducible cuspidal tempered automorphic representations of $H_{n+1}(\mathbb{A}_F)$ and $H_n(\mathbb{A}_F)$ respectively. Let $V_{n+1} = \otimes_v V_{n+1,v}$ and $V_n = \otimes_v V_{n,v}$ be the space on which π_{n+1} and π_n are realized respectively. Let $\mathcal{B}_{\pi_{n+1}}$ and $\mathcal{B}_{\pi_n}$ be the Petersson inner products on V_{n+1} (resp. V_n) respectively. We fix a decomposition

$$\mathcal{B}_{\pi_{n+1}} = \prod_v \mathcal{B}_{\pi_{n+1,v}}, \quad \mathcal{B}_{\pi_n} = \prod_v \mathcal{B}_{\pi_{n,v}}$$

where $\mathcal{B}_{\pi_{n+1,v}}$ and $\mathcal{B}_{\pi_{n,v}}$ are inner products on $V_{n+1,v}$ and $V_{n,v}$ respectively.

Let $f_{n+1} = \otimes f_{n+1,v} \in V_{n+1}$ and $f_n = \otimes f_{n,v} \in V_n$. Define

$$\mathcal{I}(f_{n+1}, f_n) = \int_{H_n(F)\backslash H_n(\mathbb{A}_F)} f_{n+1}(h) f_n(h) \mathrm{d}h \cdot \overline{\int_{H_n(F)\backslash H_n(\mathbb{A}_F)} f_{n+1}(h) f_n(h) \mathrm{d}h}. \tag{6.3.1}$$

For each place v, we define

$$\mathcal{I}_v(f_{n+1,v}, f_{n,v}) = \int_{H_n(F_v)} \mathcal{B}_{n+1,v}(\pi_{n+1,v}(h_v) f_{n+1,v}, f_{n+1,v}) \mathcal{B}_{n,v}(\pi_{n,v}(h_v) f_{n,v}, f_{n,v}) \mathrm{d}h_v. \tag{6.3.2}$$

Let S be a sufficiently large finite set of places of F containing all archimedean places so that if $v \notin S$, then $f_{n+1,v}$ (resp. $f_{n,v}$) is $H_{n+1}(\mathfrak{o}_{F,v})$ (resp. $H_n(\mathfrak{o}_{F,v})$) fixed and $\mathcal{B}_{\pi_{n+1,v}}(f_{n+1,v}, f_{n+1,v}) = \mathcal{B}_{\pi_{n,v}}(f_{n,v}, f_{n,v}) = 1$. In particular, $\pi_{n+1,v}$ and $\pi_{n,v}$ are both unramified if $v \notin S$. We define the partial L-functions

$$L^S(s, \pi_{n+1} \times \pi_n) = L^S(s, \dot{\pi}_{n+1} \times \dot{\pi}_n), \quad L^S(s, \pi_i, \mathrm{Ad}) = L^S(s, \dot{\pi}_i, \mathrm{Ad}),\ i = n, n+1,$$

where $\dot{\pi}_i$ is an irreducible constituent of π_i^0 which is invariant by the nontrivial outer automorphism ϵ. The L-functions on the right hand side of each equality is independent of the choice of this irreducible constituent.

The Ichino–Ikeda's conjecture for the full orthogonal group is the following.

Conjecture 6.3.1. *We have*

$$\mathcal{I}(f_{n+1}, f_n) = \frac{2^\gamma}{|S_{\pi_{n+1}}||S_{\pi_n}|} \Delta_{H_{n+1}}^S \frac{L^S(\frac{1}{2}, \pi_{n+1} \times \pi_n)}{L^S(1, \pi_{n+1}, \mathrm{Ad}) L^S(1, \pi_n, \mathrm{Ad})} \prod_{v \in S} \mathcal{I}_v(f_{n+1,v}, f_{n,v}). \tag{6.3.3}$$

where γ is given as follows. Suppose n is even (resp. odd). Let $\Psi_{\pi_n} = \boxplus \Pi_i$ (resp. $\Psi_{\pi_{n+1}} = \boxplus \Pi_i$) where Π_i is an irreducible cuspidal automorphic representation of $\mathrm{GL}_{N_i}(\mathbb{A}_F)$. Then $\gamma = 0$ (resp. 1) if at least one of N_i's is odd (resp. all N_i's are even).

Remark 6.3.2. We may have a neater formulation of the conjecture if we replace our definition of the centralizers S_{π_i} by the one given in [4] for parameters of full orthogonal groups. We stick to our current formulation as it is more convenient for the applications in this paper.

Similar to Conjecture 2.3.1, we may rewrite the identity (6.3.3) in an equivalent form, which does not involve the finite set S. We may define the completed L-function

$$L(s, \pi_{n+1} \times \pi_n) = \prod_v L(s, \pi_{n+1,v} \times \pi_{n,v}), \quad L(s, \pi_i, \mathrm{Ad}) = \prod_v L(s, \pi_{i,v}, \mathrm{Ad}),\ i = n, n+1.$$

The actually definition of the local Euler factors outside the set S is irrelevant to our discussion since the conjecture does not reply on how these Euler factors are defined. Let

$$\mathcal{L} = \Delta_{H_{n+1}} \frac{L(\frac{1}{2}, \pi_{n+1} \times \pi_n)}{L(1, \pi_{n+1}, \mathrm{Ad}) L(1, \pi_n, \mathrm{Ad})},$$

and by $\mathcal{L}_v$ the Euler factor of $\mathcal{L}$ at the place v. We define

$$\mathcal{I}_v^{\natural} = \mathcal{L}_v^{-1} \cdot \mathcal{I}_v.$$

Then Conjecture 6.3.1 can be written as a decomposition of linear forms

$$\mathcal{I} = \frac{2^{\gamma}}{|S_{\pi_{n+1}}||S_{\pi_n}|}\mathcal{L} \cdot \prod_v \mathcal{I}_v^{\natural}. \tag{6.3.4}$$

The product on the right hand side ranges over all places v of F. It is convergent since for almost all v, i.e. $v \notin S$, $\mathcal{I}_v^{\natural} = 1$. We may write Conjecture 6.2.1 in a similar forms.

Proposition 6.3.3. *Conjecture 6.3.1 follows from Conjecture 6.2.1.*

Proof. We assume that n is odd. The case n being even can be handled similarly, with modifications of notation at various places. Then $H_n \simeq H_n^0 \times \mu_2$. So $\pi_{n,v}^0$ is irreducible for all places v of F. Let $\mathfrak{S}$ be the set of places of F such that $\pi_{n+1,v}^0$ is reducible.

If $v \notin S$, then $f_{n+1,v} = \phi_{n+1,v}$ is fixed by $H_{n+1}(\mathfrak{o}_{F,v})$ and $f_{n,v}$ is fixed by $H_n(\mathfrak{o}_{F,v})$. We may further assume that $f_{n+1,v} = f_{n+1,v}^+ \in V_{n+1,v}^+$ if $v \in S \cap \mathfrak{S}$. Thus

$$f_{n+1,v} = \prod_{v \in S \cap \mathfrak{S}} f_{n+1,v}^+ \prod_{v \in S, v \notin \mathfrak{S}} f_{n+1,v} \prod_{v \notin S} \phi_{n+1,v}.$$

Put

$$S' = S \backslash (S \cap \mathfrak{S}), \quad s = |S \cap \mathfrak{S}|, \quad s' = |S'|.$$

For any finite set of places T of F, we define $F_T = \prod_{v \in T} F_v$.

If $\mathfrak{S} \neq \varnothing$, then

$$\begin{aligned}
&\int_{H_n(F)\backslash H_n(\mathbb{A}_F)} f_{n+1}(h) f_n(h) \mathrm{d}h \\
=&\frac{1}{2^{s+s'+1}} \sum_{\epsilon \in \mu_2(F_S)} \int_{H_n^0(F)\backslash H_n^0(\mathbb{A}_F)} f_{n+1}(h\epsilon) f_n(h\epsilon) \mathrm{d}h \\
=&\frac{1}{2^{s+s'+1}} \sum_{\epsilon \in \mu_2(F_{S'})} \int_{H_n^0(F)\backslash H_n^0(\mathbb{A}_F)} (f_{n+1}(h\epsilon) f_n(h\epsilon) + f_{n+1}(h\epsilon t) f_n(h\epsilon t)) \mathrm{d}h \\
=&\frac{1}{2^{s+s'}} \sum_{\epsilon \in \mu_2(F_{S'})} \int_{H_n^0(F)\backslash H_n^0(\mathbb{A}_F)} f_{n+1}(h\epsilon) f_n(h\epsilon) \mathrm{d}h.
\end{aligned}$$

If $\mathfrak{S} = \varnothing$, then

$$\int_{H_n(F)\backslash H_n(\mathbb{A}_F)} f_{n+1}(h) f_n(h) \mathrm{d}h = \frac{1}{2^{s'+1}} \sum_{\epsilon \in \mu_2(F_S)} \int_{H_n^0(F)\backslash H_n^0(\mathbb{A}_F)} f_{n+1}(h\epsilon) f_n(h\epsilon) \mathrm{d}h.$$

We fix a decomposition

$$\mathcal{B}_{\pi_{n+1}^+} = \prod_{v \in \mathfrak{S}} \mathcal{B}_{\pi_{n+1,v}^+} \prod_{v \notin \mathfrak{S}} \mathcal{B}_{\pi_{n+1,v}^0}, \quad \text{resp. } \mathcal{B}_{\pi_{n+1}^0} = 2 \prod_v \mathcal{B}_{\pi_{n+1,v}^0}$$

if $\mathfrak{S} \neq \varnothing$ (resp. $\mathfrak{S} = \varnothing$), so that $\mathcal{B}_{\pi_{n+1,v}} = \mathcal{B}^{\natural}_{\pi_{n+1,v}}$ if $v \in \mathfrak{S}$ (resp. $\mathcal{B}_{\pi_{n+1,v}} = \mathcal{B}_{\pi^0_{n+1,v}}$ if $v \notin \mathfrak{S}$). We fix a decomposition

$$\mathcal{B}_{\pi_n^0} = 2 \prod_v \mathcal{B}_{\pi^0_{n,v}},$$

so that $\mathcal{B}_{\pi_{n,v}} = \mathcal{B}_{\pi^0_{n,v}}$.

We say that we are in the exceptional case if the following conditions are satisfied.

- π^0_{n+1} is irreducible and appears with multiplicity two in the discrete spectrum of $H^0_{n+1}(\mathbb{A}_F)$.
- The period integral

$$\int_{H_n^0(F)\backslash H_n^0(\mathbb{A}_F)} f_{n+1}(h) f_n(h) \mathrm{d}h$$

  is identically zero on $V^0_{n+1} \times V^0_n$, where we denote as before $V_i^0 = \{f|_{H_i^0(\mathbb{A}_F)} \mid f \in V_i\}$, $i = n, n+1$.
- The period integral is not identically zero on the isotypic component of π^0_{n+1}.

Suppose that we are not in the exceptional case. Then Conjecture 6.2.1 implies that

$$\begin{aligned}\mathcal{I}(f_{n+1}, f_n) =& \frac{2^{m+\gamma'}}{2^{2s+2s'}|S_{\pi_{n+1}}||S_{\pi_n}|} \Delta^S_{H_{n+1}} \frac{L^S(\frac{1}{2}, \pi_{n+1} \times \pi_n)}{L^S(1, \pi_{n+1}, \mathrm{Ad}) L^S(1, \pi_n, \mathrm{Ad})} \\ & \sum_{\epsilon, \epsilon' \in \mu_2(F_{S'})} \prod_{v \in S} \mathcal{J}_v(\pi_{n+1}(\epsilon) f_{n+1,v}, \pi_{n+1}(\epsilon') f_{n+1,v}, \pi_n(\epsilon) f_{n,v}, \pi_n(\epsilon') f_{n,v}),\end{aligned}$$

where

- $\gamma' = 1$ (resp. 0) if π^0_{n+1} is reducible (resp. irreducible).
- $m = 1$ (resp. 0) if π^0_{n+1} is irreducible and appears with multiplicity two (resp. any irreducible constituent appears with multiplicity one) in the discrete spectrum of $H^0_{n+1}(\mathbb{A}_F)$.

We note that $\gamma = m + \gamma'$. In fact, in the first (resp. second, resp. third) case in Hypothesis O, we have $\gamma = \gamma' = m = 0$ (resp. $\gamma = 1, m = 1, \gamma' = 0$, resp. $\gamma = 1, m = 0, \gamma' = 1$). Therefore to deduce Conjecture 6.3.1 from Conjecture 6.2.1, we only need to prove the following two identities. If $v \in \mathfrak{S}$, then

$$\frac{1}{4} \mathcal{J}_v(f_{n+1,v}, f_{n+1,v}, f_{n,v}, f_{n,v}) = \mathcal{I}_v(f_{n+1,v}, f_{n,v}).$$

If $v \notin \mathfrak{S}$, then

$$\frac{1}{4} \sum_{\epsilon, \epsilon' \in \mu_2(F_v)} \mathcal{J}_v(\pi_{n+1}(\epsilon) f_{n+1,v}, \pi_{n+1}(\epsilon') f_{n+1,v}, \pi_n(\epsilon) f_{n,v}, \pi_n(\epsilon') f_{n,v}) = \mathcal{I}_v(f_{n+1,v}, f_{n,v}).$$

These two identities can be proved in the same way as [24, Lemma 3.4]. Therefore Conjecture 6.3.1 follows from Conjecture 6.2.1 if we are not in the exceptional case.

Now assume that we are in the exceptional case. Let π'^0_{n+1} be an irreducible cuspidal automorphic representation of $H^0_{n+1}(\mathbb{A}_F)$ which realizes on V'^0_{n+1} such that V'^0_{n+1} is invariant under the outer automorphism of H^0_{n+1}, $V'^0_{n+1} \neq V^0_{n+1}$ and π'^0_{n+1} is isomorphic to π^0_{n+1} (as abstract representations). Then the period integral

$$\int_{H^0_n(F)\backslash H^0_n(\mathbb{A}_F)} f_{n+1}(h) f_n(h) \mathrm{d}h$$

is not identically zero on $V'^0_{n+1} \times V^0_n$. Therefore

$$\mathrm{Hom}_{H^0_n(\mathbb{A}_F)}(\pi'^0_{n+1} \otimes \pi^0_n, \mathbb{C}) \neq 0.$$

Since V'^0_{n+1} is invariant under the outer automorphism of H^0_{n+1}, there is an automorphic representation π'_{n+1} of $H_{n+1}(\mathbb{A}_F)$ which is realized on V'_{n+1} whose restriction to $H^0_{n+1}(\mathbb{A}_F)$ is V'^0_{n+1}.

Let T be a finite subset of places of F and we let $\det_T$ be the character of $H_{n+1}(\mathbb{A}_F)$ defined by

$$(g_v) \mapsto \prod_{v \in T} \det g_v \in \{\pm 1\}, \quad (g_v) \in H_{n+1}(\mathbb{A}_F).$$

Then $\det_T$ is automorphic if and only if $|T|$ is even.

Note that $n \geq 3$ in this case. Let $Z_n \simeq \mu_2$ be the center of H_n and it is identified with a subgroup of H_{n+1} via the embedding $H_n \to H_{n+1}$. Let $l = \otimes l_v \in \mathrm{Hom}_{H^0_n(\mathbb{A}_F)}(\pi'_{n+1} \otimes \pi_n, \mathbb{C})$ and $\theta = (\theta_v) \in Z_n(\mathbb{A}_F)$. Let $l^\theta = \otimes l_v^{\theta_v} \in \mathrm{Hom}_{H^0_n(\mathbb{A}_F)}(\pi'_{n+1} \otimes \pi_n, \mathbb{C})$ be defined by

$$l_v^{\theta_v}(\xi_{n+1,v} \otimes \xi_{n,v}) = l_v(\pi_{n+1,v}(\theta_v)\xi_{n+1,v} \otimes \pi_{n,v}(\theta_v)\xi_{n,v}),$$

Since $\theta_v^2 = 1$ and $\dim \mathrm{Hom}_{H^0_n(F_v)}(\pi'_{n+1,v} \otimes \pi_{n,v}, \mathbb{C}) = 1$, we have $l_v^{\theta_v} = \pm l_v$. It follows that there is finite set T of places of F so that $l^\theta = \det_T(\theta) \cdot l$. Since π'_{n+1} and π_n are automorphic, $\det_T$ is also automorphic. It follows that $|T|$ is even.

Let $\pi''_{n+1} = \pi'_{n+1} \otimes \det_T$. Then π''_{n+1} is an automorphic representation of $H_{n+1}(\mathbb{A}_F)$ and is realized on V''_{n+1}. Its restriction to $H^0_{n+1}(\mathbb{A}_F)$ is V'^0_{n+1}. Moreover for any place v of F,

$$\mathrm{Hom}_{H_n(F_v)}(\pi''_{n+1,v} \otimes \pi_{n,v}, \mathbb{C}) \neq 0.$$

Since π_{n+1} and π''_{n+1} are not isomorphic but their restrictions to $H^0_{n+1}(\mathbb{A}_F)$ are isomorphic, there is at least one place v, such that $\pi_{n+1,v} \simeq \pi''_{n+1,v} \otimes \det_v$. We claim that

$$\mathrm{Hom}_{H_n(F_v)}(\pi_{n+1,v} \otimes \pi_{n,v}, \mathbb{C}) = 0.$$

In fact, $\mathrm{Hom}_{H_n(F_v)}(\pi''_{n+1,v} \otimes \pi_{n,v}, \mathbb{C}) \neq 0$ is the $+1$ eigenspace of $\theta_v = -1 \in Z_n(F_v)$ on $\mathrm{Hom}_{H^0_n(F_v)}(\pi^0_{n+1,v} \otimes \pi^0_{n,v}, \mathbb{C})$ while $\mathrm{Hom}_{H_n(F_v)}(\pi_{n+1,v} \otimes \pi_{n,v}, \mathbb{C})$ is the -1 eigenspace. Since $\dim \mathrm{Hom}_{H^0_n(F_v)}(\pi^0_{n+1,v} \otimes \pi^0_{n,v}, \mathbb{C}) = \dim \mathrm{Hom}_{H_n(F_v)}(\pi''_{n+1,v} \otimes \pi_{n,v}, \mathbb{C}) = 1$, we conclude that $\mathrm{Hom}_{H_n(F_v)}(\pi_{n+1,v} \otimes \pi_{n,v}, \mathbb{C}) = 0$.

It follows that the linear form $\mathcal{I}_v$ is identically zero in the exceptional case. Therefore both sides of (6.3.3) are zero. $\square$

7. Compactibility with Ichino–Ikeda's conjecture: $\mathrm{Sp}(2n) \times \mathrm{Mp}(2n)$

7.1. **The Theorem.** The goal of this section is to study Conjecture 2.3.1 for $\mathrm{Sp}(2n) \times \mathrm{Mp}(2n)$. We are going to show that Conjecture 2.3.1 is compatible with Ichino–Ikeda's conjecture for $\mathrm{SO}(2n+1) \times \mathrm{SO}(2n)$ in some cases. Result of this sort for unitary groups appeared in [50, Proposition 1.4.1]. The local counterpart of this argument has been used to establish the local Gan–Gross–Prasad conjecture for the Fourier–Jacobi models [3, 11].

Let $\lambda \in F^\times$. Let (V, q_V) be a $2n+1$-dimensional orthogonal space and V_λ is a $2n$-dimensional subspace such that $V_\lambda^\perp$ is a one dimensional orthogonal space of discriminant λ. Let $H = \mathrm{O}(V)$ and $H_\lambda = \mathrm{O}(V_\lambda)$ and $\iota_\lambda : H_\lambda \to H$ be the natural embedding.

Let W be a $2n$-dimensional symplectic space and $G = \mathrm{Sp}(W)$, $\widetilde{G} = \mathrm{Mp}(W)$. Let $\widetilde{\Omega}_\psi$ (resp. Ω_ψ) be the Weil representation of $\widetilde{G}(\mathbb{A}_F) \times H(\mathbb{A}_F)$ (resp. $G(\mathbb{A}_F) \times H_\lambda(\mathbb{A}_F)$) which is realized on $\mathcal{S}(V(\mathbb{A}_F)^n)$ (resp. $\mathcal{S}(V_\lambda(\mathbb{A}_F)^n)$). Let ω_{ψ_λ} be the Weil representation of $\widetilde{G}(\mathbb{A}_F)$ realized on $\mathcal{S}(\mathbb{A}_F^n)$. Then we have the theta series

$$\widetilde{\Theta}_\psi(\widetilde{g}, h, \Phi), \quad \Theta_\psi(g, h_\lambda, \Phi_\lambda), \quad \theta_{\psi_\lambda}(\widetilde{g}, \phi)$$

on $\widetilde{G}(\mathbb{A}_F) \times H(\mathbb{A}_F)$, $G(\mathbb{A}_F) \times H_\lambda(\mathbb{A}_F)$ and $\widetilde{G}(\mathbb{A}_F)$ respectively, where $\Phi \in \mathcal{S}(V(\mathbb{A}_F)^n)$, $\Phi_\lambda \in \mathcal{S}(V_\lambda(\mathbb{A}_F)^n)$ and $\phi \in \mathcal{S}(\mathbb{A}_F^n)$.

Let π be an irreducible cuspidal tempered genuine automorphic representation of $\widetilde{G}(\mathbb{A}_F)$. Let $\widetilde{\Theta}_\psi(\pi)$ be the theta lift of π to $H(\mathbb{A}_F)$, i.e. the automorphic representation generated by the functions of the form

$$\widetilde{\Theta}_\psi(\varphi, \Phi)(\cdot) = \int_{G(F)\backslash G(\mathbb{A}_F)} \overline{\varphi(g)} \widetilde{\Theta}_\psi(g, \cdot, \Phi) \mathrm{d}g, \quad \varphi \in \pi, \ \Phi \in \mathcal{S}(V(\mathbb{A}_F)^n).$$

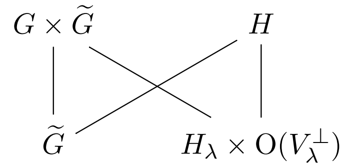
Let σ be an irreducible cuspidal tempered automorphic representation of $H_\lambda(\mathbb{A}_F)$. Let $\Theta_\psi(\sigma)$ be the theta lift of σ to $G(\mathbb{A}_F)$, i.e. the automorphic representation generated by the functions of the form

$$\Theta_\psi(f, \Phi_\lambda)(\cdot) = \int_{H_\lambda(F)\backslash H_\lambda(\mathbb{A}_F)} \overline{f(h_\lambda)} \Theta_\psi(\cdot, h_\lambda, \Phi_\lambda) \mathrm{d}h_\lambda, \quad f \in \sigma, \ \Phi_\lambda \in \mathcal{S}(V_\lambda(\mathbb{A}_F)^n).$$

Theorem 7.1.1. *Suppose that $\widetilde{\Theta}_{\psi_{-1}}(\pi)$ and $\Theta_\psi(\sigma)$ are both cuspidal (possibly zero). If Conjecture 6.3.1 holds for $(\widetilde{\Theta}_{\psi_{-1}}(\pi), \sigma)$, then Conjecture 2.3.1(3) holds for $(\pi, \Theta_\psi(\sigma))$ with the additive character $\psi_{-\lambda}$.*

Remark 7.1.2. We have shown in Proposition 6.3.3 that Conjecture 6.3.1 can be deduced from the original conjecture of Ichino–Ikeda (Conjecture 6.2.1). The theorem thus says that Conjecture 2.3.1(3) and Ichino–Ikeda's conjecture are compatible in this situation. The same remark also applies to Theorem 8.1.1 in the next section.

7.2. **A seesaw diagram.** The proof of Theorem 7.1.1 is very similar to [50, Proposition 1.4.1]. It makes use of the following seesaw diagram.



Suppose that $f=\otimes f_v\in\sigma$, $\varphi=\otimes\varphi_v\in\pi$, $\Phi_\lambda=\otimes\Phi_{\lambda,v}\in\mathcal{S}(V_\lambda(\mathbb{A}_F)^n)$ and $\phi=\otimes\phi_v\in\mathcal{S}(\mathbb{A}_F^n)$ are all factorizable.

Lemma 7.2.1. *We have*

$$\mathcal{F}\mathcal{J}_{\psi_{-\lambda}}(\varphi,\Theta_\psi(f,\Phi_\lambda),\phi)=\overline{\int_{H_\lambda(F)\backslash H_\lambda(\mathbb{A}_F)} f(h)\widetilde{\Theta}_{\psi_{-1}}(\varphi,\overline{\Phi_\lambda}\otimes\phi)(\iota_\lambda(h))\mathrm{d}h}.$$

Proof.

$$\begin{aligned}
&\mathcal{F}\mathcal{J}_{\psi_{-\lambda}}(\varphi,\Theta_\psi(f,\Phi_\lambda),\phi)\\
=&\int_{G(F)\backslash G(\mathbb{A}_F)}\int_{H_\lambda(F)\backslash H_\lambda(\mathbb{A}_F)}\varphi(g)\overline{f(h)}\Theta_\psi(g,h,\Phi_\lambda)\overline{\theta_{\psi_{-\lambda}}(g,\phi)}\mathrm{d}h\mathrm{d}g\\
=&\int_{H_\lambda(F)\backslash H_\lambda(\mathbb{A}_F)}\int_{G(F)\backslash G(\mathbb{A}_F)}\varphi(g)\widetilde{\Theta}_\psi(g,\iota_\lambda(h),\Phi_\lambda\otimes\overline{\phi})\overline{f(h)}\mathrm{d}g\mathrm{d}h\\
=&\overline{\int_{H_\lambda(F)\backslash H_\lambda(\mathbb{A}_F)} f(h)\widetilde{\Theta}_{\psi_{-1}}(\varphi,\overline{\Phi_\lambda}\otimes\phi)(\iota_\lambda(h))\mathrm{d}h}.
\end{aligned}$$

□

Let v be a place of F. We use $\mathcal{B}$ to denote the inner products on various unitary representations.

Lemma 7.2.2. *The integral*

$$\int_{H_\lambda(F_v)}\int_{G(F_v)}\overline{\mathcal{B}(\sigma_v(h)f_v,f_v)}\mathcal{B}(\Omega_{\psi_v}(g,h)\Phi_{\lambda,v},\Phi_{\lambda,v})\mathcal{B}(\pi_v(g)\varphi_v,\varphi_v)\overline{\mathcal{B}(\omega_{\psi_{-\lambda,v}}(g)\phi_v,\phi_v)}\mathrm{d}g\mathrm{d}h$$

is absolutely convergent.

Proof. To simplify notation, we suppress the subscript v from the notation in the proof. Put

$$\Upsilon(x)=\begin{cases}1, & |x|\le 1;\\ |x|^{-1}, & |x|>1.\end{cases}$$

By the weak inequality (3.1.5) and the estimates (3.1.2), (3.1.4), it is enough to prove that the double integral

$$\int_{A_{H_\lambda}^+}\int_{A_G^+}\delta_{H_\lambda}^{-\frac{1}{2}}(b)\delta_G^{-\frac{1}{2}}(a)|a_1\cdots a_n|^{\frac{2n+1}{2}}\prod_{i=1}^n\prod_{j=1}^r\Upsilon(a_ib_j^{-1})\varsigma(a)^M\varsigma(b)^M\mathrm{d}a\mathrm{d}b \tag{7.2.1}$$

is convergent, where M is some positive real number, r is the Witt index of V_λ and

$$a = \mathrm{diag}[a_1, \cdots, a_n, a_n^{-1}, \cdots, a_1^{-1}] \in A_G^+, \quad b = \mathrm{diag}[b_1, \cdots, b_r, 1, \cdots, 1, b_r^{-1}, \cdots, b_1^{-1}] \in A_{H_\lambda}^+.$$

We assume that $r < n$. The case $r = n$ is very similar and needs only a slight modification. We left it to the interested readers.

We have $|b_1| \leq \cdots \leq |b_r| \leq 1$. Let $\underline{j} = (j_1, \cdots, j_r)$ be r nonnegative integers such that $j_1 + \cdots + j_r \leq n$ and let $I_{\underline{j}}$ be the subset of $A_G^+ \times A_{H_\lambda}^+$ consisting of elements

$$a_1 \leq \cdots \leq a_{j_1} \leq b_1 \leq a_{j_1+1} \leq \cdots \leq a_{j_1+j_2} \leq b_2 \leq \cdots \leq b_r \leq a_{j_1+\cdots+j_r+1} \leq \cdots \leq a_n \leq 1.$$

Then $A_G^+ \times A_{H_\lambda}^+ = \bigcup_{\underline{j}} I_{\underline{j}}$. Thus it is enough to prove the convergence of (7.2.1) when the domain is replaced by $I_{\underline{j}}$.

Over the region $I_{\underline{j}}$, the integrand of (7.2.1) equals

$$\begin{aligned}&|a_1|^{\frac{1}{2}} \cdots |a_{j_1}|^{\frac{2j_1+1}{2}} |b_1|^{-j_1+1} |a_{j_1+1}|^{\frac{2j_1+1}{2}} \cdots |a_{j_1+j_2}|^{\frac{2j_1+2j_2-3}{2}} |b_2|^{-j_1-j_2+2} \\ &\cdots |b_r|^{-j_1-\cdots-j_r+r} |a_{j_1+\cdots+j_r+1}|^{\frac{2(j_1+\cdots+j_r)+1-2r}{2}} \cdots |a_n|^{\frac{2n-1-2r}{2}}.\end{aligned}$$

Then lemma then follows from the following elementary fact.

Fact. Fix D a positive real number. The integral

$$\int_{|x_1| \leq \cdots \leq |x_s| \leq 1} |x_1|^{n_1-1} \cdots |x_s|^{n_s-1} \left(-\sum_{i=1}^{s} \log|x_i| \right)^D \mathrm{d}x_1 \cdots \mathrm{d}x_s$$

is convergent if $n_1 + \cdots n_t > 0$ for all $1 \leq t \leq s$. □

7.3. **Proof of Theorem 7.1.1.** Let S be a sufficiently large finite set of places of F, such that if $v \notin S$, then the following conditions hold.

(1) v is non-archimedean, 2 and λ are in $\mathfrak{o}_{F,v}^\times$, the conductor of ψ_v is $\mathfrak{o}_{F,v}$.
(2) The group A is unramified with a hyperspecial subgroup $A(\mathfrak{o}_{F,v})$, where $A = H, H_\lambda, G$.
(3) f_v is $H_\lambda(\mathfrak{o}_{F,v})$ fixed and φ_v is $G(\mathfrak{o}_{F,v})$ fixed. Moreover $\mathcal{B}(f_v, f_v) = \mathcal{B}(\varphi_v, \varphi_v) = 1$.
(4) Φ_λ is the characteristic function of $V_\lambda(\mathfrak{o}_{F,v})^n$ and ϕ_v is the characteristic function of $\mathfrak{o}_{F,v}^n$.
(5) The volume of the hyperspecial subgroup K_{A_v} is 1 under the chosen measure on $A(F_v)$, where $A = H, H_\lambda, G$.

We may assume that $\widetilde{\Theta}_{\psi^{-1}}(\pi) \neq 0$. If this is not the case, it follows from the computation below that both sides of Conjecture 2.3.1(3) vanish. Applying Lemma 7.2.1, Conjecture 6.3.1

and the Rallis inner product formula (for theta lifting from $\widetilde{G}$ to H), we get

(7.3.1)

$$|\mathcal{FJ}_{\psi_\lambda}(\varphi, \Theta_\psi(f, \Phi_\lambda), \phi)|^2 = \frac{2^{\gamma-1}\Delta_H^S}{|S_{\widetilde{\Theta}_{\psi_{-1}}(\pi)}||S_\sigma|} \frac{L^S(\frac{1}{2}, \widetilde{\Theta}_{\psi_{-1}}(\pi)\times\sigma)}{L^S(1, \widetilde{\Theta}_{\psi_{-1}}(\pi), \mathrm{Ad})L^S(1,\sigma,\mathrm{Ad})} \frac{L^S_{\psi_{-1}}(\frac{1}{2}, \pi\times\chi_V)}{\prod_{i=1}^n \zeta_F^S(2i)}$$

$$\prod_{v\in S}\int_{H_\lambda(F_v)}\int_{G(F_v)} \overline{\mathcal{B}(\sigma_v(h)f_v, f_v)}\mathcal{B}(\Omega_{\psi_v}(g,h)\Phi_{\lambda,v}, \Phi_{\lambda,v})\mathcal{B}(\pi_v(g)\varphi_v, \varphi_v)\overline{\mathcal{B}(\omega_{\psi_{-\lambda,v}}(g)\phi_v, \phi_v)}\mathrm{d}g\mathrm{d}h,$$

where γ is described as in Conjecture 6.3.1. We explain the use the Rallis inner product formula here in detail. In the remaining part of this paper, we are going to apply the same sort of argument several times. We will simply say that we apply the Rallis inner product for the rest of the paper.

First by Lemma 7.2.1, we have

$$|\mathcal{FJ}_{\psi_\lambda}(\varphi, \Theta_\psi(f, \Phi_\lambda), \phi)|^2 = \mathcal{I}(f, \widetilde{\Theta}_{\psi_{-1}}(\varphi, \overline{\Phi_\lambda}\otimes\phi)),$$

where $\mathcal{I}$ is defined in Section 6.3. Apply Conjecture 6.3.1 (in the form (6.3.4)), we have

$$\mathcal{I} = \frac{2^{\gamma}\Delta_H}{|S_{\widetilde{\Theta}_{\psi_{-1}}(\pi)}||S_\sigma|} \frac{L(\frac{1}{2}, \widetilde{\Theta}_{\psi_{-1}}(\pi)\times\sigma)}{L(1, \widetilde{\Theta}_{\psi_{-1}}(\pi), \mathrm{Ad})L(1,\sigma,\mathrm{Ad})}\prod_v \mathcal{I}_v^\natural.$$

Note that here the local linear form $\mathcal{I}_v^\natural$ is defined using an inner product $\mathcal{B}_v$ on $\widetilde{\Theta}_{\psi_{-1}}(\pi)_v$ so that $\prod_v \mathcal{B}_v$ equals the Petersson inner product on $\widetilde{\Theta}_{\psi_{-1}}(\pi)$ (defined using the Tamagawa measure on $H(\mathbb{A}_F)$). We view the Rallis inner product as another decomposition of the Petersson inner product on $\widetilde{\Theta}_{\psi_{-1}}(\pi)$. The integral

$$\int_{G(F_v)} \mathcal{B}(\widetilde{\Omega}_{\psi_v}(g,1)\Phi_v, \Phi'_v)\mathcal{B}(\pi_v(g)\varphi_v, \varphi'_v)\mathrm{d}g,$$

where we have used $\mathcal{B}$ to denote inner products on $\widetilde{\Omega}_{\psi_v}$ and on π_v by abuse of notation, defines a linear form on

$$\widetilde{\Omega}_{\psi_v}\otimes\pi_v\otimes\overline{\widetilde{\Omega}_{\psi_v}\otimes\pi_v}$$

which descends to an inner product on $\widetilde{\Theta}_{\psi_{-1,v}}(\pi_v)$ which we denote by $\mathcal{B}'_v$. Put

$$\mathcal{B}'^\natural_v = \mathcal{B}'_v\left(\frac{L_{\psi_{v,-1}}(\frac{1}{2}, \pi_v\times\chi_{V,v})}{\prod_{i=1}^n \zeta_{F_v}(2i)}\right)^{-1}.$$

Then in this case, the Rallis inner product formula claims that

$$\frac{1}{2}\frac{L_{\psi_{-1}}(\frac{1}{2}, \pi\times\chi_V)}{\prod_{i=1}^n\zeta_F(2i)}\prod_v \mathcal{B}'^\natural_v$$

equals the Petersson inner product on $\widetilde{\Theta}_{\psi_{-1}}(\pi)$. Let $\mathcal{I}'_v$ be the linear form defined in the same way as $\mathcal{I}_v$ but using the inner product $\mathcal{B}'_v$. Define

$$\mathcal{I}'^{\natural}_v = \mathcal{I}'_v \cdot \left(\Delta_{H_v} \frac{L(\frac{1}{2}, \widetilde{\Theta}_{\psi_{-1},v}(\pi_v) \times \sigma_v)}{L(1, \widetilde{\Theta}_{\psi_{-1},v}(\pi_v), \mathrm{Ad}) L(1, \sigma_v, \mathrm{Ad})} \frac{L_{\psi_v,-1}(\frac{1}{2}, \pi_v \times \chi_{V,v})}{\prod_{i=1}^n \zeta_{F_v}(2i)} \right)^{-1}.$$

It follows that we have a decomposition

$$\mathcal{I} = \frac{2^{\gamma-1}\Delta_H}{|S_{\widetilde{\Theta}_{\psi_{-1}}(\pi)}||S_\sigma|} \frac{L(\frac{1}{2}, \widetilde{\Theta}_{\psi_{-1}}(\pi) \times \sigma)}{L(1, \widetilde{\Theta}_{\psi_{-1}}(\pi), \mathrm{Ad}) L(1, \sigma, \mathrm{Ad})} \frac{L_{\psi_{-1}}(\frac{1}{2}, \pi \times \chi_V)}{\prod_{i=1}^n \zeta_F(2i)} \prod_v \mathcal{I}'^{\natural}_v. \tag{7.3.2}$$

This is an identity of elements in

$$\mathrm{Hom}_{\widetilde{G}(\mathbb{A}_F)\times H_\lambda(\mathbb{A}_F)}(\widetilde{\Omega}_\psi \otimes \pi \otimes \overline{\sigma}, \mathbb{C}) \otimes \overline{\mathrm{Hom}_{\widetilde{G}(\mathbb{A}_F)\times H_\lambda(\mathbb{A}_F)}(\widetilde{\Omega}_\psi \otimes \pi \otimes \overline{\sigma}, \mathbb{C})},$$

which descends to an identity of elements in

$$\mathrm{Hom}_{H_\lambda(\mathbb{A}_F)}(\widetilde{\Theta}_{\psi_{-1}}(\pi) \otimes \overline{\sigma}, \mathbb{C}) \otimes \overline{\mathrm{Hom}_{H_\lambda(\mathbb{A}_F)}(\widetilde{\Theta}_{\psi_{-1}}(\pi) \otimes \overline{\sigma}, \mathbb{C})}.$$

We now compute $\mathcal{I}(f, \widetilde{\Theta}_{\psi_{-1}}(\varphi, \overline{\Phi_\lambda} \otimes \phi))$ using decomposition (7.3.2). Note that

$$\widetilde{\Omega}_\psi|_{\widetilde{G}(\mathbb{A}_F)\times H_\lambda(\mathbb{A}_F)} \simeq \Omega_\psi \otimes \omega_{\psi_\lambda},$$

where $\widetilde{G}(\mathbb{A}_F)$ acts on both factors on the right hand side and $H_\lambda(\mathbb{A}_F)$ acts only on Ω_ψ. We also note that if $v \notin S$, then

$$\mathcal{I}'^{\natural}_v(\overline{\Phi_{\lambda,v}} \otimes \phi_v, \overline{\varphi_v}, f_v) = 1.$$

Then the identity (7.3.1) follows.

We continue the proof of Theorem 7.1.1. The double integral on the right hand side of (7.3.1) is absolutely convergent by Lemma 7.2.2. Thus we can change the order of integration by integrating over $g \in G(F_v)$ first. Then we apply Rallis inner product formula (for theta lifting from H_λ to G), and get

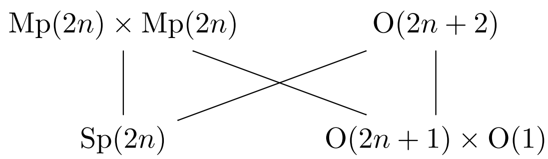
$$|\mathcal{F}\mathcal{J}_{\psi_\lambda}(\varphi, \xi, \phi)|^2 = \frac{2^{\gamma-1}\Delta_H^S}{|S_{\widetilde{\Theta}_{\psi_{-1}}(\pi)}||S_\sigma|} \cdot \frac{L^S(\frac{1}{2}, \widetilde{\Theta}_{\psi_{-1}}(\pi) \times \sigma)}{L^S(1, \widetilde{\Theta}_{\psi_{-1}}(\pi), \mathrm{Ad}) L^S(1, \sigma, \mathrm{Ad})} \left(\frac{L^S(1, \sigma)}{\prod_{i=1}^n \zeta_F^S(2i)} \right)^{-1}$$
$$\frac{L^S_{\psi_{-1}}(\frac{1}{2}, \pi \times \chi_V)}{\prod_{i=1}^n \zeta_F^S(2i)} \prod_{v \in S} \int_{H_\lambda(F_v)} \mathcal{B}(\Theta_{\psi_v}(\sigma_v)(g)\xi_v, \xi_v) \mathcal{B}(\pi_v(g)\varphi_v, \varphi_v) \overline{\mathcal{B}(\omega_{\psi_{-\lambda}}(g)\phi_v, \phi_v)} \mathrm{d}g,$$

where $\Theta_\psi(f, \Phi_\lambda) = \xi = \otimes \xi_v \in \Theta_\psi(\sigma)$. Here we fixed a surjective map $\vartheta_v : \overline{\sigma}_v \otimes \Omega_{\psi_v} \to \Theta_{\psi_v}(\sigma_v)$ for each v and put $\vartheta_v(f_v, \Phi_{\lambda,v}) = \xi_v$, so that $\xi = \otimes \xi_v$ holds. By Lemma 5.2.3, $|S_{\widetilde{\Theta}_{\psi_{-1}}(\pi)}||S_\sigma| = 2^{\gamma-1}|S_\pi||S_{\Theta_\psi(\sigma)}|$. Theorem 7.1.1 then follows from Lemma 5.2.2.

7.4. **Some remarks.** We end this section by some remarks on Theorem 7.1.1.

Remark 7.4.1. We have proved in the theorem that we can deduce Conjecture 2.3.1(3) from Conjecture 6.3.1 under the assumptions of the theorem. Similarly, we may also deduce Conjecture 6.3.1 from Conjecture 2.3.1(3). We only need to run the above argument backwards.

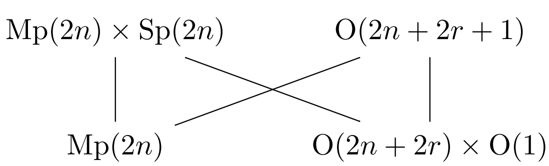
Remark 7.4.2. Instead of the seesaw diagram that has been used in the proof of Theorem 7.1.1, we may consider the following seesaw diagram.



Then we can go back and forth between Conjecture 2.3.1(3) for $\mathrm{Sp}(2n) \times \mathrm{Mp}(2n)$ and Ichino–Ikeda's conjecture for $\mathrm{SO}(2n+2) \times \mathrm{SO}(2n+1)$.

In particular, if $n = 1$, then the Ichino–Ikeda's conjecture, hence Conjecture 6.3.1 is known. In this case, without assuming Hypothesis LLC, GLC and O, [39, Theorem 4.5] proved Conjecture 2.3.1(3) with $|S_{\pi_2}||S_{\pi_0}|$ replaced by $\frac{1}{4}$. This result is compatible with our conjecture if we assume LLC, GLC and O.

Remark 7.4.3. Instead of the seesaw diagrams above, we may consider



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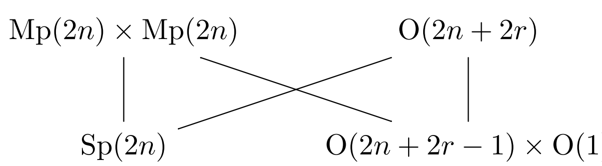

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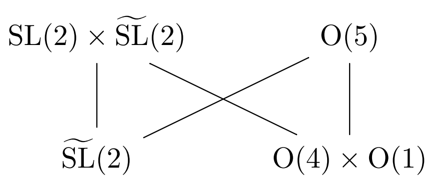
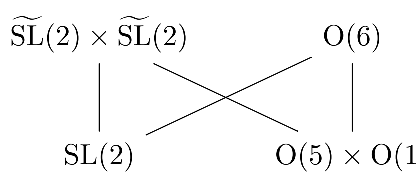
In this way, the Conjecture 2.3.1(3) for tempered representations on $\mathrm{Sp}(2n) \times \mathrm{Mp}(2n)$ will be related to the Ichino–Ikeda's conjecture for nontempered representations. Ichino [22] and Ichino–Ikeda [23] made use of the following seesaw diagrams respectively.



,


At this moment, there is no precise form of the refined Gan–Gross–Prasad conjecture for nontempered representations. We hope that Conjecture 2.3.1(3) together with the seesaw diagrams as above could shed some light on the formulation of this conjecture.

8. Compactibility with Ichino–Ikeda's conjecture: $\mathrm{Sp}(2n+2)\times\mathrm{Mp}(2n)$

8.1. **The theorem.** The goal of this section is to study Conjecture 2.3.1(3) for $\mathrm{Sp}(2n+2)\times\mathrm{Mp}(2n)$.

Let W be an $2n+2$ dimensional symplectic space and $G=\mathrm{Sp}(W)$. We choose a basis $\{e_1,\cdots,e_{n+1},e_1^*,\cdots,e_{n+1}^*\}$ of W so that symplectic form on W is given by the matrix

$$\begin{pmatrix} & 1_n \\ -1_n & \end{pmatrix}.$$

Let $X=\langle e_{n+1}\rangle$, $X^*=\langle e_{n+1}^*\rangle$ and $W_0=\langle e_1,\cdots,e_n,e_1^*,\cdots,e_n^*\rangle$. With this choice of basis, we identify W with F^{2n+2} and W_0 with F^{2n}. Let $L=\langle e_1,\cdots,e_n\rangle\simeq F^n$ and $L^*=\langle e_1^*,\cdots,e_n^*\rangle\simeq F^n$. Then $W_0=L+L^*$ is a complete polarization of W_0. We represent elements in G as matrices.

Let $R=R(W_0)=NG_0$ be the Jacobi group associated to W_0, where N is the unipotent radical and $G_0\simeq\mathrm{Sp}(W_0)$. The group R takes the form

$$\begin{pmatrix} 1_n & & & {}^ty \\ x & 1 & y & \kappa \\ & & 1_n & {}^tx \\ & & & 1\end{pmatrix}\begin{pmatrix} a & & b & \\ & 1 & & \\ c & & d & \\ & & & 1\end{pmatrix}$$

where $x,y\in F^n$, $\kappa\in F$ and $\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in G_0$. We write the first matrix as $n=n(x,y,\kappa)$. Let $\widetilde{G_0}=\mathrm{Mp}(W_0)$ and $\widetilde{R}=R\widetilde{G_0}$.

Let (V,q_V) be a $2n+2$ dimensional orthogonal space and $H=\mathrm{O}(V)$. Let $\lambda\in F^\times$ and $v_\lambda^0\in V$ such that $q_V(v_\lambda^0,v_\lambda^0)=\lambda$. Let V_λ be the orthogonal complement of $\langle v_\lambda^0\rangle$ and $H_\lambda=\mathrm{O}(V_\lambda)$.

Let ω_{ψ_λ} be the Weil representation of $\widetilde{R}(\mathbb{A}_F)$ which is realized on $\mathcal{S}(\mathbb{A}_F^n)$. Let Ω_ψ be the Weil representation of $G(\mathbb{A}_F)\times H(\mathbb{A}_F)$ which is realized on $\mathcal{S}(V(\mathbb{A}_F)^{n+1})$. Let Ω_ψ^0 be the Weil representation of $G_0(\mathbb{A}_F)\times H(\mathbb{A}_F)$ which is realized on $\mathcal{S}(V(\mathbb{A}_F)^n)$. Let $\widetilde{\Omega}_\psi$ be the Weil representation of $\widetilde{G_0}(\mathbb{A}_F)\times H_\lambda(\mathbb{A}_F)$ which is realized on $\mathcal{S}(V_\lambda(\mathbb{A}_F)^n)$. Suppose that $\phi\in\mathcal{S}(\mathbb{A}_F^n)$ (resp. $\Phi\in\mathcal{S}(V^{n+1}(\mathbb{A}_F))$, resp. $\widetilde{\Phi}\in\mathcal{S}(V_\lambda(\mathbb{A}_F)^n)$). Then we have the theta series on $\widetilde{R}(\mathbb{A}_F)$ (resp. $G(\mathbb{A}_F)\times H(\mathbb{A}_F)$, resp. $\widetilde{G_0}(\mathbb{A}_F)\times H_\lambda(\mathbb{A}_F)$)

$$\theta_{\psi_\lambda}(r,\phi),\quad \text{resp. } \Theta_\psi(g,h,\Phi),\quad \text{resp. } \widetilde{\Theta}_\psi(\widetilde{g},h_\lambda,\widetilde{\Phi}).$$

Let π be an irreducible cuspidal tempered automorphic representation of $H(\mathbb{A}_F)$. We denote by $\Theta_\psi(\pi)$ the global theta lifting of π to $G(\mathbb{A}_F)$, i.e. the automorphic representation of $G(\mathbb{A}_F)$ generated by the functions of the form

$$\Theta_\psi(f,\Phi)(\cdot)=\int_{H(F)\backslash H(\mathbb{A}_F)}\overline{f(h)}\Theta_\psi(\cdot,h,\Phi)\mathrm{d}h,\quad f\in\pi,\ \Phi\in\mathcal{S}(V(\mathbb{A}_F)^{n+1}).$$

Let σ be an irreducible cuspidal tempered genuine automorphic representation of $\widetilde{G_0}(\mathbb{A}_F)$ and $\widetilde{\Theta}_\psi(\sigma)$ be the theta lifting of σ to $H_\lambda(\mathbb{A}_F)$, i.e. the automorphic representation of $H_\lambda(\mathbb{A}_F)$ generated by the functions of the form

$$\widetilde{\Theta}_\psi(\varphi, \widetilde{\Phi})(\cdot) = \int_{G_0(F)\backslash G_0(\mathbb{A}_F)} \overline{\varphi(g)} \widetilde{\Theta}_\psi(g, \cdot, \widetilde{\Phi}) \mathrm{d}g.$$

Theorem 8.1.1. *Assume that $\Theta_\psi(\pi)$ and $\widetilde{\Theta}_\psi(\sigma)$ are both cuspidal. If Conjecture 6.3.1 holds for $(\pi, \widetilde{\Theta}_\psi(\sigma))$, then Conjecture 2.3.1(3) holds for $(\Theta_\psi(\pi), \sigma)$ (with the additive character ψ_λ). In particular, if $n = 1$, then Conjecture 2.3.1(3) holds for $(\Theta_\psi(\pi), \sigma)$ (with the additive character ψ_λ).*

The proof of this theorem will occupy the following four subsections. The last assertion follows from the fact that Ichino–Ikeda's conjecture is known for $\mathrm{SO}(4) \times \mathrm{SO}(3)$. Thus Conjecture 6.3.1 holds for $\mathrm{O}(4) \times \mathrm{O}(3)$.

Remark 8.1.2. We don't assume that $\widetilde{\Theta}_\psi(\sigma)$ is not zero. In fact, if $\widetilde{\Theta}_\psi(\sigma)$ is zero, then it follows from the computation below that both sides of the identity in Conjecture 2.3.1(3) are zero.

Remark 8.1.3. By assumption, there is a $v_\lambda^0 \in V$ such that $q_V(v_\lambda^0, v_\lambda^0) = \lambda$. If follows from the computation below that if such a v_λ^0 does not exist, then both sides of the identity in Conjecture 2.3.1(3) are zero.

8.2. **Measures.** Without saying to the contrary, we always take the Tamagawa measure on the group of adelic points of an algebraic group. Note that $\mathrm{vol}\, A(F)\backslash A(\mathbb{A}_F) = 1$ where $A = G, G_0, H, H_\lambda$. Note also that $\mathrm{vol}\, G_0(F)\backslash \widetilde{G_0}(\mathbb{A}_F) = 1$. Suppose that $A = G, G_0, H, H_\lambda$ or $\widetilde{G_0}$. We fix a decomposition $\mathrm{d}g = \prod_v \mathrm{d}g_v$ where $\mathrm{d}g_v$ is a measure on $A(F_v)$ so that for almost all places v, $\mathrm{vol}\, K_v = 1$ where $K_v = A(\mathfrak{o}_{F,v})$ is a hyperspecial maximal compact subgroup of $A(F_v)$.

Lemma 8.2.1. *Let $f \in \mathcal{S}(V(\mathbb{A}_F))$. Then*

$$\int_{\mathbb{A}_F} \left(\int_{V(\mathbb{A}_F)} f(v)\psi(\kappa q_V(v,v)) \mathrm{d}v \right) \psi(-\lambda\kappa) \mathrm{d}\kappa = \int_{H_\lambda(\mathbb{A}_F)\backslash H(\mathbb{A}_F)} f(h^{-1}v_\lambda^0) \mathrm{d}h. \tag{8.2.1}$$

Proof. Suppose that V is not a four dimensional split quadratic space. Then the lemma follows from the Siegel–Weil formula for $\mathrm{SL}_2 \times H$. Let $E(g, \Phi_f^{(s)})$ be the Eisenstein series on $\mathrm{SL}_2(\mathbb{A}_F)$ where $\Phi_f^{(s)} \in \mathrm{Ind}_B^{\mathrm{SL}_2(\mathbb{A}_F)} \chi_V |\cdot|^s$ is the Siegel–Weil section where B is the standard upper triangular Borel subgroup of SL_2. Then the left hand side of (8.2.1) is the ψ_λ-Fourier coefficient of $E(g, \Phi_f^{(s)})$ at $s = s_0 = n$. The right hand side of (8.2.1) is the ψ_λ-Fourier coefficient of the theta integral

$$\int_{H(F)\backslash H(\mathbb{A}_F)} \theta_\psi(g, h, f) \mathrm{d}h,$$

where $\theta_\psi(g,h,f)$ is the theta series on $\mathrm{SL}_2(\mathbb{A}_F)\times H(\mathbb{A}_F)$. The lemma then follows from the (convergent) Siegel–Weil formula

$$E(g,\Phi_f^{(s)})|_{s=s_0}=\int_{H(F)\backslash H(\mathbb{A}_F)}\theta_\psi(g,h,f)\mathrm{d}h.$$

Suppose that V is split and $\dim V=4$. Without loss of generality, we may assume that $\lambda=1$. Then V is identified with the space of 2×2 matrices over F and the quadratic form is given by the determinant. We may assume $v_1^0=1_2\in V$. Under this identification, $H_1(\mathbb{A}_F)\backslash H(\mathbb{A}_F)$ is identified with $\mathrm{SL}_2(\mathbb{A}_F)$ and the quotient measure is identified with the Tamagawa measure on $\mathrm{SL}_2(\mathbb{A}_F)$. This is because the volume of $H(F)H_1(\mathbb{A}_F)\backslash H(\mathbb{A}_F)$ equals one.

We write an element in V as $\begin{pmatrix}x_1 & x_2\\ x_3 & x_4\end{pmatrix}$. The left hand side of the desired identity equals

$$\int_{\mathbb{A}_F}\int_{\mathbb{A}_F^4}f(x_1,x_2,x_3,x_4)\psi(\kappa(x_1x_4-x_2x_3)-\kappa)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4\mathrm{d}\kappa.$$

By the Fourier inversion formula, it equals

$$\int_{\mathbb{A}_F^2}\int_{\mathbb{A}_F}f(x_1^0+ax_3,x_2^0+ax_4,x_3,x_4)\mathrm{d}a\mathrm{d}x_3\mathrm{d}x_4,$$

where $(x_1^0,x_2^0)\in\mathbb{A}_F^2$ is a fixed vector of norm one and perpendicular to (x_3,x_4) under the usual Euclidean inner product on $\mathbb{A}_F^2$. The choice of (x_1^0,x_2^0) is not unique, but the above formula does not depend on the choice. The measure $\mathrm{d}a\mathrm{d}x_3\mathrm{d}x_4$ gives a measure on $\mathrm{SL}_2(\mathbb{A}_F)$ which is invariant under the right multiplication of $\mathrm{SL}_2(\mathbb{A}_F)$. It is clear that it gives $\mathrm{SL}_2(F)\backslash\mathrm{SL}_2(\mathbb{A}_F)$ volume one, hence it is the Tamagawa measure on $\mathrm{SL}_2(\mathbb{A}_F)$. The lemma then follows. □

8.3. **Global Fourier–Jacobi periods of Theta liftings.** The goal of this subsection is to compute

$$\int_{G_0(F)\backslash G_0(\mathbb{A}_F)}\int_{N(F)\backslash N(\mathbb{A}_F)}\int_{H(F)\backslash H(\mathbb{A}_F)}\overline{f(h)}\Theta_\psi(ng,h,\Phi)\overline{\theta_{\psi_\lambda}(ng,\phi)}\varphi(g)\mathrm{d}h\mathrm{d}n\mathrm{d}g. \tag{8.3.1}$$

The idea of the computation is putting in the definition of the theta series and unfolding the integrals. The essential step is the identity (8.3.2). In this identity, the summation over rational points in V of norm λ is replaced by the summation over $H_\lambda(F)\backslash H(F)$. This is the key step which enable us to unfold the integrals. We divide the computation in several steps.

Step 1. The goal is to unwinding the definition of the theta functions.

Suppose $n=n(x,y,\kappa)$, $\kappa\in F\backslash\mathbb{A}_F$, $x=(x_1,\cdots,x_n)\in(F\backslash\mathbb{A}_F)^n$ and $y=(y_1,\cdots,y_n)\in(F\backslash\mathbb{A}_F)^n$. By definition, we have

$$\theta_{\psi_\lambda}(ng,\phi)=\sum_{l_1,\cdots,l_n\in F}\omega_{\psi_\lambda}(g)\phi(l_1+x_1,\cdots,l_n+x_n)\psi(\lambda y_1(x_1+2l_1)+\cdots+\lambda y_n(x_n+2l_n)+\lambda\kappa).$$

Suppose $\Phi = \Phi^0 \otimes \Phi_{n+1}$ where $\Phi^0 \in \mathcal{S}(V(\mathbb{A}_F)^n)$ and $\Phi_{n+1} \in \mathcal{S}(V(\mathbb{A}_F))$. We have an $H(\mathbb{A}_F) \times G_0(\mathbb{A}_F)$ equivariant isomorphism

$$\mathcal{S}(V(\mathbb{A}_F)^{n+1}) \simeq \mathcal{S}(V(\mathbb{A}_F)^n) \otimes \mathcal{S}(V(\mathbb{A}_F)),$$

where the left hand side is the Weil representation Ω_ψ restricted to $H(\mathbb{A}_F) \times G_0(\mathbb{A}_F)$ and this group acts on the first factor via the Weil representation Ω^0_ψ and on the second factor via projection to $H(\mathbb{A}_F)$ and multiplication from the left.

Then we have

$$\Theta(ng, h, \Phi) = \sum_{v_1,\cdots,v_n,v_{n+1}\in V} \Omega^0_\psi(g)\Phi^0(h^{-1}(v_1 + x_1v_{n+1}), \cdots, h^{-1}(v_n + x_nv_{n+1}))\Phi_{n+1}(h^{-1}v_{n+1})$$
$$\psi(2y_1q_V(v_1, v_{n+1}) + \cdots + 2y_nq_V(v_n, v_{n+1}) + (\kappa + y\,{}^tx)q_V(v_{n+1}, v_{n+1})).$$

Therefore

$$\int_{F\backslash\mathbb{A}_F} \Theta(ng, h, \Phi)\overline{\psi_\lambda(\kappa)}\mathrm{d}\kappa = \sum_{\substack{v_1,\cdots,v_n\in V\\ q_V(v_{n+1},v_{n+1})=\lambda}} \Omega^0_\psi(g)\Phi^0(h^{-1}(v_1 + x_1v_{n+1}), \cdots, h^{-1}(v_n + x_nv_{n+1}))$$
$$\Phi_{n+1}(h^{-1}v_{n+1})\psi(2y_1q_V(v_1, v_{n+1}) + \cdots 2y_nq_V(v_n, v_{n+1}) + y\,{}^tx\lambda)).$$

From this we get

$$\int_{N(F)\backslash N(\mathbb{A}_F)} \Theta(ng, h, \Phi)\overline{\theta_{\psi_\lambda}(n\iota(g), \phi)}\mathrm{d}n$$
$$= \sum_{\substack{v_1,\cdots,v_n\in V\\ q_V(v_{n+1},v_{n+1})=\lambda\\ l_1,\cdots,l_n\in F}} \int_{(F\backslash\mathbb{A}_F)^{2n}} \Omega^0_\psi(g)\Phi^0(h^{-1}(v_1 + x_1v_{n+1}), \cdots, h^{-1}(v_n + x_nv_{n+1}))\Phi_{n+1}(h^{-1}v_{n+1})$$
$$\overline{\omega_{\psi_\lambda}(\iota(g))\phi(l_1 + x_1, \cdots, l_n + x_n)}\psi(2y_1(q_V(v_1, v_{n+1}) - l_1\lambda) + \cdots 2y_n(q_V(v_n, v_{n+1}) - l_n\lambda)))\mathrm{d}x\mathrm{d}y.$$

Recall that if $g \in G_0$, then we define $\iota(g) = (g, 1) \in \widetilde{G_0}$.

Step 2. This is the key step. We replace the summation over rational points in V of norm λ by the summation over $H_\lambda(F)\backslash H(F)$.

Let $\Lambda_\lambda = \{v \in V \mid q_V(v, v) = \lambda\}$. Then the group $H(F)$ acts transitively on $\Lambda_\lambda(F)$ and identifies $H_\lambda(F)\backslash H(F)$ with $\Lambda_\lambda(F)$ by $h \mapsto h^{-1}v^0_\lambda$. It follows that

(8.3.2)

$$(8.3.1) = \sum_{\substack{v_1,\cdots,v_n\in V\\ l_1,\cdots,l_n\in F}} \int_{(F\backslash\mathbb{A}_F)^{2n}} \int_{G_0(F)\backslash G_0(\mathbb{A}_F)} \int_{H_\lambda(F)\backslash H(\mathbb{A}_F)} \overline{f(h)}$$
$$\Omega^0_\psi(g)\Phi^0(h^{-1}(v_1 + x_1v^0_\lambda), \cdots, h^{-1}(v_n + x_nv^0_\lambda))\Phi_{n+1}(h^{-1}v^0_\lambda)\overline{\omega_{\psi_\lambda}(g)\phi(l_1 + x_1, \cdots, l_n + x_n)}$$
$$\psi(2y_1(q_V(v_1, v^0_\lambda) - l_1\lambda) + \cdots + 2y_n(q_V(v_n, v^0_\lambda) - l_n\lambda)))\varphi(g)\mathrm{d}h\mathrm{d}g\mathrm{d}x\mathrm{d}y.$$

Then

$$(8.3.1) = \sum_{\substack{v_1,\cdots,v_n\in V\\ l_1,\cdots,l_n\in F}} \int_{(F\backslash\mathbb{A}_F)^{2n}} \int_{G_0(F)\backslash G_0(\mathbb{A}_F)} \int_{H_\lambda(\mathbb{A}_F)\backslash H(\mathbb{A}_F)} \int_{H_\lambda(F)\backslash H_\lambda(\mathbb{A}_F)} \overline{f(h_\lambda h)}$$

$$\Omega^0_\psi(g)\Phi^0(h^{-1}h_\lambda^{-1}v_1 + x_1h^{-1}v^0_\lambda, \cdots, h^{-1}h_\lambda^{-1}v_n + x_nh^{-1}v^0_\lambda)\Phi_{n+1}(h^{-1}v^0_\lambda)\overline{\omega_{\psi_\lambda}(g)\phi(l_1+x_1,\cdots,l_n+x_n)}$$

$$\psi(2y_1(q_V(v_1,v^0_\lambda)-l_1\lambda)+\cdots+2y_n(q_V(v_n,v^0_\lambda)-l_n\lambda)))\varphi(g)\mathrm{d}h_\lambda\mathrm{d}h\mathrm{d}g\mathrm{d}x\mathrm{d}y.$$

Step 3. Simplifying the expression. This step is mostly formal.

Integrations over y_i's yield

$$(8.3.1) = \sum_{\substack{v_1,\cdots,v_n\in V\\ l_1,\cdots,l_n\in F\\ q_V(v_i,v^0_\lambda)=l_i\lambda,\forall i}} \int_{(F\backslash\mathbb{A}_F)^{n}} \int_{G_0(F)\backslash G_0(\mathbb{A}_F)} \int_{H_\lambda(\mathbb{A}_F)\backslash H(\mathbb{A}_F)} \int_{H_\lambda(F)\backslash H_\lambda(\mathbb{A}_F)}$$

$$\overline{f(h_\lambda h)}\Omega^0_\psi(g)\Phi^0(h^{-1}h_\lambda^{-1}v_1 + x_1h^{-1}v^0_\lambda, \cdots, h^{-1}h_\lambda^{-1}v_n + x_nh^{-1}v^0_\lambda)$$

$$\Phi_{n+1}(h^{-1}v^0_\lambda)\overline{\omega_{\psi_\lambda}(g)\phi(l_1+x_1,\cdots,l_n+x_n)}\varphi(g)\mathrm{d}h_\lambda\mathrm{d}h\mathrm{d}g\mathrm{d}x.$$

The variables v_i have to be of the form $l_iv^0_\lambda + w_i$ where $w_i \in V_\lambda$. Therefore

$$(8.3.1) = \sum_{\substack{w_1,\cdots,w_n\in V_\lambda\\ l_1,\cdots,l_n\in F}} \int_{(F\backslash\mathbb{A}_F)^{n}} \int_{G_0(F)\backslash G_0(\mathbb{A}_F)} \int_{H_\lambda(\mathbb{A}_F)\backslash H(\mathbb{A}_F)} \int_{H_\lambda(F)\backslash H_\lambda(\mathbb{A}_F)}$$

$$\overline{f(h_\lambda h)}\Omega^0_\psi(g)\Phi^0(h^{-1}h_\lambda^{-1}w_1 + (l_1+x_1)h^{-1}v^0_\lambda, \cdots, h^{-1}h_\lambda^{-1}w_n + (l_n+x_n)h^{-1}v^0_\lambda)$$

$$\Phi_{n+1}(h^{-1}v^0_\lambda)\overline{\omega_{\psi_\lambda}(g)\phi(l_1+x_1,\cdots,l_n+x_n)}\varphi(g)\mathrm{d}h_\lambda\mathrm{d}h\mathrm{d}g\mathrm{d}x.$$

Thus

$$(8.3.1) = \sum_{w_1,\cdots,w_n\in V_\lambda} \int_{\mathbb{A}_F^n} \int_{G_0(F)\backslash G_0(\mathbb{A}_F)} \int_{H_\lambda(\mathbb{A}_F)\backslash H(\mathbb{A}_F)} \int_{H_\lambda(F)\backslash H_\lambda(\mathbb{A}_F)} \overline{f(h_\lambda h)}$$

$$\Omega^0_\psi(g)\Phi^0(h^{-1}h_\lambda^{-1}w_1 + x_1h^{-1}v^0_\lambda, \cdots, h^{-1}h_\lambda^{-1}w_n + x_nh^{-1}v^0_\lambda)$$

$$\Phi_{n+1}(h^{-1}v^0_\lambda)\overline{\omega_{\psi_\lambda}(g)\phi(x_1,\cdots,x_n)}\varphi(g)\mathrm{d}h_\lambda\mathrm{d}h\mathrm{d}g\mathrm{d}x.$$

We define

$$\Phi^0 * \overline{\phi}(w_1,\cdots,w_n) = \int_{\mathbb{A}_F^n} \Phi^0(w_1+x_1v^0_\lambda,\cdots,w_n+x_nv^0_\lambda)\overline{\phi(x_1,\cdots,x_n)}\mathrm{d}x_1\cdots\mathrm{d}x_n. \tag{8.3.3}$$

Then $\Phi^0 * \overline{\phi} \in \mathcal{S}(V_\lambda(\mathbb{A}_F)^n)$.

It is straight forward to check that

$$\widetilde{\Omega}_\psi(\widetilde{g},h_\lambda)(\Phi^0 * \overline{\phi}) = (\Omega^0_\psi(g,h_\lambda)\Phi^0) * \overline{(\omega_{\psi_\lambda}(\widetilde{g})\phi)}, \quad \widetilde{g}\in\widetilde{G_0}(\mathbb{A}_F), h_\lambda \in H_\lambda(\mathbb{A}_F),$$

where g is the image of $\widetilde{g}$ in $G_0(\mathbb{A}_F)$. With this definition, we have

$$(8.3.1) = \int_{H_\lambda(\mathbb{A}_F)\backslash H(\mathbb{A}_F)} \left(\int_{H_\lambda(F)\backslash H_\lambda(\mathbb{A}_F)} \overline{f(h_\lambda h)}\widetilde{\Theta}_\psi(\overline{\varphi}, (\Omega^0_\psi(h)\Phi^0) * \overline{\phi})(h_\lambda)\mathrm{d}h_\lambda \right) \Phi_{n+1}(h^{-1}v^0_\lambda)\mathrm{d}h.$$

We summarize the above computation in the following lemma.

Lemma 8.3.1.

$$\int_{G_0(F)\backslash G_0(\mathbb{A}_F)}\int_{N(F)\backslash N(\mathbb{A}_F)}\int_{H(F)\backslash H(\mathbb{A}_F)}\overline{f(h)}\Theta_\psi(ng,h,\Phi^0\otimes\Phi_{n+1})\overline{\theta_{\psi_\lambda}(ng,\phi)}\varphi(g)\mathrm{d}h\mathrm{d}n\mathrm{d}g$$
$$=\int_{H_\lambda(\mathbb{A}_F)\backslash H(\mathbb{A}_F)}\left(\int_{H_\lambda(F)\backslash H_\lambda(\mathbb{A}_F)}\overline{f(h_\lambda h)}\widetilde{\Theta}_\psi(\overline{\varphi},(\Omega^0_\psi(h)\Phi^0)*\overline{\phi})(h_\lambda)\mathrm{d}h_\lambda\right)\Phi_{n+1}(h^{-1}v^0_\lambda)\mathrm{d}h.$$

8.4. **Local Fourier–Jacobi periods of theta liftings.** We now switch to the local situation. We fix a place v of F and suppress it from all notation. So F stands for a local field of characteristic zero. We have the local version of all the previous objects, e.g. Weil representations, the representations π, σ, and the theta liftings $\Theta_\psi(\pi)$, $\widetilde{\Theta}_\psi(\sigma)$, the orbit Λ_λ of v^0_λ under the action of $H(F)$, which is identified with $H_\lambda(F)\backslash H(F)$, etc. We denote by $\mathcal{B}$ the inner products on various unitary representations.

The goal is to compute

$$\int_{G_0(F)}\int_{N(F)}\int_{H(F)}\overline{\mathcal{B}(\pi(h)f,f)}\mathcal{B}(\Omega_\psi(ng,h)\Phi,\Phi)\overline{\mathcal{B}(\omega_{\psi_\lambda}(ng)\phi,\phi)}\mathcal{B}(\sigma(g)\varphi,\varphi)\mathrm{d}h\mathrm{d}n\mathrm{d}g, \tag{8.4.1}$$

where $\Phi=\Phi^0\otimes\Phi_{n+1}$ where $\Phi^0\in\mathcal{S}(V^n)$ and $\Phi_{n+1}\in\mathcal{S}(V)$.

The computation is parallel to the global computation as given in the previous subsection. The idea is again to unwind the definition of the Weil representations. The unfolding argument in the global situation is replaced by several integration formulas in the local case. The computation, however, is messy and technical. We list the main steps.

(1) Showing that the integral (8.4.1) is absolutely convergent. Thus we may change the order of integration.
(2) Computation of the integral over $N(F)$, namely,
$$\int_{N(F)}\mathcal{B}(\Omega_\psi(ng,h)\Phi,\Phi)\overline{\mathcal{B}(\omega_{\psi_\lambda}(n\iota(g))\phi,\phi)}\mathrm{d}n$$
for $g\in G_0(F)$ and $h\in H(F)$. The goal is to unwinding the definition of the Weil representations and show that this integral equals (8.4.6). The key point in this step is the integral formula Lemma 8.4.3.
(3) Simplifying the results from the previous step. Here we make use of the integration formula Lemma 8.4.4 which is a variant of the fact that Fourier transform preserves L^2 norm of Schwartz functions. The final outcome is a clean expression (8.4.7) of the integral over $N(F)$.
(4) Computing (8.4.1) using (8.4.7). The final result is summarized in Lemma 8.4.5. This steps requires no more than making change of variables.

We organize the following computation in the above described steps.

Step 1. Absolute convergence.

Lemma 8.4.1. *The integral* (8.4.1) *is absolutely convergent.*

Proof. In view of Proposition 2.2.1 (the case $r = 1$), we only need to prove that for some $A > 0$, we have

$$\int_{H(F)} \Xi(h)|\mathcal{B}(\Omega_\psi(g,h)\Phi,\Phi)|\mathrm{d}h \ll \Xi(g)(1+\varsigma(g))^A, \quad g \in G(F). \tag{8.4.2}$$

Note that

$$\left|\int_{H(F)} \Xi(h)\mathcal{B}(\Omega_\psi(g,h)\Phi,\Phi)\mathrm{d}h\right| \ll \Xi(g)(1+\varsigma(g))^A, \quad g \in G(F),$$

since the left hand side is a matrix coefficient of a tempered representation.

Even though in general $|\mathcal{B}(\Omega_\psi(g,h)\Phi,\Phi)|$ is not a matrix coefficient of the Weil representation, we claim that it is dominated by a matrix coefficient of the Weil representation. In fact, by the Cartan decomposition, we only need to prove this when $g = a \in A_G^+$ and $h = b \in A_H^+$. Then

$$|\mathcal{B}(\Omega_\psi(g,h)\Phi,\Phi)| \leq \int_{V(F)^{n+1}} |\Phi(b^{-1}va)\Phi(v)|\mathrm{d}v.$$

We may find a Schwartz function Φ^+ so that $|\Phi| \leq \Phi^+$ (pointwise). We have proved the claim and hence the lemma. □

Step 2. Computing the integral over $N(F)$.

We recall the following well-known lemma.

Lemma 8.4.2 ([36, Lemma 3.18])**.** *There is a unique measure* $\underline{\mathrm{d}}h$ *on* $H_\lambda(F)\backslash H(F)$*, such that for any* $f \in \mathcal{S}(V)$*, we have*

$$\int_V f(v)\mathrm{d}v = \int_{F^\times}\int_{H_\lambda(F)\backslash H(F)} f(h^{-1}v_\lambda^0)\underline{\mathrm{d}}h\mathrm{d}\lambda,$$

where $\mathrm{d}v$ *is the self-dual measure on* V *and* $\mathrm{d}\lambda$ *is the self-dual measure on* F*.*

For the rest of this section, when we use the notation $\underline{\mathrm{d}}$ to denote a measure on $H_\lambda(F)\backslash H(F)$, we always mean the measure defined in this lemma.

We need the following integration formula.

Lemma 8.4.3. *Let* $f \in \mathcal{S}(V)$*. Then* $\int_V f(v)\psi(\kappa q_V(v,v))\mathrm{d}v$ *is absolutely integrable as a function of* κ*. Moreover,*

$$\int_F \left(\int_V f(v)\psi(\kappa q_V(v,v))\mathrm{d}v\right)\psi(-\lambda\kappa)\mathrm{d}\kappa = \int_{H_\lambda(F)\backslash H(F)} f(h^{-1}v_\lambda^0)\underline{\mathrm{d}}h. \tag{8.4.3}$$

Proof. The integral $\int_V f(v)\psi(\kappa q_V(v,v))\mathrm{d}v$ equals

$$\Phi_f^n\left(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} 1 & \kappa \\ & 1\end{pmatrix}\right),$$

where Φ_f^n is the Siegel–Weil section of $\mathrm{Ind}^{\mathrm{SL}_2(F)}\chi_V|\cdot|^s$ at $s = s_0 = n$. Then by the decomposition

$$\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} 1 & \kappa \\ & 1 \end{pmatrix} = \begin{pmatrix} -\kappa^{-1} & 1 \\ & -\kappa \end{pmatrix}\begin{pmatrix} 1 & \\ \kappa^{-1} & 1 \end{pmatrix},$$

the order of magnitude of $\int_V f(v)\psi(\kappa q_V(v,v))\mathrm{d}v$ is $|\kappa|^{-n-1}$ when $|\kappa|$ is large. The integrability then follows.

By Lemma 8.4.2,

$$\int_V f(v)\psi(\kappa q_V(v,v))\mathrm{d}v = \int_{F^\times}\left(\int_{H_{\lambda'}(F)\backslash H(F)} f(h^{-1}v^0_{\lambda'})\underline{\mathrm{d}}h\right)\psi(-\lambda'\kappa)\mathrm{d}\lambda'.$$

Since f is Schwartz, $\int_{H_{\lambda'}(F)\backslash H(F)} f(h^{-1}v^0_{\lambda'})\underline{\mathrm{d}}h$ is integrable as a function of λ' and is continuous on $F^\times$. The lemma then follows from the Fourier inversion formula. $\square$

Thanks to Lemma 8.4.1, we may change the order of integrations in (8.4.1). We integrate over $N(F)$ first. By definition,

$$\begin{aligned}\mathcal{B}(\Omega_\psi(ng,h)\Phi,\Phi) = \int_{V^{n+1}} &\Omega^0_\psi(g)\Phi^0(h^{-1}(v_1+x_1v_{n+1}),\cdots,h^{-1}(v_n+x_nv_{n+1}))\overline{\Phi^0(v_1,\cdots,v_n)} \\ &\psi(2y_1q_V(v_1,v_{n+1})+\cdots 2y_nq_V(v_n,v_{n+1})+(\kappa+y\,{}^tx)q_V(v_{n+1},v_{n+1})) \\ &\Phi_{n+1}(h^{-1}v_{n+1})\overline{\Phi_{n+1}(v_{n+1})}\mathrm{d}v_1\cdots\mathrm{d}v_{n+1}.\end{aligned}$$

Here $n = n(x,y,\kappa)$ and $x = (x_1,\cdots,x_n)\in F^n$, $y = (y_1,\cdots,y_n)\in F^n$, $\kappa\in F$. It follows from Lemma 8.4.3 that

(8.4.4)

$$\begin{aligned}&\int_F\int_{V^{n+1}}\Omega_\psi(ng,h)\Phi(v_1,\cdots,v_n,v_{n+1})\overline{\Phi(v_1,\cdots,v_n,v_{n+1})}\psi(-\lambda\kappa)\mathrm{d}v_1\cdots\mathrm{d}v_n\mathrm{d}v_{n+1}\mathrm{d}\kappa \\ =&\int_{H_\lambda(F)\backslash H(F)}\int_{V^n}\Omega^0_\psi(g)\Phi^0(h^{-1}(v_1+x_1h'^{-1}v^0_\lambda),\cdots,h^{-1}(v_n+x_nh'^{-1}v^0_\lambda))\overline{\Phi^0(v_1,\cdots,v_n)} \\ &\psi(2y_1q_V(v_1,h'^{-1}v^0_\lambda)+\cdots 2y_nq_V(v_n,h'^{-1}v^0_\lambda)+(x_1y_1+\cdots x_ny_n)\lambda) \\ &\Phi_{n+1}(h^{-1}h'^{-1}v^0_\lambda)\overline{\Phi_{n+1}(h'^{-1}v^0_\lambda)}\mathrm{d}v_1\cdots\mathrm{d}v_n\underline{\mathrm{d}}h'.\end{aligned}$$

The integral on the right hand side is absolutely convergent. In fact, the integrand is bounded by

$$C|\Phi^0(v_1,\cdots,v_n)\Phi_{n+1}(h'^{-1}v^0_\lambda)|,$$

where C is a constant which is independent of x and y.

By definition,

$$\begin{aligned}&\mathcal{B}(\omega_{\psi_\lambda}(n(x,y,0)\widetilde{g})\phi,\phi) \\ =&\int_{F^n}\omega_{\psi_\lambda}(\widetilde{g})\phi(l_1+x_1,\cdots,l_n+x_n)\overline{\phi(l_1,\cdots,l_n)}\psi(\lambda y_1(x_1+2l_1)+\cdots+\lambda y_n(x_n+2l_n))\mathrm{d}l_1\cdots\mathrm{d}l_n,\end{aligned}$$

where $\widetilde{g}\in\widetilde{G_0}$.

We claim that

$$\int_{F^{2n}}\int_{H_\lambda(F)\backslash H(F)}\int_{V^n}|*||\mathcal{B}(\omega_{\psi_\lambda}(n(x,y,0)\widetilde{g})\phi,\phi)|\mathrm{d}v_1\cdots\mathrm{d}v_n\underline{\mathrm{d}}h'\mathrm{d}x\mathrm{d}y \tag{8.4.5}$$

is convergent, where $*$ stands for the integrand of the right hand side of (8.4.4). Indeed, this integral is bounded by the convergent integral

$$\begin{aligned}C\times\int_{H_\lambda(F)\backslash H(F)}\int_{V^n}|\Phi(v_1,\cdots,v_n,v_{n+1})\Phi_{n+1}(h'^{-1}v_\lambda^0)|\mathrm{d}v_1\cdots\mathrm{d}v_n\underline{\mathrm{d}}h'\\ \times\int_{F^{2n}}|\mathcal{B}(\omega_\lambda(n(x,y,0)\widetilde{g})\phi,\phi)|\mathrm{d}x\mathrm{d}y,\end{aligned}$$

where C is some constant.

Thanks to the convergence of (8.4.5), we can change the order of the integration of $x,y\in F^n$ and $h'\in H_\lambda(F)\backslash H(F)$. We end up with

$$\int_{N(F)}\mathcal{B}(\Omega_\psi(ng,h)\Phi,\Phi)\overline{\mathcal{B}(\omega_{\psi_\lambda}(n\iota(g))\phi,\phi)}\mathrm{d}n$$

equals the following integral:

(8.4.6)

$$\begin{aligned}&\int_{H_\lambda(F)\backslash H(F)}\int_{F^{2n}}\int_{V^n}\int_{F^n}\Omega_\psi^0(g)\Phi^0(h^{-1}(v_1+x_1h'^{-1}v_\lambda^0),\cdots,h^{-1}(v_n+x_nh'^{-1}v_\lambda^0))\overline{\Phi^0(v_1,\cdots,v_n)}\\&\psi(2y_1q_V(v_1,h'^{-1}v_\lambda^0)+\cdots+2y_nq_V(v_n,h'^{-1}v_\lambda^0)+(x_1y_1+\cdots x_ny_n)\lambda)\\&\overline{\omega_{\psi_\lambda}(\iota(g))\phi(l_1+x_1,\cdots,l_n+x_n)}\phi(l_1,\cdots,l_n)\psi(-\lambda y_1(x_1+2l_1)-\cdots-\lambda y_n(x_n+2l_n))\\&\Phi_{n+1}(h^{-1}h'^{-1}v_\lambda^0)\overline{\Phi_{n+1}(h'^{-1}v_\lambda^0)}\mathrm{d}l_1\cdots\mathrm{d}l_n\mathrm{d}v_1\cdots\mathrm{d}v_n\mathrm{d}y_1\cdots\mathrm{d}y_n\mathrm{d}x_1\cdots\mathrm{d}x_n\underline{\mathrm{d}}h'.\end{aligned}$$

Step 3. Simplifying the three inner integrals of (8.4.6).

We need the following integration formula.

Lemma 8.4.4. *Let f be a Schwartz function on V^n and ϕ a Schwartz function on F^n. Let $v^0\in V$ with $q_V(v^0,v^0)=\lambda$ and $\{v^0\}^\perp$ be its orthogonal complement. Then*

$$\begin{aligned}\int_{F^n}\int_{V^n}\int_{F^n}\psi(2y_1q_V(v_1,v^0)+\cdots2y_nq_V(v_n,v^0)-2y_1l_1\lambda-\cdots-2y_nl_n\lambda)\\ f(v_1,\cdots,v_n)\overline{\phi(l_1,\cdots,l_n)}\mathrm{d}l_1\cdots\mathrm{d}l_n\mathrm{d}v_1\cdots\mathrm{d}v_n\mathrm{d}y_1\cdots\mathrm{d}y_n.\end{aligned}$$

equals

$$|2\lambda|^{-n}\int_{(\{v^0\}^\perp)^n}\int_{F^n}f(l_1v^0+w_1,\cdots l_nv^0+w_n)\overline{\phi(l_1,\cdots,l_n)}\mathrm{d}l_1\cdots\mathrm{d}l_n\mathrm{d}w_1\cdots\mathrm{d}w_n.$$

Proof. Let $\widehat{f}$ and $\widehat{\phi}$ be the Fourier transform of f and ϕ respectively (with respect to ψ). Then the first integral in the lemma equals

$$\int_{F^n}\widehat{f}(2y_1v^0,\cdots,2y_nv^0)\overline{\widehat{\phi}(2y_1\lambda,\cdots,2y_n\lambda)}\mathrm{d}y_1\cdots\mathrm{d}y_n.$$

The lemma then follows from the fact that the Fourier transform preserves the inner product of Schwartz functions. $\square$

Applying this lemma, we see that

$$\begin{aligned}&\text{Inner three integrals of (8.4.6)}\\ =&|2\lambda|^{-n}\int_{V_\lambda^n}\int_{F^n}\int_{F^n}\Omega_\psi^0(g)\Phi^0(h^{-1}h'^{-1}(w_1+l_1v_\lambda^0+x_1v_\lambda^0),\cdots,h^{-1}h'^{-1}(w_n+l_nv_\lambda^0+x_nv_\lambda^0))\\ &\overline{\Phi^0(w_1+l_1h'^{-1}v_\lambda^0,\cdots,w_n+l_nh'^{-1}v_\lambda^0)}\overline{\omega_{\psi_\lambda}(\iota(g))\phi(l_1+x_1,\cdots,l_n+x_n)}\phi(l_1,\cdots,l_n)\\ &\mathrm{d}w_1\cdots\mathrm{d}w_n\mathrm{d}l_1\cdots\mathrm{d}l_n\mathrm{d}x_1\cdots\mathrm{d}x_n.\end{aligned}$$

This integral is absolutely convergent. We then make change of variables $x_i\mapsto x_i-l_i$. Then

$$\begin{aligned}&\text{Inner three integrals of (8.4.6)}\\ =&|2\lambda|^{-n}\int_{V_\lambda^n}\int_{F^n}\int_{F^n}\Omega_\psi^0(g)\Phi^0(h^{-1}h'^{-1}(w_1+x_1v_\lambda^0),\cdots,h^{-1}h'^{-1}(w_n+x_nv_\lambda^0))\\ &\overline{\Phi^0(h'^{-1}(w_1+l_1v_\lambda^0),\cdots,h'^{-1}(w_n+l_nv_\lambda^0))}\\ &\overline{\omega_{\psi_\lambda}(\iota(g))\phi(x_1,\cdots,x_n)}\phi(l_1,\cdots,l_n)\mathrm{d}l_1\cdots\mathrm{d}l_n\mathrm{d}w_1\cdots\mathrm{d}w_n\mathrm{d}x_1\cdots\mathrm{d}x_n.\end{aligned}$$

We define a local analogue of (8.3.3), i.e.

$$\Phi^0*\overline{\phi}(v_1,\cdots,v_n)=\int_{F^n}\Phi^0(v_1+x_1v_\lambda^0,\cdots,v_n+x_nv_\lambda^0)\overline{\phi(x_1,\cdots,x_n)}\mathrm{d}x_1\cdots\mathrm{d}x_n.$$

Then $\Phi^0*\overline{\phi}\in\mathcal{S}(V_\lambda^n)$ and

$$\widetilde{\Omega}_\psi(\widetilde{g},h_\lambda)(\Phi^0*\overline{\phi})=(\Omega_\psi^0(g,h_\lambda)\Phi^0)*\overline{(\omega_{\psi_\lambda}(\widetilde{g})\phi)},\quad \widetilde{g}\in\widetilde{G_0}(F),h_\lambda\in H_\lambda(F),$$

where g is the image of $\widetilde{g}$ in $G_0(F)$.

We conclude that

$$\begin{aligned}\text{(8.4.7)}\qquad (8.4.6)=|2\lambda|^{-n}\int_{H_\lambda(F)\backslash H(F)}&\mathcal{B}(\Omega_\psi^0(g,h'h)\Phi*\overline{\omega_{\psi_\lambda}(g)\phi},\Omega_\psi^0(h')\Phi*\overline{\phi})\\ &\Phi_{n+1}(h^{-1}h'^{-1}v_\lambda^0)\overline{\Phi_{n+1}(h'^{-1}v_\lambda^0)}\underline{\mathrm{d}}h'.\end{aligned}$$

Step 4. Computing (8.4.1) using (8.4.7).

Recall that we have fixed a measure on $H(F)$ and $H_\lambda(F)$ respectively. Let $\mathrm{d}h'$ be the quotient measure on $H_\lambda(F)\backslash H(F)$ and c a constant so that $c\cdot\mathrm{d}h'=\underline{\mathrm{d}}h'$ where $\underline{\mathrm{d}}h'$ is the measure on $H_\lambda(F)\backslash H(F)$ defined in Lemma 8.4.2. Then we get

$$\begin{aligned}(8.4.1)=&c\cdot|2\lambda|^{-n}\int_{H_\lambda\backslash H}\int_H\int_{G_0}\overline{\mathcal{B}(\pi(h)f,f)}\mathcal{B}(\sigma(g)\varphi,\varphi)\mathcal{B}(\Omega_\psi^0(g,h'h)\Phi*\overline{\omega_{\psi_\lambda}(g)\phi},\Omega_\psi^0(h')\Phi*\overline{\phi})\\ &\Phi_{n+1}(h^{-1}h'^{-1}v_\lambda^0)\overline{\Phi_{n+1}(h'^{-1}v_\lambda^0)}\mathrm{d}g\mathrm{d}h\mathrm{d}h'.\end{aligned}$$

We make a change of variable $h \mapsto h'^{-1}h$ and get

$$(8.4.1) = c \cdot |2\lambda|^{-n} \iint_{H_\lambda \backslash H \times H} \int_{G_0} \overline{\mathcal{B}(\pi(h)f, \pi(h')f)} \mathcal{B}(\sigma(g)\varphi, \varphi) \mathcal{B}(\Omega^0_\psi(g,h)\Phi * \overline{\omega_{\psi_\lambda}(g)\phi}, \Omega^0_\psi(h')\Phi * \overline{\phi}) \Phi_{n+1}(h^{-1}v^0_\lambda)\overline{\Phi_{n+1}(h'^{-1}v^0_\lambda)} \mathrm{d}g\mathrm{d}h\mathrm{d}h'.$$

The group H_λ embeds in $H \times H$ diagonally. This integral is absolutely convergent.

We further split the integration over h as $h_\lambda h$ where $h_\lambda \in H_\lambda$ and $h \in H_\lambda \backslash H$. Then

$$(8.4.1) = c \cdot |2\lambda|^{-n} \int_{(H_\lambda \backslash H)^2} \int_{H_\lambda} \int_{G_0} \overline{\mathcal{B}(\pi(h_\lambda h)f, \pi(h')f)} \mathcal{B}(\sigma(g)\varphi, \varphi) \mathcal{B}(\widetilde{\Omega}_\psi(g, h_\lambda)(\Omega^0_\psi(h)\Phi * \overline{\phi}), (\Omega^0_\psi(h')\Phi * \overline{\phi}))\Phi_{n+1}(h^{-1}v^0_\lambda)\overline{\Phi_{n+1}(h'^{-1}v^0_\lambda)} \mathrm{d}g\mathrm{d}h_\lambda\mathrm{d}h'\mathrm{d}h.$$

We summarize the above computation into the following lemma.

Lemma 8.4.5. *Suppose $\Phi = \Phi^0 \otimes \Phi_{n+1}$ where $\Phi^0 \in \mathcal{S}(V^n)$ and $\Phi_{n+1} \in \mathcal{S}(V)$. Then*

$$\int_{G_0(F)} \int_{N(F)} \int_{H(F)} \overline{\mathcal{B}(\pi(h)f, f)} \mathcal{B}(\Omega_\psi(ng, h)\Phi, \Phi) \overline{\mathcal{B}(\omega_{\psi_\lambda}(ng)\phi, \phi)} \mathcal{B}(\sigma(g)\varphi, \varphi) \mathrm{d}h\mathrm{d}n\mathrm{d}g$$
$$= c \cdot |2\lambda|^{-n} \int_{(H_\lambda \backslash H)^2} \int_{H_\lambda} \left(\int_{G_0} \mathcal{B}(\sigma(g)\varphi, \varphi) \mathcal{B}(\widetilde{\Omega}_\psi(g, h_\lambda)(\Omega^0_\psi(h)\Phi * \overline{\phi}), (\Omega^0_\psi(h')\Phi * \overline{\phi}))\mathrm{d}g \right) \overline{\mathcal{B}(\pi(h_\lambda h)f, \pi(h')f)} \Phi_{n+1}(h^{-1}v^0_\lambda)\overline{\Phi_{n+1}(h'^{-1}v^0_\lambda)} \mathrm{d}h_\lambda\mathrm{d}h\mathrm{d}h'.$$

8.5. **Proof of Theorem 8.1.1.** By Lemma 8.3.1, we have

$$|\mathcal{F}\mathcal{J}_{\psi_\lambda}(\Theta_\psi(f, \Phi), \varphi, \phi)|^2 = \iint_{(H_\lambda(\mathbb{A}_F) \backslash H(\mathbb{A}_F))^2} \Phi_{n+1}(h^{-1}v^0_\lambda)\overline{\Phi_{n+1}(h'^{-1}v^0_\lambda)} \left(\int_{H_\lambda(F) \backslash H_\lambda(\mathbb{A}_F)} \overline{f(h_\lambda h)} \widetilde{\Theta}_\psi(\overline{\varphi}, (\Omega^0_\psi(h)\Phi^0) * \overline{\phi})(h_\lambda)\mathrm{d}h_\lambda \right) \overline{\left(\int_{H_\lambda(F) \backslash H_\lambda(\mathbb{A}_F)} \overline{f(h'_\lambda h')} \widetilde{\Theta}_\psi(\overline{\varphi}, (\Omega^0_\psi(h')\Phi^0) * \overline{\phi})(h'_\lambda)\mathrm{d}h'_\lambda \right)} \mathrm{d}h\mathrm{d}h'.$$

We fix a sufficiently large finite set of places S of F so that if $v \notin S$, then the following conditions hold.

(1) v is non-archimedean, 2 and λ are in $\mathfrak{o}^\times_{F,v}$, the conductor of ψ is $\mathfrak{o}_{F,v}$.
(2) The group A is unramified with a hyperspecial maximal compact subgroup $K_{A_v} = A(\mathfrak{o}_{F,v})$ where $A = G, G_0, H, H_\lambda$.
(3) f_v and φ_v are K_{H_v} and $K_{G_{0,v}}$ fixed respectively. Moreover, they are normalized so that $\mathcal{B}(f_v, f_v) = \mathcal{B}(\varphi_v, \varphi_v) = 1$. In particular, π_v and σ_v are both unramified.
(4) Φ_v is the characteristic function of $V(\mathfrak{o}_{F,v})^{n+1}$, ϕ_v is the characteristic function of $\mathfrak{o}^n_{F,v}$.
(5) The volume of the hyperspecial maximal compact subgroup K_{A_v} is 1 under the chosen measure on $A(F_v)$, where $A = G, G_0, H, H_\lambda$.

Lemma 8.5.1. *If $v \notin S$, then $c_v = L_v(n+1, \chi_{V_v})^{-1}$. Recall that $\underline{\mathrm{d}}h_v = c_v \cdot \mathrm{d}h_{\lambda,v}\backslash \mathrm{d}h_v$ where $\underline{\mathrm{d}}h_v$ is the measure defined in Lemma 8.4.2.*

Proof. We denote temporarily by f_v the characteristic function of $V(\mathfrak{o}_{F,v})$. Recall from the proof of Lemma 8.4.3 that

$$\int_{H_\lambda(F_v)\backslash H(F_v)} f_v(h^{-1}v_\lambda^0)\underline{\mathrm{d}}h = \int_{F_v} \Phi_{f_v}^n\left(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} 1 & \kappa \\ & 1 \end{pmatrix}\right)\psi_v(-\lambda\kappa)\mathrm{d}\kappa,$$

where $\Phi_{f_v}^n$ is the Siegel–Weil section of $\mathrm{Ind}^{\mathrm{SL}_2(F_v)}\chi_{V_v}|\cdot|^s$ at $s = s_0 = n$. It is well-known that the right hand side equals $L_v(n+1, \chi_{V_v})^{-1}$.

We note that since $\lambda \in \mathfrak{o}_{F,v}^\times$, the orbit Λ_λ of v_λ^0 is defined over $\mathfrak{o}_{F,v}$. The group $H(\mathfrak{o}_{F,v})$ acts transitively on $V_\lambda(\mathfrak{o}_{F,v})$. Therefore $H_\lambda(\mathfrak{o}_{F,v})\backslash H(\mathfrak{o}_{F,v}) \to \Lambda_\lambda(\mathfrak{o}_{F,v})$ is a bijection. Thus $f_v(h^{-1}v_\lambda^0) = \mathbf{1}_{H_\lambda(\mathfrak{o}_{F,v})\backslash H(\mathfrak{o}_{F,v})}(h)$. Therefore under the quotient measure $\mathrm{d}h_{\lambda,v}\backslash \mathrm{d}h_v$, the left hand side equals one. The lemma then follows. □

Lemma 8.5.2.

$$\prod_{v\in S} c_v = L^S(n+1, \chi_V).$$

Proof. It follows from Lemma 8.2.1 that $\prod_v c_v = 1$. Then

$$\prod_{v\in S} c_v = \prod_{v\notin S} c_v^{-1} = L^S(n+1, \chi_V).$$

□

Conjecture 6.3.1, the Rallis' inner product formula (for theta lifting from $\widetilde{G_0}$ to H_λ) and Lemma 8.4.5 lead to

$$|\mathcal{F}\mathcal{J}_{\psi_\lambda}(\Theta_\psi(f,\Phi),\varphi,\phi)|^2 = \frac{2^{\gamma-1}}{|S_\pi||S_{\widetilde{\Theta}_\psi(\sigma)}|}\frac{L^S(\frac{1}{2}, \pi \times \widetilde{\Theta}_\psi(\sigma))}{L^S(1,\pi,\mathrm{Ad})L^S(1,\widetilde{\Theta}_\psi(\sigma),\mathrm{Ad})}$$
$$\Delta_{H(V)}^S \cdot \frac{L_{\psi_\lambda}^S(\frac{1}{2}, \sigma \times \chi_{V_\lambda})}{\prod_{j=1}^n \zeta_F^S(2j)} \prod_{v\in S} c_v^{-1} \int_{G_0(F_v)}\int_{N(F_v)}\int_{H(F_v)}$$
$$\mathcal{B}_v(\overline{\pi(h_v)f_v, f_v})\mathcal{B}_v(\Omega_{\psi_v}(h_v, n_v g_v)\Phi_v, \Phi_v)\overline{\mathcal{B}_v(\omega_{\psi_{\lambda,v}}(n_v g_v)\phi_v, \phi_v)}\mathcal{B}_v(\sigma_v(g_v)\varphi_v, \varphi_v)\mathrm{d}h_v\mathrm{d}n_v\mathrm{d}g_v,$$

where γ is described as in Conjecture 6.3.1.

We then apply the Rallis inner product formula for the theta lifting from H to G. We conclude that

$$|\mathcal{F}\mathcal{J}_{\psi_\lambda}(\xi,\varphi,\phi)|^2 = \frac{2^{\gamma-1}}{|S_\pi||S_{\widetilde{\Theta}_\psi(\sigma)}|}\frac{L^S(\frac{1}{2},\pi\times\widetilde{\Theta}_\psi(\sigma))}{L^S(1,\pi,\mathrm{Ad})L^S(1,\widetilde{\Theta}_\psi(\sigma),\mathrm{Ad})}$$

$$\Delta_H^S\cdot\frac{L_\psi^S(\frac{1}{2},\sigma\times\chi_{V_\lambda})}{\prod_{j=1}^n\zeta_F^S(2j)}\left(\frac{L^S(1,\pi)}{\prod_{i=1}^n\zeta_F^S(2i)L^S(n+1,\chi_V)}\right)^{-1}L^S(n+1,\chi_V)^{-1}$$

$$\prod_{v\in S}\int_{G_0(F_v)}\int_{N(F_v)}\mathcal{B}_v(\Theta_{\psi_v}(\pi_v)(n_vg_v)\xi_v,\xi_v)\overline{\mathcal{B}_v(\omega_{\psi_{\lambda,v}}(n_vg_v)\phi_v,\phi_v)}\mathcal{B}_v(\sigma_v(g_v)\varphi_v,\varphi_v)\mathrm{d}n_v\mathrm{d}g_v,$$

where $\xi=\otimes\xi_v\in\Theta_\psi(\pi)$. Note that $|S_\pi||S_{\widetilde{\Theta}_\psi(\sigma)}| = 2^{\gamma-1}|S_{\Theta_\psi(\pi)}||S_\sigma|$ by Lemma 5.2.3. Conjecture 2.3.1(3) then follows from Lemma 5.2.2.

8.6. **A variant.** So far we considered the case $\mathrm{Sp}(2n+2)\times\mathrm{Mp}(2n)$. The case $\mathrm{Mp}(2n+2)\times\mathrm{Sp}(2n)$ is similar. We only mention the following theorem.

Let (V,q_V) be a $2n+3$ dimensional orthogonal space and $H=\mathrm{O}(V)$. Suppose that $\lambda\in F^\times$ and there is an element $v_\lambda^0\in V$ such that $q_V(v_\lambda^0,v_\lambda^0)=\lambda$. Let V_λ be the orthogonal complement of v_λ^0 and $H_\lambda=\mathrm{O}(V_\lambda)$. Let π be an irreducible cuspidal tempered automorphic representation of $H(\mathbb{A}_F)$ and $\Theta_\psi(\pi)$ its theta lift to $\mathrm{Mp}(2n+2)(\mathbb{A}_F)$ (with additive character ψ). Let σ be an irreducible cuspidal tempered automorphic representation of $\mathrm{Sp}(2n)(\mathbb{A}_F)$ and $\Theta_\psi(\sigma)$ its theta lift to $H_\lambda(\mathbb{A}_F)$.

Theorem 8.6.1. *Suppose that $\Theta_\psi(\pi)$ and $\Theta_\psi(\sigma)$ are both cuspidal. If Conjecture 6.3.1 holds for $(\pi,\Theta_\psi(\sigma))$, then Conjecture 2.3.1(3) holds for $(\Theta_\psi(\pi),\sigma)$ (with the additive character ψ_λ).*

The proof of Theorem 8.6.1 is analogues to Theorem 8.1.1 and we leave the details to the interested reader.

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