FOURIER–JACOBI PERIODS AND THE CENTRAL VALUE OF RANKIN–SELBERG L-FUNCTIONS

HANG XUE

Contents

1. Introduction

1.1. **The refined Gan–Gross–Prasad conjecture.** In this paper, as a sequel of [Xue14], we formulate a refinement to the global Gan–Gross–Prasad conjecture for the Fourier–Jacobi periods on $\mathrm{U}(n) \times \mathrm{U}(n)$ and prove it under some local conditions, assuming some expected properties of the L-packets and some parts of the local Gan–Gross–Prasad conjecture.

This refinement is modeled on some recent work on the refined Gan–Gross–Prasad conjecture for the Bessel periods. In a seminal paper [II10], Ichino and Ikeda formulated a refinement of the Gan–Gross–Prasad conjecture for orthogonal groups $\mathrm{SO}(n+1) \times \mathrm{SO}(n)$. The $n = 2$ case corresponds to Waldspurger's formula [Wal85] and the $n = 3$ case corresponds to the triple product formula [Ich08]. Gan and Ichino proved some cases of $n = 4$ using theta correspondences. But little is known beyond this. N. Harris then formulated the refined conjecture for the unitary groups $\mathrm{U}(n+1)\times\mathrm{U}(n)$ in his Ph.D. thesis at the University of California, San Diego [Har12]. Harris also verified the case $n = 1$ and some special cases of $n = 2$. W. Zhang proved the refinement for the unitary groups $\mathrm{U}(n + 1) \times \mathrm{U}(n)$ under some local conditions [Zha14b]. Zhang's work is built on the relative trace formulae formulated by Jacquet–Rallis [JR11], the relevant fundamental lemma established by Yun [Yun11] and the existence of smooth transfer established by himself [Zha14a].

There is a parallel theory for the Fourier–Jacobi periods on $\mathrm{U}(n) \times \mathrm{U}(n)$. The non-refined conjecture for the Fourier–Jacobi periods was formulated in [GGP12]. A relative trace formulae approach to these conjectures for unitary groups was then proposed by Liu [Liu14]. The relevant fundamental lemma for $\mathrm{U}(n) \times \mathrm{U}(n)$ was also proved by Liu. Building on the work of Liu, we proved the non-refined Gan–Gross–Prasad conjecture for $\mathrm{U}(n) \times \mathrm{U}(n)$ in [Xue14] under some local conditions. The techniques in [Xue14] were largely inspired by [Zha14a]. In fact, we proved the existence of smooth transfer by reducing it to the case established in [Zha14a]. In this paper, we first formulate a refinement to the Gan–Gross–Prasad conjecture for $\mathrm{U}(n) \times \mathrm{U}(n)$, and then prove it under some local conditions. This is an analogue of [Zha14b] for the case of Fourier–Jacobi periods.

Date: July 28, 2014.

We first introduce the Fourier–Jacobi periods on $\mathrm{U}(n)\times\mathrm{U}(n)$. Let k' be a number field and k a quadratic field extension of k'. Let τ be the non-trivial element in the Galois group $\mathrm{Gal}(k/k')$. Let $\mathbb{A}'$ (resp. $\mathbb{A}$) be the ring of adeles of k' (resp. k), η the quadratic character of $k'^\times\backslash\mathbb{A}'^\times$ associated to k/k' by the global class field theory. We fix a character $\mu : k^\times\backslash\mathbb{A}^\times\to\mathbb{C}^\times$ such that $\mu|_{\mathbb{A}'^\times}=\eta$. We also fix a nontrivial additive character $\psi' : k'\backslash\mathbb{A}'\to\mathbb{C}^\times$. If v is a place of k', we write k'_v the completion of k' at v and $k_v = k\otimes_{k'} k'_v$.

Let V be an n-dimensional skew-Hermitian space over k, $\mathrm{U}(V)$ the corresponding unitary group defined over k'. Let $V^\vee$ be the dual space of V with the (dual) skew-hermitian form $\langle -,-\rangle$. Let $\mathrm{Res}\, V^\vee$ be the symplectic space over k' of dimension $2n$ whose underline vector space is $V^\vee$ viewed as a vector space over k' and whose symplectic form is given by

$$\langle\langle w,v\rangle\rangle = \frac{1}{2}\mathrm{Tr}_{k/k'}\langle w,v\rangle$$

Choose a Lagrangian subspace $\mathbf{L}\subset \mathrm{Res}\, V^\vee$ over k', i.e. a maximal isotropic subspace of $\mathrm{Res}\, V^\vee$. Then we have the Weil representation $\omega_{\psi',\mu}$ of $\mathrm{U}(V)(\mathbb{A}')$ which is realized on $\mathcal{S}(\mathbf{L}(\mathbb{A}'))$, the space of Schwartz functions on $\mathbf{L}(\mathbb{A}')$ (c.f. Appendix A for the definition). We form the theta series on the unitary group $\mathrm{U}(V)(\mathbb{A}')$ by

$$\theta_{\psi',\mu}(g,\phi) = \sum_{x\in\mathbf{L}(k')}\omega_{\psi',\mu}(g)\phi(x),\quad g\in\mathrm{U}(V)(\mathbb{A}'),\quad \phi\in\mathcal{S}(\mathbf{L}(\mathbb{A}')).$$

Let π_1 and π_2 be two irreducible cuspidal automorphic representations of $\mathrm{U}(V)(\mathbb{A}')$. Let $\varphi_i\in\pi_i$ for $i=1,2$. We define the Fourier–Jacobi period to be the integral

$$\mathcal{FJ}_{\psi',\mu}(\varphi_1,\varphi_2,\phi) = \int_{\mathrm{U}(V)(k')\backslash\mathrm{U}(V)(\mathbb{A}')}\varphi_1(g)\varphi_2(g)\overline{\theta_{\psi',\mu}(g,\phi)}\mathrm{d}g. \tag{1.1.1}$$

This integral is absolutely convergent since φ_1, φ_2 are rapid decaying and $\theta(g,\phi)$ is of moderate growth.

We are interested in determining whether the integral (1.1.1) is identically zero. The (non-refined) Gan–Gross–Prasad conjecture predicts that the nonvanishing of the linear functional $\mathcal{FJ}_{\psi',\mu}$ (possibly by varying the space V and switching to another member in the Vogan packet of π_1 and π_2) is equivalent to the nonvanishing of the central L-value $L(\frac{1}{2},\mathrm{BC}(\pi_1)\times\mathrm{BC}(\pi_2)\otimes\mu^{-1})$, where $\mathrm{BC}(\pi_i)$ $(i=1,2)$ stands for the base change of π_i to $\mathrm{GL}_n(\mathbb{A})$ (c.f. [HL04] for the notion of base change). We refer the readers to [GGP12, Section 25] for a discussion of Vogan packets and [GGP12, Conjecture 26.1] for the precise statement of this conjecture. This conjectural equivalence has been proved in [Xue14] if π_1 and π_2 satisfy some local conditions.

For some arithmetic applications, it is usually more desirable to have a precise relation between the period integral (1.1.1) and the central L-value $L(\frac{1}{2},\mathrm{BC}(\pi_1)\times\mathrm{BC}(\pi_2)\otimes\mu^{-1})$. We now formulate this conjectural relation.

We take the measure $\mathrm{d}g$ in (1.1.1) to be the Tamagawa measure of $\mathrm{U}(V)(\mathbb{A}')$. We fix a measure $\mathrm{d}g_v$ on $\mathrm{U}(V)(k'_v)$ for each v so that $\mathrm{d}g=\prod_v \mathrm{d}g_v$.

For $i=1,2$, let $\langle -,-\rangle_{\pi_i}$ be the Petersson inner product on π_i defined by

$$\langle\varphi,\varphi^\vee\rangle = \int_{\mathrm{U}(V)(k')\backslash\mathrm{U}(V)(\mathbb{A}')}\varphi(g)\overline{\varphi^\vee(g)}\mathrm{d}g,\quad \varphi,\varphi^\vee\in\pi_i.$$

We fix a pairing $\langle -,-\rangle_{\pi_{i,v}}$ on $\pi_{i,v}$ so that $\langle -,-\rangle_{\pi_i}=\prod_v\langle -,-\rangle_{\pi_{i,v}}$.

There is a natural hermitian inner product on $\omega_{\psi',\mu}$ given by

$$\langle\phi,\phi^\vee\rangle_{\omega_{\psi',\mu}} = \int_{\mathbf{L}(\mathbb{A}')}\phi(x)\overline{\phi^\vee(x)}\mathrm{d}x,\quad \phi,\phi^\vee\in\mathcal{S}(\mathbf{L}(\mathbb{A}')),$$

where we fix an isomorphism $\mathbf{L}\simeq k'_n$ and take $\mathrm{d}x$ to be the self-dual measure on $\mathbf{L}(\mathbb{A}')$ (with respect to ψ'). Let $\mathrm{d}x_v$ be the self-dual measure on $\mathbf{L}(k'_v)$ (with respect to ψ'_v). Then $\mathrm{d}x=\prod \mathrm{d}x_v$. This hermitian pairing decomposes as a product $\langle -,-\rangle_{\omega_{\psi',\mu}}=\prod_v\langle -,-\rangle_{\omega_{\psi'_v,\mu_v}}$ under the decomposition $\omega_{\psi',\mu}=\otimes_v\omega_{\psi'_v,\mu_v}$, where

$$\langle\phi_v,\phi_v^\vee\rangle_{\omega_{\psi'_v,\mu_v}} = \int_{\mathbf{L}(k'_v)}\phi_v(x)\overline{\phi_v^\vee(x)}\mathrm{d}x,\quad \phi_v,\phi_v^\vee\in\mathcal{S}(\mathbf{L}(k'_v)).$$

Let $\varphi_{1,v}, \varphi_{1,v}^\vee \in \pi_{1,v}$, $\varphi_{2,v}, \varphi_{2,v}^\vee \in \pi_{2,v}$, $\phi_v, \phi_v^\vee \in \mathcal{S}(\mathbf{L}(k'_v))$ be K_v-finite vectors where K_v is a maximal compact subgroup of $\mathrm{U}(V)(k')$. Define

$$\begin{aligned}&\alpha_v(\varphi_{1,v}, \varphi_{1,v}^\vee, \varphi_{2,v}, \varphi_{2,v}^\vee, \phi_v, \phi_v^\vee)\\ &= \int_{\mathrm{U}(V)(k'_v)} \langle \pi_{1,v}(g_v)\varphi_{1,v}, \varphi_{1,v}^\vee\rangle_{\pi_{1,v}} \langle \pi_{2,v}(g_v)\varphi_{2,v}, \varphi_{2,v}^\vee\rangle_{\pi_{2,v}} \overline{\langle \omega_{\psi'_v,\mu_v}(g_v)\phi_v, \phi_v^\vee\rangle_{\omega_{\psi',\mu}}} \mathrm{d}g_v.\end{aligned} \tag{1.1.2}$$

We also put $\alpha_v(\varphi_{1,v}, \varphi_{2,v}, \phi_v) = \alpha_v(\varphi_{1,v}, \varphi_{1,v}, \varphi_{2,v}, \varphi_{2,v}, \phi_v, \phi_v)$.

We prove in Appendix D that the integral (1.1.2) has the following nice properties.

Proposition 1.1.1. *Assume that $\pi_{1,v}$ and $\pi_{2,v}$ are tempered.*

1. *The integral (1.1.2) is convergent.*
2. *If $\mathrm{Hom}_{\mathrm{U}(V)(k'_v)}(\pi_{1,v} \otimes \pi_{2,v} \otimes \overline{\omega_{\psi'_v,\mu_v}}, \mathbb{C}) \neq 0$, then the linear form α_v is not identically zero. Moreover, $\alpha_v(\varphi_{1,v}, \varphi_{2,v}, \phi_v) \geq 0$.*
3. *Let v be a non-archimedean place of k'. Assume that $\pi_{1,v}$, $\pi_{2,v}$, ψ_v, μ_v are all unramified. Recall that this means that the quadratic extension k_v/k'_v is unramified, the group $\mathrm{U}(V)(k'_v)$ has a hyperspecial subgroup $K_v = \mathrm{U}(V)(\mathfrak{o}'_v)$, the conductor of ψ'_v (resp. μ_v) is $\mathfrak{o}'_v$ (resp. $\mathfrak{o}_v^\times$) and $\pi_{i,v}$ $(i = 1, 2)$ has a K_v fixed vector. Then*
$$\frac{\alpha_v(\varphi_{1,v}^0, \varphi_{2,v}^0, \phi_v^0)}{\langle \varphi_{1,v}^0, \varphi_{1,v}^0\rangle \langle \varphi_{2,v}^0, \varphi_{2,v}^0\rangle \langle \phi_v^0, \phi_v^0\rangle} = \mathrm{vol}\, K_v \times \prod_{i=1}^n L(i, \eta_v^i) \times \frac{L(\frac{1}{2}, \mathrm{BC}(\pi_{1,v}) \times \mathrm{BC}(\pi_{2,v}) \otimes \mu_v^{-1})}{L(1, \pi_{1,v}, \mathrm{Ad})L(1, \pi_{2,v}, \mathrm{Ad})},$$
where $\varphi_{i,v}^0$ $(i = 1, 2)$ and ϕ_v^0 are K_v-fixed vectors in $\pi_{i,v}$ and $\mathcal{S}(\mathbf{L}(k'_v))$ respectively.

Because of Proposition 1.1.1 (3), we define a normalized linear form $\alpha_v^\natural$ by

$$\begin{aligned}&\alpha_v^\natural(\varphi_{1,v}, \varphi_{1,v}^\vee, \varphi_{2,v}, \varphi_{2,v}^\vee, \phi_v, \phi_v^\vee)\\ &= \left(\prod_{i=1}^n L(i, \eta_v^i) \times \frac{L(\frac{1}{2}, \mathrm{BC}(\pi_{1,v}) \times \mathrm{BC}(\pi_{2,v}) \otimes \mu_v^{-1})}{L(1, \pi_{1,v}, \mathrm{Ad})L(1, \pi_{2,v}, \mathrm{Ad})}\right)^{-1} \alpha_v(\varphi_{1,v}, \varphi_{1,v}^\vee, \varphi_{2,v}, \varphi_{2,v}^\vee, \phi_v, \phi_v^\vee).\end{aligned} \tag{1.1.3}$$

As explained in [II10], the automorphic representations π_1 and π_2 should come from some elliptic Arthur parameters

$$\Psi_i : L_{k'} \to {}^L\mathrm{U}(V) = \widehat{\mathrm{U}(V)} \rtimes \mathrm{Gal}(\overline{k'}/k'),\ i = 1, 2,$$

where $L_{k'}$ is the (hypothetical) Langlands group of k' and $\widehat{\mathrm{U}(V)}$ is the Langlands dual group of $\mathrm{U}(V)$. We define S_{π_i} $(i = 1, 2)$ to be the centralizer of the image Ψ_i in $\widehat{\mathrm{U}(V)}$. They are finite abelian 2-groups.

We define the following constant which is associated to the dual Gross motive defined in [Gro97]

$$\Delta_{\mathrm{U}(V)} = \prod_{i=1}^n L(i, \eta^i).$$

Conjecture 1.1.2. *Assume that π_1 and π_2 are irreducible cuspidal tempered automorphic representations of $\mathrm{U}(V)(\mathbb{A}')$. If $\varphi_1 = \otimes\varphi_{1,v} \in \pi_1, \varphi_2 = \otimes\varphi_{2,v} \in \pi_2, \phi = \otimes\phi_v \in \mathcal{S}(\mathbf{L}(\mathbb{A}'))$, then*

$$|\mathcal{FJ}_{\psi',\mu}(\varphi_1, \varphi_2, \phi)|^2 = \frac{\Delta_{\mathrm{U}(V)}}{|S_{\pi_1}||S_{\pi_2}|} \frac{L(\frac{1}{2}, \mathrm{BC}(\pi_1) \times \mathrm{BC}(\pi_2) \otimes \mu^{-1})}{L(1, \pi_1, \mathrm{Ad})L(1, \pi_2, \mathrm{Ad})} \prod_v \alpha_v^\natural(\varphi_{1,v}, \varphi_{2,v}, \phi_v). \tag{1.1.4}$$

Remark 1.1.3. If $n = 1$, Conjecture 1.1.2 has been proved in Yang's thesis [Yan97]. Such a formula was then applied in [RVY99, Yan99, MY11] to show that certain elliptic curves studied in [Gro80] has rank zero. This formula has also found its applications in the Iwasawa theory for CM fields, see [Fin06a, Fin06b, Hsi14].

If $n = 2$, Conjecture 1.1.2 essentially follows from Ichino's triple product formula [Har12, Ich08]. The triple product formula has lots of applications in the analytic theory of L-functions. See [BR10, MV10, NPS14], to list a few. This formula has also been used in [DR14] to construct triple product p-adic L-functions.

Remark 1.1.4. As explained in [LM15, Section 5], we may avoid the use of Arthur parameters in the formulation of Conjecture 1.1.2. The quantity $|S_{\pi_i}|$ can also be defined via the base change of π_i to the general linear group GL_n.

Remark 1.1.5. Let S be a finite set of places such that if $v \notin S$, then $\pi_{i,v}$ $(i = 1, 2)$, ψ'_v and μ_v are unramified, the volume of the hyperspecial maximal compact subgroup K_v equals 1, $\varphi_{1,v}$, $\varphi_{2,v}$ and ϕ_v are K_v fixed, $\langle \varphi_{1,v}, \varphi_{1,v}\rangle = \langle \varphi_{2,v}, \varphi_{2,v}\rangle = \langle \phi_v, \phi_v\rangle = 1$. Then the right hand side of the conjecture can be written as

$$\frac{\Delta^S_{\mathrm{U}(V)}}{|S_{\pi_1}||S_{\pi_2}|} \frac{L^S(\frac{1}{2}, \mathrm{BC}(\pi_1) \times \mathrm{BC}(\pi_2) \otimes \mu^{-1})}{L^S(1, \pi_1, \mathrm{Ad})L^S(1, \pi_2, \mathrm{Ad})} \prod_{v \in S} \alpha_v(\varphi_{1,v}, \varphi_{2,v}, \phi_v),$$

where $\Delta^S_{\mathrm{U}(V)}$ is defined in the same way as $\Delta_{\mathrm{U}(V)}$, except that the partial L-functions are used. Thus the definition of the local L-factors at the ramified places plays no role on the right hand side of (1.1.4).

Remark 1.1.6. We can formulate a similar conjecturse for the more general Fourier–Jacobi periods for $\mathrm{U}(n) \times \mathrm{U}(m)$ where $n - m$ is even. We will do this in a forthcoming paper and verify some low rank cases. The relative trace formulae in the general case seems much harder to establish. However, there is another approach via the automorphic descent method, see [LM] for the case of the Whittaker–Fourier coefficients on the metaplectic groups.

We now look at Conjecture 1.1.2 from the representation theoretic point of view. The integral (1.1.1) defines an element in

$$\mathrm{Hom}_{\mathrm{U}(V)(\mathbb{A}')}(\pi_1 \otimes \pi_2 \otimes \overline{\omega_{\psi',\mu}}, \mathbb{C}).$$

Therefore the left hand side of (1.1.4) defines an element in

$$\mathrm{Hom}_{\mathrm{U}(V)(\mathbb{A}')\times \mathrm{U}(V)(\mathbb{A}')}(\pi_1 \otimes \pi_2 \otimes \overline{\omega_{\psi',\mu}} \otimes \overline{\pi_1} \otimes \overline{\pi_2} \otimes \omega_{\psi',\mu}, \mathbb{C}).$$

The product $\prod_v \alpha^\natural_v$ on the right hand side of (1.1.4) defines an element in the same space. By Proposition 1.1.1 (2), if $\mathrm{Hom}_{\mathrm{U}(V)(\mathbb{A}')}(\pi_1 \otimes \pi_2 \otimes \overline{\omega_{\psi',\mu}}, \mathbb{C}) \neq 0$, then $\prod_v \alpha^\natural_v$ is nonzero.

It follows from the multiplicity one theorems [Sun12, SZ12] that

$$\dim \mathrm{Hom}_{\mathrm{U}(V)(\mathbb{A}')\times \mathrm{U}(V)(\mathbb{A}')}(\pi_1 \otimes \pi_2 \otimes \overline{\omega_{\psi',\mu}} \otimes \overline{\pi_1} \otimes \overline{\pi_2} \otimes \omega_{\psi',\mu}, \mathbb{C}) \leq 1.$$

Thus there is a constant C, such that as linear functionals,

$$\mathcal{FJ}_{\psi',\mu} \cdot \overline{\mathcal{FJ}_{\psi',\mu}} = C \times \prod_v \alpha^\natural_v. \tag{1.1.5}$$

Conjecture 1.1.2 then computes the constant C. This conjecture takes a very similar form to the existing conjectures for the Bessel periods [II10, Har12, Liu].

1.2. **Main results.** We first make a series of hypotheses that we need throughout this paper. The first is some expected properties of the L-packets, analogue to the work of Arthur on orthogonal and symplectic groups. See [Mok] for some recent progresses.

Hypothesis 1.2.1. *Let π be an irreducible cuspidal automorphic representation of $\mathrm{U}(V)(\mathbb{A}')$. Assume that at some place v of k' that splits in k, the local component π_v is supercuspidal.*

1. *The base change $\mathrm{BC}(\pi)$ of π to $\mathrm{GL}_n(\mathbb{A})$ exists and is cuspidal with unitary central character. Moreover, the Asai L-function $L(s, \mathrm{BC}(\pi), \mathrm{As}^{(-1)^{n+1}})$ has a simple pole at $s = 1$ (c.f.* [GGP12, Section 7] *for a discussion of Asai L-functions).*
2. *The multiplicity of π in $L^2(\mathrm{U}(V)(k')\backslash \mathrm{U}(V)(\mathbb{A}'))$ is one.*
3. *For almost all places v, π_v lies in a generic local Vogan packet.*

Let V and W be two skew-hermitian spaces of dimension n. Let π (resp. σ) be an irreducible cuspidal automorphic representation of $\mathrm{U}(V)(\mathbb{A}')$ (resp $\mathrm{U}(W)(\mathbb{A}')$). Assume that at some split place v of k', both π_v and σ_v are supercuspidal. Assume that π and σ are nearly equivalent, i.e. for almost all places v of k', we have $\pi_v \simeq \sigma_v$ with respect to a fixed isomorphism $V_v \simeq W_v$.

4. *For all places v, π_v and σ_v lie in the same local Vogan packet and this packet is generic.*

The second hypothesis is a part of the local Gan–Gross–Prasad conjecture. Gan and Ichino [GI] has recently proved (a stronger version) of the local Gan–Gross–Prasad conjecture for $\mathrm{U}(n) \times \mathrm{U}(n)$ over non-archimedean local fields, assuming some expected properties of the L-packets.

Hypothesis 1.2.2. *Let v be a place of k'. Then in a generic local Vogan packet of $\mathrm{U}(V)(k'_v) \times \mathrm{U}(V)(k'_v)$, there is at most one representation $\sigma_{1,v} \otimes \sigma_{2,v}$ of $\mathrm{U}(W)(k'_v) \times \mathrm{U}(W)(k'_v)$, where W is an n-dimensional skew-hermitian space, such that*

$$\mathrm{Hom}_{\mathrm{U}(W)(k'_v)}(\sigma_{1,v} \otimes \sigma_{2,v} \otimes \overline{\omega_{\psi'_v,\mu_v}}, \mathbb{C}) \neq 0.$$

We are going to use the fundamental lemma of Jacquet–Rallis [JR11, Yun11] and its variant for the relative trace formulae on $\mathrm{U}(n) \times \mathrm{U}(n)$ [Liu14]. Liu proved that there is a constant $c(n)$ depending only on n, such that if v is a place of k' whose residue characteristic is larger than $c(n)$, then the fundamental lemma holds. C.f. Theorem 4.2.4 for a precise statement.

Theorem 1.2.3. *Let π_1 and π_2 be two irreducible cuspidal tempered automorphic representations of $\mathrm{U}(V)(\mathbb{A}')$. Assume Hypotheses 1.2.1 and 1.2.2. Denote by Σ the finite set of non-split places of k' such that μ_v, $\pi_{1,v}$ and $\pi_{2,v}$ are unramified if $v \notin \Sigma$. Assume the following conditions.*

1. *There is a split non-archimedean place v_0, such that π_{1,v_0} and π_{2,v_0} are supercuspidal.*
2. *If $v \in \Sigma$, then both $\pi_{1,v}$ and $\pi_{2,v}$ are supercuspidal.*
3. *The set Σ contains all the places of k' whose residue characteristic is smaller than $c(n)$.*
4. *All archimedean places of k' split in k.*

Then

$$|\mathcal{FJ}_{\psi',\mu}(\varphi_1,\varphi_2,\phi)|^2 = \frac{\Delta_{\mathrm{U}(V)}}{4} \frac{L(\frac{1}{2}, \mathrm{BC}(\pi_1) \times \mathrm{BC}(\pi_2) \otimes \mu^{-1})}{L(1,\pi_1,\mathrm{Ad})L(1,\pi_2,\mathrm{Ad})} \prod_v \alpha_v^\natural(\varphi_{1,v},\varphi_{2,v},\phi_v). \tag{1.2.1}$$

Corollary 1.2.4. *Under the conditions of Theorem 1.2.3, we have*

$$L\left(\frac{1}{2}, \mathrm{BC}(\pi_1) \times \mathrm{BC}(\pi_2) \otimes \mu^{-1}\right) \geq 0.$$

Remark 1.2.5. It follows from Hypothesis 1.2.1 that the base change of π_i ($i = 1, 2$) to GL_n are cuspidal. The order of the component group $|S_{\pi_1}||S_{\pi_2}| = 4$. This is compatible with Conjecture 1.1.2.

Remark 1.2.6. We clarify here what the conditions in the theorem are used for. Condition (1) and (4) are included so that we can apply the simple relative trace formulae developed in [Liu14, Xue14]. Condition (2) is included because we are only able to establish the local germ expansion for the distribution on the unitary groups for supercuspidal representations. This seems to be only a technical condition, c.f. Lemma 6.4.1. Note that Lemma 6.4.1 is the only place where condition (2) is used. Condition (3) is included because we need to use the fundamental lemma to establish the local distribution identity at the unramified places.

Remark 1.2.7. The simple relative trace formula in [Xue14] requires that there are two split places v_1 and v_2 of k', such that π_1 and π_2 are both supercuspidal at v_1 and v_2. We prove in Appendix C that this hypothesis can be weakened to that π_1 and π_2 are supercuspidal at v_1 and tempered at v_2.

Remark 1.2.8. In [Zha14b, Theorem 1.2 (2)], Zhang established a formula for the compact periods (i.e. the unitary groups at archimedean places are all compact) up to an undetermined constant depending only on the archimedean components of the automorphic representations. Our situation is slightly more complicated. In order to establish the relative trace formulae when the unitary groups are all compact at archimedean places, we need slightly more than the existence of the smooth matching of test functions to handle of the spectral side of the trace formulae, see Remark 4.2.2. Moreover, we need "Schwartz test functions" instead of only the compactly supported ones. We will handle this in another paper.

1.3. **Outline of the proof.** The proof of Theorem 1.2.3 is based on the relative trace formula developed in [Liu14, Xue14] and the local harmonic analysis technique of [Zha14b].

Definition 1.3.1. Let $f_1, f_2 \in \mathcal{C}_c^\infty(\mathrm{U}(V)(\mathbb{A}'))$ and $\phi_1, \phi_2 \in \mathcal{S}(\mathbf{L}(\mathbb{A}'))$. Define the global distribution

$$J_{\pi_1,\pi_2}(f_1,f_2,\phi_1,\phi_2) = \sum_{\varphi_1,\varphi_2} \mathcal{FJ}_{\psi',\mu}(\pi_1(f_1)\varphi_1, \pi_2(f_2)\varphi_2, \phi_1)\overline{\mathcal{FJ}_{\psi',\mu}(\varphi_1,\varphi_2,\phi_2)}, \tag{1.3.1}$$

where φ_1 (resp. φ_2) runs through an orthonormal basis of π_1 (resp. π_2).

Let v be a place of k'. Recall that we have defined a linear functional α_v and its normalized version $\alpha_v^\natural$. Let $f_{1,v}, f_{2,v} \in \mathcal{C}_c^\infty(\mathrm{U}(V)(k'_v))$ and $\phi_{1,v}, \phi_{2,v} \in \mathcal{S}(\mathbf{L}(k'_v))$.

Definition 1.3.2. Define

$$J_{\pi_{1,v},\pi_{2,v}}(f_{1,v},f_{2,v},\phi_{1,v},\phi_{2,v}) = \sum_{\varphi_{1,v},\varphi_{2,v}} \alpha_v(\pi_{1,v}(f_{1,v})\varphi_{1,v},\varphi_{1,v},\pi_{2,v}(f_{2,v})\varphi_{2,v},\varphi_{2,v},\phi_{1,v},\phi_{2,v}), \tag{1.3.2}$$

and its normalized version

$$J^{\natural}_{\pi_{1,v},\pi_{2,v}}(f_{1,v},f_{2,v},\phi_{1,v},\phi_{2,v}) = \sum_{\varphi_{1,v},\varphi_{2,v}} \alpha^{\natural}_v(\pi_{1,v}(f_{1,v})\varphi_{1,v},\varphi_{1,v},\pi_{2,v}(f_{2,v})\varphi_{2,v},\varphi_{2,v},\phi_{1,v},\phi_{2,v}), \tag{1.3.3}$$

where in both expressions, $\varphi_{1,v}$ (resp. $\varphi_{2,v}$) runs over an orthonormal basis of $\pi_{1,v}$ (resp. $\pi_{2,v}$).

Remark 1.3.3. The local distributions J_v and $J^{\natural}_v$ do not depend on the choice of $\langle -,-\rangle_{\pi_{i,v}}$ $(i=1,2)$.

Conjecture 1.3.4. *Assume that π_1 and π_2 are irreducible cuspidal tempered automorphic representations of $\mathrm{U}(V)(\mathbb{A}')$. If $f_1=\otimes f_{1,v}, f_2=\otimes f_{2,v}, \phi_1=\otimes\phi_{1,v}, \phi_2=\otimes\phi_{2,v}$ are all factorizable, then*

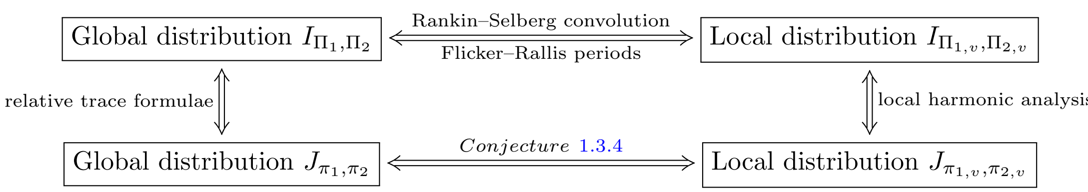
$$J_{\pi_1,\pi_2}(f_1,f_2,\phi_1,\phi_2) = \frac{\Delta_{\mathrm{U}(V)}}{|S_{\pi_1}||S_{\pi_2}|}\frac{L(\frac{1}{2},\mathrm{BC}(\pi_1)\times\mathrm{BC}(\pi_2)\otimes\mu^{-1})}{L(1,\pi_1,\mathrm{Ad})L(1,\pi_2,\mathrm{Ad})}\prod_v J^{\natural}_{\pi_{1,v},\pi_{2,v}}(f_{1,v},f_{2,v},\phi_{1,v},\phi_{2,v}). \tag{1.3.4}$$

Lemma 1.3.5. *Conjecture 1.3.4 is equivalent to Conjecture 1.1.2.*

Proof. By summing over orthonormal bases of π_1 and π_2 in the identity (1.2.1), we see that Conjecture 1.1.2 implies Conjecture 1.3.4.

Conversely, if for some v, $\mathrm{Hom}(\pi_{1,v}\otimes\pi_{2,v}\otimes\overline{\omega_{\psi'_v,\mu_v}},\mathbb{C})=0$, then Conjecture 1.1.2 holds since both sides of (1.1.4) vanish. Assume that for any v, $\mathrm{Hom}(\pi_{1,v}\otimes\pi_{2,v}\otimes\overline{\omega_{\psi'_v,\mu_v}},C)\neq 0$. Then $\alpha^{\natural}_v\neq 0$ for all v by Proposition 1.1.1. By (1.1.5), there is a constant C, such that $\mathcal{FJ}_{\psi',\mu}\cdot\overline{\mathcal{FJ}_{\psi',\mu}} = C\times\prod\alpha^{\natural}_v$. Summing over orthonormal bases of π_1 and π_2 on both sides of (1.1.5), we conclude that $J_{\pi_1,\pi_2}=C\times\prod J^{\natural}_{\pi_{1,v},\pi_{2,v}}$. Conjecture 1.3.4 then implies Conjecture 1.1.2. ☐

We can define an analogue global distribution I_{Π_1,Π_2} and a local distribution $I_{\Pi_{1,v},\Pi_{2,v}}$ on the general linear group where Π_i is the base change of π_i $(i=1,2)$, c.f. Section 3.2 for the precise definition. It can be proved without much difficulty that I_{Π_1,Π_2} decomposes as a product of $I_{\Pi_{1,v},\Pi_{2,v}}$, a relation analogous to Conjecture 1.3.4. Then to prove Conjecture 1.1.2, with the relative trace formulae and the decomposition of I_{Π_1,Π_2} at hand, all we have to prove is a relation between the local distributions $J_{\pi_{1,v},\pi_{2,v}}$ and $I_{\Pi_{1,v},\Pi_{2,v}}$. This process can be summarized as follows.



After setting up the notation and measures in Section 2, we handle the top horizontal arrow in Section 3. The desired decomposition of the distribution I_{Π_1,Π_2} follows without much difficulty from known results on the Rankin–Selberg convolution and the Flicker–Rallis periods. The left vertical arrow is the relative trace formulae developed in [Liu14, Xue14]. We recall some key ingredients from [Xue14] in Section 4: orbits, orbital integrals, smooth matching of test functions and the relative trace identity. The right vertical arrow is the most technical part and it forms the last two sections of the paper. The key observation, which has already appeared in [Liu14, Xue14] is the following. The global Gan–Gross–Prasad conjectures for $\mathrm{U}(n)\times\mathrm{U}(n)$ and for $\mathrm{U}(n+1)\times\mathrm{U}(n)$ are quite different. However, it is quite remarkable that Liu's relative trace formulae and Jacquet–Rallis relative trace formula lead to very similar local harmonic analysis problems. In fact, when suitably modified, most of the local techniques in [Zha14b] can be adapted to our situation.

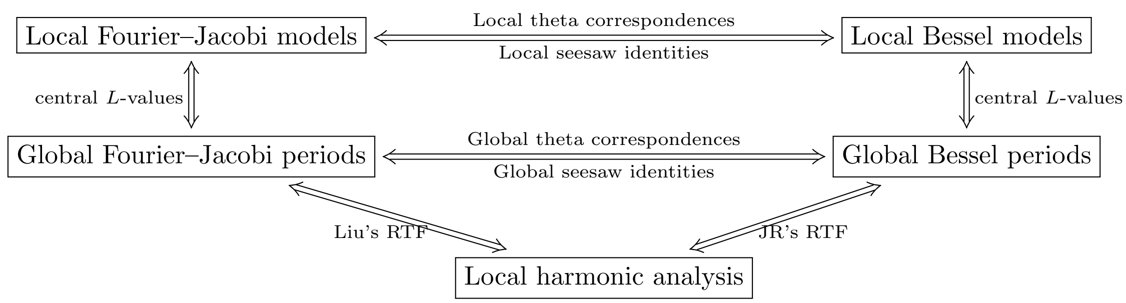
There are four appendices which supplement the main body of this paper. In Appendix A, we summarize the definitions and facts from the theory of theta correspondences. In appendix B, we summarize some estimates that are needed in Appendix C and Appendix D. The results in these two appendices are well-known and are included only for the convenience of the readers. In Appendix C, we strengthen [Xue14, Proposition 6.1.2] and thus weaken some local conditions of [Xue14]. In Appendix D, we prove Proposition 1.1.1.

1.4. **Bessel periods versus Fourier–Jacobi periods.** As already observed in [GGP12], the Gan–Gross–Prasad conjecture for $\mathrm{U}(n+1) \times \mathrm{U}(n)$ and for $\mathrm{U}(n) \times \mathrm{U}(n)$ are related by the theta correspondences. In particular, Gan and Ichino [GI] have proved the local Gan–Gross–Prasad conjecture for $\mathrm{U}(n) \times \mathrm{U}(n)$ in the non-archimedean case by reducing it to the local Gan–Gross–Prasad conjecture for $\mathrm{U}(n+1) \times \mathrm{U}(n)$ which has been proved by Beuzart-Plessis [BP].

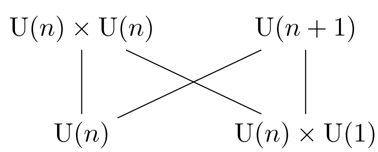
The proof of Proposition 1.1.1 follows the similar strategy. By using the local theta correspondences and some local seesaw identities, we prove Proposition 1.1.1 by reducing it to the analogue known statements for $\mathrm{U}(n+1) \times \mathrm{U}(n)$.

We apply Theorem 1.2.3 to establish some new endoscopic cases of the refined Gan–Gross–Prasad conjecture for $\mathrm{U}(n+1) \times \mathrm{U}(n)$ (c.f. [Har12]). Note that this case cannot be dealt with directly using the simple relative trace formulae developed in [Zha14a, Zha14b] until more information on the spectral side is obtained.

The relation between the conjectures for $\mathrm{U}(n) \times \mathrm{U}(n+1)$ and for $\mathrm{U}(n) \times \mathrm{U}(n)$ can be summarized as follows. We write "RTF" for the relative trace formulae for short. The left (resp. right) slanted arrow stands for Liu's relative trace formulae (resp. Jacquet–Rallis relative trace formulae). As we have noted above, both relative trace formulae lead to very similar local harmonic analysis problems.



The seesaw identities we use (both local and global) are given by the following seesaw diagram (we are not specifying the spaces involved).



We now describe the precise results. Let $W_n \subset W_{n+1}$ be a pair of hermitian spaces of dimension n and $n+1$ respectively. Let π_n and π_{n+1} be irreducible cuspidal automorphic representations of $\mathrm{U}(W_n)(\mathbb{A}')$ and $\mathrm{U}(W_{n+1})(\mathbb{A}')$ respectively. Let $\varphi_{n+1} \in \pi_{n+1}$, $\varphi_n \in \pi_n$. We define the Bessel period by

$$\mathcal{B}(\varphi_{n+1}, \varphi_n) = \int_{\mathrm{U}(W_n)(k') \backslash \mathrm{U}(W_n)(\mathbb{A}')} \varphi_{n+1}(\iota(h))\varphi_n(h)\mathrm{d}h,$$

where $\iota : \mathrm{U}(W_n) \to \mathrm{U}(W_{n+1})$ is the embedding induced by the inclusion of the hermitan spaces $W_n \subset W_{n+1}$. The measure $\mathrm{d}h$ is the Tamagawa measure on $\mathrm{U}(W_n)(\mathbb{A}')$. We choose a measure $\mathrm{d}h_v$ on $\mathrm{U}(W_n)(k'_v)$ for each place v of k' so that $\mathrm{d}h = \prod_v \mathrm{d}h_v$. Let v be a place of k'. Let $\varphi_{n,v}, \varphi^\vee_{n,v} \in \pi_{n,v}$ and $\varphi_{n+1,v}, \varphi^\vee_{n+1,v} \in \pi_{n+1,v}$. Then we define

$$\beta_v(\varphi_{n+1,v}, \varphi^\vee_{n+1,v}, \varphi_{n,v}, \varphi^\vee_{n,v}) = \int_{\mathrm{U}(W_n)(k'_v)} \langle \pi_{n+1,v}(\iota(h_v))\varphi_{n+1,v}, \varphi^\vee_{n+1,v}\rangle \langle \pi_{n,v}(h_v)\varphi_{n,v}, \varphi^\vee_{n,v}\rangle \mathrm{d}h_v.$$

This integral is convergent if $\pi_{n,v}$ and $\pi_{n+1,v}$ are both tempered [Har12, Proposition 2.1]. We put $\beta_v(\varphi_{n+1,v}, \varphi_{n,v}) = \beta_v(\varphi_{n+1,v}, \varphi_{n+1,v}, \varphi_{n,v}, \varphi_{n,v})$. Then $\beta_v(\varphi_{n+1,v}, \varphi_{n,v}) \geq 0$. Assume that $\pi_{n,v}$ and $\pi_{n+1,v}$ are both unramified, i.e. $\pi_{n,v}$ (resp. $\pi_{n+1,v}$) has a K_n (resp. K_{n+1}) fixed vector $\varphi^0_{n,v}$ (resp. $\varphi^0_{n+1,v}$), where $K_n = \mathrm{U}(W_n)(\mathfrak{o}'_v)$ (resp. $K_{n+1} = \mathrm{U}(W_{n+1})(\mathfrak{o}'_v)$) is a hyperspecial maximal compact subgroup of $\mathrm{U}(W_n)(k'_v)$ (resp. $\mathrm{U}(W_{n+1})(k'_v)$).

Then

$$\frac{\beta_v(\varphi^0_{n+1},\varphi^0_n)}{\langle\varphi^0_{n+1,v},\varphi^0_{n+1,v}\rangle\langle\varphi^0_{n,v},\varphi^0_{n,v}\rangle}=\frac{L\left(\frac12,\Pi_{n+1,v}\times\Pi_{n,v}\right)}{L(1,\pi_{n+1,v},\mathrm{Ad})L(1,\pi_{n,v},\mathrm{Ad})}\times\prod_{j=1}^{n+1}L(j,\eta_v^j)\times\mathrm{vol}\,K_n.$$

We thus define

$$\begin{aligned}&\beta_v^\natural(\varphi_{n+1,v},\varphi^\vee_{n+1,v},\varphi_{n,v},\varphi^\vee_{n,v})\\ =&\beta_v(\varphi_{n+1,v},\varphi^\vee_{n+1,v},\varphi_{n,v},\varphi^\vee_{n,v})\times\left(\frac{L\left(\frac12,\Pi_{n+1,v}\times\Pi_{n,v}\right)}{L(1,\pi_{n+1,v},\mathrm{Ad})L(1,\pi_{n,v},\mathrm{Ad})}\times\prod_{j=1}^{n+1}L(j,\eta_v^j)\right)^{-1},\end{aligned}$$

where $\Pi_{n+1,v}$ and $\Pi_{n,v}$ stand for the base change of $\pi_{n+1,v}$ and $\pi_{n,v}$ respectively.

The following proposition confirms some cases of [Har12, Conjecture 1.3]. We refer the readers to Appendix A for the definitions and facts about the theta correspondences.

Proposition 1.4.1. *Let π_{n+1} and π_n be irreducible cuspidal tempered automorphic representations of $\mathrm{U}(W_{n+1})(\mathbb{A}')$ and $\mathrm{U}(W_n)(\mathbb{A}')$ respectively. Assume that π_{n+1} (resp. π_n) is the theta lifting of σ_1 (resp. σ_2), where σ_1 and σ_2 are two irreducible cuspidal tempered automorphic representations of $\mathrm{U}(V_n)(\mathbb{A}')$ for some n-dimensional skew-hermitian space V_n. Assume that σ_1 and σ_2 satisfy the conditions of Theorem 1.2.3. Then*

$$|\mathcal{B}(\varphi_{n+1},\varphi_n)|^2=\frac18\Delta_{\mathrm{U}(W_{n+1})}\frac{L\left(\frac12,\mathrm{BC}(\pi_{n+1})\times\mathrm{BC}(\pi_n)\right)}{L(1,\pi_{n+1},\mathrm{Ad})L(1,\pi_n,\mathrm{Ad})}\times\prod_v\beta_v^\natural(\varphi_{n+1,v},\varphi_{n,v}).$$

Proof. Suppose that $\pi_{n+1}=\theta_{\psi',\mu}(\sigma_1)$ and $\pi_n=\theta_{\psi',\mu}(\sigma_2)$ by Proposition A.4.1. Note that $\sigma_2=\theta_{\psi'^{-1},\mu^{-1}}(\pi_n)$. We omit the subscripts on the spaces in the notation of the Weil representations in this proof, keeping only the characters.

Let S be a sufficiently large finite set of places of k' so that if $v\notin S$, then $\pi_{n+1,v},\pi_{n,v},\psi'_v,\mu'_v$ (hence $\sigma_{1,v},\sigma_{2,v}$) are unramified, the hyperspecial maximal compact subgroup $K_{n,v}$ (resp. $K_{n+1,v}$) of $\mathrm{U}(W_n)(F_v)$ (resp. $\mathrm{U}(W_{n+1})(F_v)$) is of volume 1, $\varphi_{n,v}$ (resp. $\varphi_{n+1,v}$) is $K_{n,v}$ (resp. $K_{n+1,v}$) fixed, $\langle\varphi_{n+1,v},\varphi_{n+1,v}\rangle=\langle\varphi_{n,v},\varphi_{n,v}\rangle=1$. The desired identity is equivalent to the following

$$|\mathcal{B}(\varphi_{n+1},\varphi_n)|^2=\frac18\Delta^S_{\mathrm{U}(W_{n+1})}\frac{L^S\left(\frac12,\mathrm{BC}(\pi_{n+1})\times\mathrm{BC}(\pi_n)\right)}{L^S(1,\pi_{n+1},\mathrm{Ad})L^S(1,\pi_n,\mathrm{Ad})}\times\prod_{v\in S}\beta_v(\varphi_{n+1,v},\varphi_{n,v}).$$

By looking at the Satake parameters of the theta lifting of unramified representations [Liu, Appendix], one sees that if $v\notin S$, then $\mathrm{BC}(\pi_{n+1,v})=\mathrm{BC}(\sigma_{1,v})\otimes\mu_v^{-1}\boxplus\mu_v^n$ and $\mathrm{BC}(\sigma_{2,v})=\mathrm{BC}(\pi_{n,v})$.

Let $f_1\in\sigma_1$, $f_2\in\sigma_2$ and $\phi_1\in\mathcal{S}(\mathbf{L}_n(\mathbb{A}'))$ where $\mathbf{L}_n\subset\mathrm{Res}\,V_n^\vee$ is a Lagrangian subspace. Assume that $f_2=\theta_{\psi'^{-1},\mu^{-1}}(\varphi_n,\overline{\phi_n})$ with $\varphi_n\in\pi_n$ and $\phi_n\in\mathcal{S}(\mathbf{L}_{nn}(\mathbb{A}'))$ where $\mathbf{L}_{nn}\subset\mathrm{Res}(V_n\otimes W_n)^\vee$ is a Lagrangian subspace. Then $\mathbf{L}_{n+1,n}=\mathbf{L}_n+\mathbf{L}_{nn}$ is a Lagrangian subspace of $\mathrm{Res}(V_{n+1}\otimes W_n)^\vee$. Then it is not hard to see that

$$\mathcal{FJ}(f_1,\theta_{\psi'^{-1},\mu^{-1}}(\varphi_n,\phi_n),\phi_1)=\overline{\mathcal{B}(\theta_{\psi',\mu}(f_1,\phi_n\otimes\phi_1),\varphi_n)},$$

where $\phi_n\otimes\phi_1\in\mathcal{S}(\mathbf{L}_{n+1,n}(\mathbb{A}'))$, $\theta_{\psi',\mu}(f_1,\phi_n\otimes\phi_1)\in\pi_{n+1}$.

Then Theorem 1.2.3 together with the Rallis inner product formula (c.f. Theorem A.4.2) imply that

$$\begin{aligned}&|\mathcal{B}(\theta_{\psi',\mu}(f_1,\phi_n\otimes\phi_1),\varphi_n)|^2\\ =&\frac14\times\frac{L^S\left(\frac12,\mathrm{BC}(\sigma_1)\times\mathrm{BC}(\sigma_2)\otimes\mu^{-1}\right)}{L^S(1,\sigma_1,\mathrm{Ad})L^S(1,\sigma_2,\mathrm{Ad})}\times\prod_{j=1}^nL^S(j,\eta^j)\times\frac{L^S\left(\frac12,\mathrm{BC}(\pi_n)\otimes\mu^n\right)}{\prod_{j=1}^nL^S(j,\eta^{n-j})}\\ &\prod_{v\in S}\int_{\mathrm{U}(V_n)(k')}\int_{\mathrm{U}(W_n)(k')}\overline{\langle\sigma_{1,v}(g)f_{1,v},f_{1,v}\rangle}\langle\omega_{\psi'_v,\mu_v}(g,\iota(h))\phi_n\otimes\phi_1,\phi_n\otimes\phi_1\rangle\langle\sigma_n(h)\varphi_{n,v},\varphi_{n,v}\rangle\mathrm{d}h\mathrm{d}g.\end{aligned}$$

The double integral is absolutely convergent by Lemma A.3.1. Therefore we can integrate over g first and then over h. Applying the explicit local theta lifting (c.f. Lemma A.2.2) and the Rallis inner product formula again, we get

$$
\begin{aligned}
|\mathcal{B}(\varphi_{n+1},\varphi_n)|^2 =& \frac{1}{4}\times\frac{L^S\left(\frac{1}{2},\mathrm{BC}(\sigma_1)\times\mathrm{BC}(\sigma_2)\otimes\mu^{-1}\right)}{L^S(1,\sigma_1,\mathrm{Ad})L^S(1,\sigma_2,\mathrm{Ad})}\times\prod_{j=1}^{n}L^S(j,\eta^j)\times\frac{\prod_{j=1}^{n}L^S(j+1,\eta^{n-j+1})}{2\times L^S(1,\mathrm{BC}(\sigma_1)\times\mu^n)}\\
&\times\frac{L^S\left(\frac{1}{2},\mathrm{BC}(\sigma_2)\times\mu^n\right)}{\prod_{j=1}^{n}L^S(j,\eta^{n-j})}\times\prod_{v\in S}\beta_v(\varphi_{n+1,v},\varphi_{n,v})\\
=&\frac{1}{8}\times\Delta^S_{\mathrm{U}(W_{n+1})}\frac{L^S\left(\frac{1}{2},\mathrm{BC}(\pi_{n+1})\times\mathrm{BC}(\pi_n)\right)}{L^S(1,\pi_{n+1},\mathrm{Ad})L^S(1,\pi_n,\mathrm{Ad})}\times\prod_{v\in S}\beta_v(\varphi_{n+1,v},\varphi_{n,v}).
\end{aligned}
$$

□

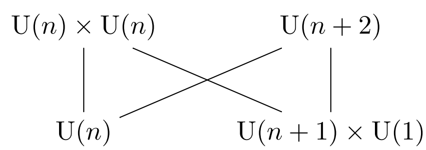
Conversely, if one knows enough about the Bessel periods, one can deduce some cases of Conjecture 1.1.2.

Proposition 1.4.2. *Assume* [Har12, Conjecture 1.3]. *Let V be an n-dimensional skew-hermitian space over k. Let π_1 and π_2 be irreducible cuspidal automorphic tempered representations of* $\mathrm{U}(V)(\mathbb{A}')$. *If π_1 (resp. π_2) admits a nonzero cuspidal theta lift to $\mathrm{U}(W_n)(\mathbb{A}')$ (resp. $\mathrm{U}(W_{n+1})(\mathbb{A}')$) where $W_n\subset W_{n+1}$ are hermitian spaces of dimension n and $n+1$ respectively, then Conjecture 1.1.2 holds.*

The proof is similar to the previous proposition, except that we run the argument backwards. We omit the details.

Remark 1.4.3. By the Rallis inner product formula and the theta dichotomy for the unitary groups [GI, Appendix A; HKS96], one sees that there is an n-dimensional hermitian space W so that the theta lift of π_i to $\mathrm{U}(W)(\mathbb{A}')$ is nonzero if and only if $L(\frac{1}{2},\mathrm{BC}(\pi_i)\otimes\mu^n)\neq 0$. So the condition of this proposition is not verified if $L(\frac{1}{2},\mathrm{BC}(\pi_1)\otimes\mu^n)=L(\frac{1}{2},\mathrm{BC}(\pi_2)\otimes\mu^n)=0$.

Now assume that $L(\frac{1}{2},\mathrm{BC}(\pi_1)\otimes\mu^n)=L(\frac{1}{2},\mathrm{BC}(\pi_2)\otimes\mu^n)=0$. Then via the seesaw diagram



,

Conjecture 1.1.2 is related to the Gan–Gross–Prasad conjecture for $\mathrm{U}(n+1)\times\mathrm{U}(n+2)$ with the automorphic representation of $\mathrm{U}(n+2)(\mathbb{A}')$ being cuspidal but non-tempered. In this case, the defining integral of the local linear form β_v is not convergent and needs to be regularized. It is not clear how to formulate the refined Gan–Gross–Prasad conjecture for the non-tempered automorphic representations. As pointed out in [II10], the proportionality of $\mathcal{B}\cdot\overline{\mathcal{B}}$ and $\prod_v\beta_v^\natural$ depends not only on the global data, but also on the local data. See [II10, Section 9, 10, 11] and [Qiu] for some low rank examples for the orthogonal groups. We hope that Conjecture 1.1.2 together with the theta correspondences could shed some light on the formulation of the refined Gan–Gross–Prasad conjecture for the non-tempered automorphic representations and the regularization of the defining integral of β_v.

Acknowledgement. I thank Y. Liu and W. Zhang for many helpful discussions. I also thank S. Zhang for his constant support and interest in this work.

2. Notation and measures

2.1. **General notation.** Let k' be a number field and k a quadratic field extension of k'. Let τ be the non-trivial element in the Galois group $\mathrm{Gal}(k/k')$. Let $\mathbb{A}'$ (resp. $\mathbb{A}$) be the ring of adeles of k' (resp. k).

Let $\eta=\otimes\eta_v$ the quadratic character of $k'^{\times}\backslash\mathbb{A}'^{\times}$ associated to k/k' by the global class field theory. We fix a character $\mu=\otimes\mu_v$ of $k^\times\backslash\mathbb{A}^\times$ such that $\mu|_{\mathbb{A}'^\times}=\eta$. We also fix a nontrivial additive character $\psi'=\otimes\psi'_v$ of $k'\backslash\mathbb{A}'$ and extend it to an additive character of $k\backslash\mathbb{A}$ by setting $\psi(x)=\psi'(\frac{1}{2}\operatorname{Tr}_{k/k'}x)$.

We denote by k^- the set of purely imaginary elements in k and we have the decomposition $k=k'+k^-$. Similarly we have a decomposition $k_v=k'_v+k^-_v$ for any place v of k' and $\mathbb{A}=\mathbb{A}'+\mathbb{A}^-$. We fix a nonzero purely imaginary element $\delta\in k^-$ throughout.

We write $M_{m,n}$ for the additive group scheme of $m \times n$ matrices. We write M_n for $M_{n,n}$. We denote by $k_n = M_{1,n}(k)$ and $k^n = M_{n,1}(k)$. Put $e_n = (0, \cdots, 0, 1) \in k_n$. The identity matrix in $M_n(k)$ is denoted by 1_n.

Let M be a smooth manifold. We denote by $\mathcal{C}_c^\infty(M)$ the space of compactly supported smooth functions on M. We denote by $\mathcal{S}(\mathbb{A}_n)$ the space of Schwartz functions on $\mathbb{A}_n$. Similarly we have the spaces of Schwartz functions on other spaces, e.g. $\mathcal{S}(\mathbb{A}'_n)$, $\mathcal{S}(k_{v,n})$, etc.

Let V be a vector space. We denote by $V^\vee$ its dual space. Let V be a skew-hermitian space. Suppose under some basis of V, the skew-hermitian form of V is given by a skew-hermitian matrix β. Then under the dual basis in $V^\vee$, the skew-hermitian form of $V^\vee$ is given by β^{-1}.

Let X be any set and $U \subset X$ a subset. We denote the characteristic function of U by $\mathbf{1}_U$.

2.2. **Groups.** Let GL_n be the general linear group. We denote it by $\mathrm{GL}_{n,F}$ if we need to specify the field F over which GL_n is defined. We denote by B_n the standard upper triangular Borel subgroup, N_n its unipotent radical, and A_n the diagonal torus. If $a_1, \cdots, a_n \neq 0$, we write $\mathrm{diag}[a_1, \cdots, a_n] = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}$. More generally if A_i is a $k_i \times k_i$ matrix ($i = 1, \cdots, r$), we write

$$\mathrm{diag}[A_1, \cdots A_r] = \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_r \end{pmatrix}.$$

Let Z_n be the center of GL_n. Let $P_n \subset \mathrm{GL}_n$ be the mirabolic subgroup consisting of matrices whose last row is $(0, \cdots, 0, 1)$.

Let $H \subset \mathrm{GL}_n$ be a subgroup. We denote by H_- the subgroup of GL_n consisting of the transpose of the elements in H. For example, $B_{n,-}$ is the group of lower triangular matrices and $N_{n,-}$ is its unipotent radical.

We use Gothic letters to denote the Lie algebra of the corresponding group. For instance, $\mathfrak{gl}_n$ stands for the Lie algebra of GL_n, $\mathfrak{b}_n$ stands for the Lie algebra of B_n, etc..

We usually consider GL_{n-1} as a subgroup of GL_n via the embedding $\iota : \mathrm{GL}_{n-1} \to \mathrm{GL}_n$, $\iota(g) = \mathrm{diag}[g, 1]$.

We denote by $\mathsf{Herm}_n(k)$ the set of isometric classes of skew-hermitian spaces over k of dimension n. Let $(V, \langle -, - \rangle) \in \mathsf{Herm}_n(k)$. Let $\mathrm{U}(V)$ be the corresponding unitary group over k' and $\mathfrak{u}(V)$ its Lie algebra. Denote by $Z_{\mathrm{U}(V)}$ the center of $\mathrm{U}(V)$. By choosing a basis of V, we write the skew-hermitian form $\langle -, - \rangle$ as a nonsingular skew-hermitian matrix β. Then $\mathrm{U}(V)(k')$ can be viewed as a subgroup of $\mathrm{GL}_n(k)$ consisting of matrices satisfying ${}^t g^\tau \beta g = \beta$. The Lie algebra $\mathfrak{u}(V)(k')$ consists of matrices $X \in M_n(k)$ with

$${}^t X^\tau \beta + \beta X = 0.$$

We define the Caylay transform on the unitary groups

$$\mathfrak{c} : \mathfrak{u}(V) \to \mathrm{U}(V), \quad X \to (1 + X)(1 - X)^{-1}, \tag{2.2.1}$$

whenever the expression makes sense. It defines a homeomorphism from a neighborhood of $0 \in \mathfrak{u}(V)$ to a neighborhood of $1 \in \mathrm{U}(V)$.

Let $\mathrm{Res}\, V$ be the symplectic space over k' of dimension $2n$ whose underline vector space is V viewed as a vector space over k' and whose symplectic form is given by

$$\langle\langle w, v \rangle\rangle = \frac{1}{2} \mathrm{Tr}_{k/k'} \langle w, v \rangle$$

By a Lagrangian subspace of $\mathrm{Res}\, V$, we mean a maximal isotropic subspace of $\mathrm{Res}\, V$.

2.3. **Self-dual measures on the additive groups.** Let v be a place of k' and $k_v = k \otimes_{k'} k'_v$. Let f be a Schwartz function on k_v. Let

$$\widehat{f}(y) = \int_{k_v} f(x) \psi(xy) \mathrm{d}x,$$

be its Fourier transform. Let $\mathrm{d}x$ be the self-dual measure on k_v, i.e. the unique Haar measure on k_v so that $\widehat{\widehat{f}}(x) = f(-x)$. We put a measure $\mathrm{d}x$ on $M_{m,n}(k_v)$ via the identification $M_{m,n}(k_v) \simeq k_v^{mn}$. Let f is a

Schwartz function on $M_{m,n}(k_v)$ and $y \in M_{n,m}(k_v)$. Define the Fourier transform

$$\widehat{f}(y) = \int_{M_{m,n}(k_v)} f(x)\psi(\mathrm{Tr}\, xy)\mathrm{d}x.$$

Then

$$\int_{M_{n,m}(k_v)} \widehat{f}(y)\psi(\mathrm{Tr}\, xy)\mathrm{d}y = f(-x).$$

We define the measures on k'_v, $M_{m,n}(k'_v)$, k^- and $M_{m,n}(k^-)$ in a similar way. To do so, we only have to note that $\psi|_{k'} = \psi'$ by definition and $(x, y) \mapsto \mathrm{Tr}\, xy$ is a nondegenerate pairing when restricted to each of the above spaces. We call the measures obtained in this way the self-dual measures on the corresponding spaces.

We then normalize our absolute value on k'_v by

$$\mathrm{d}(ax) = |a|_{k'_v}\mathrm{d}x, \quad a \in k'_v,$$

and similarly for the absolute value on k_v. Note that for $x \in k_v$, we have $|x|_{k_v} = |\mathrm{N}_{k/k'}\, x|_{k'_v}$.

Recall that we have fixed a purely imaginary element δ. We sometimes make the identification $k_v^- \simeq k'_v$ by sending δ to 1. Transporting the self-dual measure of k_v^- to k'_v via this identification, we get the measure $|\delta|_{k_v}^{-\frac{1}{2}}\mathrm{d}x$ on k'_v, where $\mathrm{d}x$ is the self-dual measure on k'_v.

We take the product measure $\mathrm{d}x = \prod_v \mathrm{d}x_v$ on $\mathbb{A}$ where $\mathrm{d}x_v$ is the self-dual measure on k_v. This is the self-dual measure for the Fourier transform

$$\widehat{f}(y) = \int_{\mathbb{A}} f(x)\psi(xy)\mathrm{d}x,$$

where f is a Schwartz function on $\mathbb{A}$. Similarly we define the self-dual measures on $\mathbb{A}'$, $\mathbb{A}^-$, $M_{m,n}(\mathbb{A})$, etc..

2.4. **Measures on the general linear group.** Let v be a place of k'. Define the normalized multiplicative measure on $k'^{\times}_v$ by

$$\mathrm{d}^\times x = \zeta_{k'_v}(1)\frac{\mathrm{d}x}{|x|_{k'_v}},$$

where $\mathrm{d}x$ is the self-dual measure on k'_v. We also define the unnormalized one by

$$\mathrm{d}^* x = \frac{\mathrm{d}x}{|x|_{k'_v}}.$$

Similarly we define the normalized and unnormalized multiplicative measures on $k_v^\times$. On $\mathrm{GL}_n(k'_v)$, we take the measure

$$\mathrm{d}g = \zeta_{k'_v}(1)\frac{\prod_{ij}\mathrm{d}x_{ij}}{|\det g|_{k'_v}^n}, \quad g = (x_{ij}) \in \mathrm{GL}_n(k'_v),$$

and similarly on $\mathrm{GL}_n(k_v)$. If ψ'_v is unramified, then the volume of $\mathrm{GL}_n(\mathfrak{o}'_v)$ under the measure $\mathrm{d}g$ is $\zeta_{k'_v}(2)^{-1}\cdots\zeta_{k'_v}(n)^{-1}$. On $N_n(k'_v)$, we take the measure

$$\mathrm{d}n = \prod_{1\le i<j\le n}\mathrm{d}u_{ij}, \quad u = (u_{ij}) \in N_n(k'_v),$$

and similarly on other unipotent groups, e.g. $N_n(k_v)$, $N_{n,-}(k'_v)$, etc..

Let $A_n \subset \mathrm{GL}_n$ be the diagonal torus. We put a measure $\mathrm{d}a$ on $A_n(k_v)$ by identifying $A_n(k_v)$ with $A_n(k_v) \simeq (k_v^\times)^n$ and taking the measure $\mathrm{d}^* x$ on $k^\times$. Let $a = \mathrm{diag}[a_1, \cdots, a_n] \in A_n(k_v)$. We define the modulus character δ_n of $B_n(k_v)$ by

$$\delta_n(a) = \prod_{i=1}^n |a_i|^{n-2i+1}.$$

The measure $\mathrm{d}n$ satisfies the following change of variable formula

$$\mathrm{d}(ana^{-1}) = \delta_n(a)\mathrm{d}n.$$

The subset $N_n(k_v)A_n(k_v)N_{n,-}(k_v)$ is an open subset of $\mathrm{GL}_n(k_v)$. The restriction of $\mathrm{d}g$ to this subset decomposes as

$$\mathrm{d}g = \zeta_{k_v}(1)|\delta_n^{-1}(a)|\mathrm{d}a\mathrm{d}n\mathrm{d}n_-,$$

where $a \in A_n(k_v)$, $n \in N_n(k_v)$ and $n_- \in N_{n,-}(k_v)$. We have a similar decomposition for the group $\mathrm{GL}_n(k'_v)$.

We take the product measure

$$\mathrm{d}g = \prod_v \mathrm{d}g_v$$

on $\mathrm{GL}_n(\mathbb{A}')$ where $\mathrm{d}g_v$ is the measure on $\mathrm{GL}_n(k'_v)$ defined above. Similarly we take the product measure on $N_n(\mathbb{A}')$, $\mathrm{GL}_n(\mathbb{A})$, etc..

Let Π be an irreducible cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A})$ with unitary central character. We define the Petersson inner product on Π by

$$\langle \varphi_1, \varphi_2 \rangle_{\mathrm{Pet}} = \int\limits_{Z_n(\mathbb{A})\,\mathrm{GL}_n(k)\backslash \mathrm{GL}_n(\mathbb{A})} \varphi_1(g)\overline{\varphi_2(g)}\mathrm{d}g, \quad \varphi_1, \varphi_2 \in \Pi.$$

2.5. **Measures on the unitary groups.** Let V be a skew-hermitian space over k. Let $\mathrm{U}(V)$ be the corresponding unitary group and $\mathfrak{u}(V)$ its Lie algebra. Let $\mathrm{d}X$ on $\mathfrak{u}(V)(\mathbb{A}')$ be the self-dual measure for the Fourier transform

$$\widehat{f}(Y) = \int\limits_{\mathfrak{u}(V)(\mathbb{A}')} f(X)\psi(\mathrm{Tr}\, XY)\mathrm{d}X,$$

where f is a Schwartz function on $\mathfrak{u}(V)(\mathbb{A}')$. Similarly we can define the self-dual measure $\mathrm{d}X_v$ on $\mathfrak{u}(V)(k'_v)$ for any place v of k'.

Let ω be a top invariant differential form on $\mathrm{U}(V)$ so that its pullback $\mathfrak{c}^*\omega$ via the Caylay transform (2.2.1) gives rise to the self-dual measure on $\mathfrak{u}(V)$. We choose the measure on $\mathrm{U}(V)(k'_v)$ as $|\omega|_{k'_v}$. It is the unique Haar measure on $\mathrm{U}(V)(k'_v)$ such that the Caylay transform is measure preserving when restricted to a small neighborhood of $0 \in \mathfrak{u}(V)(k'_v)$. If $\mathrm{U}(V)$ is unramified at v, ψ'_v is unramified, then the volume of the maximal hyperspecial subgroup $\mathrm{U}(V)(\mathfrak{o}'_v)$ under the measure $|\omega|_{k'_v}$ is $L(1,\eta_v)^{-1}\zeta_{k'_v}(2)^{-1}\cdots L(n,\eta_v^n)^{-1}$.

We define the measure on $\mathrm{U}(V)(\mathbb{A}')$ by

$$\mathrm{d}g = \prod_v L(1,\eta_v)|\omega|_{k'_v}.$$

The Tamagawa measure on $\mathrm{U}(V)(\mathbb{A}')$ is $L(1,\eta)^{-1}\mathrm{d}g$. However, we use only the measure $\mathrm{d}g$ in the rest of the paper.

Let π be an irreducible cuspidal automorphic representation of $\mathrm{U}(V)(\mathbb{A}')$. We define the Petersson inner product on π by

$$\langle \varphi_1, \varphi_2 \rangle_{\mathrm{Pet}} = \int\limits_{\mathrm{U}(V)(k')\backslash \mathrm{U}(V)(\mathbb{A}')} \varphi_1(g)\overline{\varphi_2(g)}\mathrm{d}g, \quad \varphi_1, \varphi_2 \in \pi.$$

3. Global distributions on the general linear groups

3.1. **Period integrals on the general linear groups.** Let $\Pi = \otimes'\Pi_v$ be an irreducible cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A})$ with unitary central character Ω_Π. We extend ψ to a character of $N_n(\mathbb{A})$ by setting

$$\psi(u) = \psi\left(\sum_{i=1}^{n-1} u_{i,i+1}\right), \quad u = (u_{ij}) \in N_n(\mathbb{A}).$$

Similar conventions apply to other unipotent groups such as $N_n(\mathbb{A}')$, $N_n(k_v)$, etc..

Let v be a place of k'. Till the end of this section, we consider $\mathrm{GL}_{n,k}$ as an algebraic group over k'. Then $k_v = k \otimes_{k'} k'_v$ an etale algebra over k'_v of degree two. By the local component Π_v of Π at v, we mean $\Pi_v = \prod_{w|v} \Pi_w$ as a representation of $\mathrm{GL}_n(k_v)$, where w runs over all the places (one or two) of k over v.

Let $\mathcal{C}^\infty(N_n(\mathbb{A})\backslash \mathrm{GL}_n(\mathbb{A}), \psi)$ be the space of smooth functions f on $\mathrm{GL}_n(\mathbb{A})$ such that

$$f(ug) = \psi(u)f(g), \quad u \in N_n(\mathbb{A}),\ g \in \mathrm{GL}_n(\mathbb{A}).$$

We also define its local counterpart $\mathcal{C}^\infty(N_n(k_v)\backslash \mathrm{GL}_n(k_v), \psi_v)$ for any place v of k'.

Let $\varphi \in \Pi$. Define its Whittaker–Fourier coefficient by

$$W_\varphi(g) = \int_{N_n(k)\backslash N_n(\mathbb{A})} \varphi(ug)\overline{\psi(u)}\mathrm{d}u \in \mathcal{C}^\infty(N_n(\mathbb{A})\backslash \mathrm{GL}_n(\mathbb{A}), \psi).$$

The map $\Pi \to \mathcal{C}^\infty(N_n(\mathbb{A})\backslash \mathrm{GL}_n(\mathbb{A}), \psi)$, $\varphi \mapsto W_\varphi$ is $\mathrm{GL}_n(\mathbb{A})$-equivariant and injective. Its image is called the (global) Whittaker model of Π, denoted by $\mathcal{W}(\Pi, \psi)$.

Similarly, for each place v of k', the local component Π_v is generic and we have the local Whittaker model of Π_v which is a subspace of $\mathcal{C}^\infty(N_n(k_v)\backslash \mathrm{GL}_n(k_v), \psi)$ and is denoted by $\mathcal{W}(\Pi_v, \psi_v)$. Note that $\mathcal{W}(\Pi_v, \psi_v)$ contains $\mathcal{C}_c^\infty(N_{n-1}(k_v)\backslash \mathrm{GL}_{n-1}(k_v), \psi_v)$, where GL_{n-1} is viewed as a subgroup of GL_n via the embedding $g \mapsto \mathrm{diag}[g, 1]$.

There is an inner product $\mathcal{W}(\Pi_v, \psi_v)$ defined by

$$\langle W_1, W_2\rangle_{\mathcal{W}_v} = \int_{N_{n-1}(k_v)\backslash \mathrm{GL}_{n-1}(k_v)} W_1(\iota(g))\overline{W_2(\iota(g))}\mathrm{d}g. \tag{3.1.1}$$

This integral is convergent since Π_v is unitary. This pairing is indeed $\mathrm{GL}_n(k_v)$-invariant. This follows from [Ber84] if v is non-archimedean and from [Bar03] if v is archimedean. In this paper, when we speak of the inner product on $\mathcal{W}(\Pi_v, \psi_v)$, we mean the one defined by (3.1.1). If Π_v and ψ_v are unramified, W_1, W_2 are both $\mathrm{GL}_n(\mathfrak{o}_v)$-fixed and normalized such that $W_1(1) = W_2(1) = 1$ and the measure $\mathrm{d}g$ satisfies $\mathrm{vol}\,\mathrm{GL}_n(\mathfrak{o}_v) = 1$, then

$$\langle W_1, W_2\rangle_{\mathcal{W}_v} = L(1, \Pi_v \times \widetilde{\Pi_v}). \tag{3.1.2}$$

Let $\langle -, -\rangle_{\mathrm{Pet}}$ be the Petersson inner product on Π.

Proposition 3.1.1 ([Zha14b, Proposition 3.1])**.** *Let $\varphi_1, \varphi_2 \in \Pi$. Assume $W_{\varphi_1} = \otimes_v W_{1,v}$ and $W_{\varphi_2} = \otimes_v W_{2,v}$. Then*

$$\langle \varphi_1, \varphi_2\rangle_{\mathrm{Pet}} = \frac{n \times \mathrm{Res}_{s=1} L(s, \Pi \times \widetilde{\Pi})}{\mathrm{vol}(k^\times\backslash \mathbb{A}^1)} \prod_v \frac{\langle W_{1,v}, W_{2,v}\rangle_{\mathcal{W}_v}}{L(1, \Pi_v \times \widetilde{\Pi_v})},$$

where $\mathbb{A}^1$ is the subring of $\mathbb{A}$ consisting of adeles of absolute value one.

Assume that $\Omega_\Pi|_{\mathbb{A}'^\times} = 1$. Let $\varphi \in \Pi$. We define the global Flicker–Rallis period by

$$\beta_n(\varphi) = \int_{Z_n(\mathbb{A}')\,\mathrm{GL}_n(k')\backslash \mathrm{GL}_n(\mathbb{A}')} \varphi(h)\eta^{n+1}(\det h)\mathrm{d}h. \tag{3.1.3}$$

Put

$$\epsilon_{n-1} = \mathrm{diag}[\delta^{n-1}, \cdots, \delta] \in \mathrm{GL}_{n-1}(k).$$

Let v be a place of k'. Let $W_v \in \mathcal{W}(\Pi_v, \psi_v)$. Define the local Flicker–Rallis period by

$$\beta_{n,v}(W_v) = \int_{N_{n-1}(k'_v)\backslash \mathrm{GL}_{n-1}(k'_v)} W_v(\iota(\epsilon_{n-1}h))\eta_v^{n+1}(\det h)\mathrm{d}h. \tag{3.1.4}$$

This integral is absolutely convergent [Fli91]. Note that $\beta_{n,v}$ depends on δ. If Π_v and ψ_v are unramified, W_v is $\mathrm{GL}_n(\mathfrak{o}_v)$-fixed and normalized so that $W_v(1) = 1$, $\delta \in \mathfrak{o}_v^\times$, the measure $\mathrm{d}h$ satisfies $\mathrm{vol}\,\mathrm{GL}_{n-1}(\mathfrak{o}'_v) = 1$. Then

$$\beta_{n,v}(W_v) = L(1, \Pi_v, \mathrm{As}^{(-1)^{n+1}}). \tag{3.1.5}$$

Define

$$\beta^\natural_{n,v}(W_v) = \frac{\beta_{n,v}(W_v)}{L(1, \Pi_v, \mathrm{As}^{(-1)^{n+1}})}. \tag{3.1.6}$$

Proposition 3.1.2 ([GJR01, Section 2; Zha14b, Proposition 3.2])**.** *Let $\varphi \in \Pi$ and assume that $W_\varphi = \otimes W_v$. Then*

$$\beta_n(\varphi) = \frac{n \times \mathrm{Res}_{s=1} L(s, \Pi, \mathrm{As}^{(-1)^{n+1}})}{\mathrm{vol}(k'^\times\backslash \mathbb{A}'^1)} \prod_v \beta^\natural_{n,v}(W_v),$$

where $\mathbb{A}'^1$ is the subring of $\mathbb{A}'$ consisting of adeles of absolute value one.

Let $\Pi_1 = \otimes' \Pi_{1,v}$ and $\Pi_2 = \otimes' \Pi_{2,v}$ be irreducible cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A})$ with unitary central characters Ω_1 and Ω_2 respectively. For simplicity, we assume that $\Omega_1\Omega_2|_{\mathbb{A}'^\times}$ is trivial. Let Φ be a Schwartz function on $\mathbb{A}_n$. Define the mirabolic Eisenstein series by

$$E_{\mu,\chi}(s,g,\Phi) = \int_{k^\times\backslash\mathbb{A}^\times} \sum_{\gamma\in P_n(k)\backslash \mathrm{GL}_n(k)} \Phi(e_n\gamma zg)\mu(\det zg)\overline{\chi(z)}|\det zg|^s \mathrm{d}z, \quad g\in \mathrm{GL}_n(\mathbb{A}).$$

Recall that P_n is the mirabolic subgroup of GL_n consisting of matrices whose last row is $(0,\cdots,0,1)$. The integral is absolutely convergent for $\mathrm{Re}\, s \gg 0$ and has a meromorphic continuation to the whole complex plane. It is holomorphic at $s=\frac{1}{2}$ (c.f. [JS81]). For each holomorphic point s, $E_{\mu,\chi}(s,g,\Phi)$ is an automorphic form on $\mathrm{GL}_n(\mathbb{A})$ with central character χ.

Let $\varphi_1 \in \Pi_1$ and $\varphi_2\in\Pi_2$. Define the Rankin–Selberg integral by

$$Z(s,\varphi_1,\varphi_2,\mu,\Phi) = \int_{Z_n(\mathbb{A})\,\mathrm{GL}_n(k)\backslash \mathrm{GL}_n(\mathbb{A})} \varphi_1(g)\varphi_2(g)\overline{E_{\mu,\Omega_{\Pi_1}\Omega_{\Pi_2}}(s,g,\Phi)}\mathrm{d}g.$$

Then $Z(s,\varphi_1,\varphi_2,\mu,\Phi)$ is convergent and holomorphic in s whenever $E_{\mu,\Omega_{\Pi_1}\Omega_{\Pi_2}}(s,g,\Phi)$ is holomorphic. Define the global Rankin–Selberg period by

$$\lambda(\varphi_1,\varphi_2,\mu,\Phi) = Z\left(\frac{1}{2},\varphi_1,\varphi_2,\mu,\Phi\right). \tag{3.1.7}$$

Now suppose $W_{\varphi_1} = \otimes W_{1,v} \in \mathcal{W}(\Pi_1,\psi)$, $W_{\varphi_2} = \otimes W_{2,v}\in\mathcal{W}(\Pi_2,\overline{\psi})$ and $\Phi = \otimes\Phi_v \in \mathcal{S}(\mathbb{A}_n)$ are all factorizable. For any place v of k', we define the local Rankin–Selberg integral by

$$Z_v(s,W_{1,v},W_{2,v},\mu_v,\Phi_v) = \int_{N_n(k_v)\backslash \mathrm{GL}_n(k_v)} W_{1,v}(g)W_{2,v}(g)\overline{\mu_v(\det g)\Phi_v(e_ng)}|\det g|^s\mathrm{d}g.$$

This integral is convergent when $\Re s \gg 0$ and has a meromorphic continuation to the whole complex plane. The quotient

$$\frac{Z_v(s,W_{1,v},W_{2,v},\mu_v,\Phi_v)}{L(s,\Pi_{1,v}\times\Pi_{2,v}\otimes\mu_v^{-1})}$$

is holomorphic for all s. We define

$$\lambda_v^\natural(W_{1,v},W_{2,v},\mu_v,\Phi_v) = \left.\frac{Z_v(s,W_{1,v},W_{2,v},\mu_v,\Phi_v)}{L(s,\Pi_{1,v}\times\Pi_{2,v}\otimes\mu_v^{-1})}\right|_{s=\frac{1}{2}}.$$

If $Z_v(s,W_{1,v},W_{2,v},\mu_v,\Phi_v)$ is holomorphic at $s=\frac{1}{2}$ (e.g. $\Pi_{1,v}$ and $\Pi_{2,v}$ are both tempered, c.f. [JPSS83]), we define

$$\lambda_v(W_{1,v},W_{2,v},\mu_v,\Phi_v) = Z_v\left(\frac{1}{2},W_{1,v},W_{2,v},\mu_v,\Phi_v\right). \tag{3.1.8}$$

Then

$$\lambda_v^\natural(W_{1,v},W_{2,v},\mu_v,\Phi_v) = \frac{\lambda_v(W_{1,v},W_{2,v},\mu_v,\Phi_v)}{L(\frac{1}{2},\Pi_{1,v}\times\Pi_{2,v}\otimes\mu_v^{-1})}.$$

If Π_1, Π_2, ψ, μ are unramified at v, $W_{1,v}$ and $W_{2,v}$ are $\mathrm{GL}_n(\mathfrak{o}_v)$-fixed, $W_{1,v}(1)=W_{2,v}(1)=1$ and the measure satisfies $\mathrm{vol}\,\mathrm{GL}_n(\mathfrak{o}_v)=1$, then

$$\lambda_v^\natural(W_{1,v},W_{2,v},\mu_v,\Phi_v) = 1. \tag{3.1.9}$$

Proposition 3.1.3 ([JPSS83]). *We have*

$$\lambda(\varphi_1,\varphi_2,\mu,\Phi) = L\left(\frac{1}{2},\Pi_1\times\Pi_2\otimes\mu^{-1}\right)\prod_v \lambda_v^\natural(W_{1,v},W_{2,v},\mu_v,\Phi_v).$$

3.2. **Global distributions on the general linear groups and their decomposition.** Let Π_1 and Π_2 be two irreducible cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A})$ with unitary central characters Ω_{Π_1} and Ω_{Π_2} respectively. Assume that $\Omega_{\Pi_1}|_{\mathbb{A}'^\times}$ and $\Omega_{\Pi_2}|_{\mathbb{A}'^\times}$ are trivial. Let $F_1, F_2 \in \mathcal{C}_c^\infty(\mathrm{GL}_n(\mathbb{A}))$ and $\Phi \in \mathcal{S}(\mathbb{A}_n)$. Define

$$I_{\Pi_1,\Pi_2}(F_1,F_2,\Phi) = \sum_{\varphi_1,\varphi_2} \lambda(\Pi_1(F_1)\varphi_1, \Pi_2(F_2)\varphi_2, \mu, \Phi)\overline{\beta_n(\varphi_1)\beta_n(\varphi_2)}, \tag{3.2.1}$$

where the φ_1 (resp. φ_2) runs through an orthonormal basis of Π_1 (resp. Π_2).

Let v be a place of k'. Let $F_{1,v}, F_{2,v} \in \mathcal{C}_c^\infty(\mathrm{GL}_n(k_v))$ and $\Phi_v \in \mathcal{S}(k_{v,n})$. Then for any $s \in \mathbb{C}$ we define

$$I_{\Pi_{1,v},\Pi_{2,v},s}(F_{1,v},F_{2,v},\Phi_v) = \sum_{W_1,W_2} Z_v(s, \Pi_{1,v}(F_{1,v})W_1, \Pi_{2,v}(F_{2,v})W_2, \mu_v, \Phi_v)\overline{\beta_{n,v}(W_1)\beta_{n,v}(W_2)}, \tag{3.2.2}$$

where the sum W_1 (resp. W_2) runs through an orthonormal basis of $\mathcal{W}(\Pi_{1,v},\psi_v)$ (resp. $\mathcal{W}(\Pi_{2,v},\overline{\psi_v})$). We write $I_{\Pi_{1,v},\Pi_{2,v}} = I_{\Pi_{1,v},\Pi_{2,v},\frac{1}{2}}$ if $I_{\Pi_{1,v},\Pi_{2,v},s}$ is holomorphic at $s=\frac{1}{2}$. We also define its normalized version

$$I^\natural_{\Pi_{1,v},\Pi_{2,v}}(F_{1,v},F_{2,v},\Phi_v) = \sum_{W_1,W_2} \lambda^\natural_v(\Pi_{1,v}(F_{1,v})W_1, \Pi_{2,v}(F_{2,v})W_{2,v}, \mu_v, \Phi_v)\overline{\beta^\natural_{n,v}(W_1)\beta^\natural_{n,v}(W_2)}. \tag{3.2.3}$$

where the sum W_1 (resp. W_2) runs through an orthonormal basis of $\mathcal{W}(\Pi_{1,v},\psi_v)$ (resp. $\mathcal{W}(\Pi_{2,v},\overline{\psi_v})$).

Proposition 3.2.1. *Assume that Π_1 (resp. Π_2) is the base change of an irreducible cuspidal automorphic representation π_1 (resp. π_2) of $\mathrm{U}(V_1)(\mathbb{A}')$ (resp. $\mathrm{U}(V_2)(\mathbb{A}')$) for some skew-hermitian space V_1 (resp. V_2). Suppose $F_1 = \otimes F_{1,v}, F_2 = \otimes F_{2,v} \in \mathcal{C}_c^\infty(\mathrm{GL}_n(\mathbb{A}))$ and $\Phi = \otimes \Phi_v \in \mathcal{S}(\mathbb{A}_n)$ are all decomposable. Then*

$$I_{\Pi_1,\Pi_2}(F_1,F_2,\Phi) = L(1,\eta)^2 \frac{L\left(\frac{1}{2}, \Pi_1 \times \Pi_2 \otimes \mu^{-1}\right)}{L(1,\pi_1,\mathrm{Ad})L(1,\pi_2,\mathrm{Ad})} \prod_v I^\natural_{\Pi_{1,v},\Pi_{2,v}}(F_{1,v},F_{2,v},\Phi_v). \tag{3.2.4}$$

Proof. By definition

$$L(s,\Pi_i \times \Pi_i^\tau) = L(s,\Pi_i,\mathrm{As}^+)L(s,\Pi_i,\mathrm{As}^-).$$

Since Π_i $(i=1,2)$ is the base change from a unitary group, we must have that Π_i $(i=1,2)$ is conjugate self-dual, i.e. $\Pi_i^\tau \simeq \widetilde{\Pi_i}$. The L-function $L(s,\Pi_i \times \widetilde{\Pi_i})$ has a simple pole at $s=1$. By Hypothesis 1.2.1, $L(s,\Pi_i,\mathrm{As}^{(-1)^{n+1}})$ has a simple pole at $s=1$. It follows that $L(s,\Pi_i,\mathrm{As}^{(-1)^n})$ is holomorphic and nonzero at $s=1$. We also note that $L(s,\Pi_i,\mathrm{As}^{(-1)^n}) = L(s,\pi_i,\mathrm{Ad})$.

The Proposition then follows from Propositions 3.1.1, 3.1.2 and 3.1.3. We only have to note that

$$\frac{\mathrm{vol}(k^\times\backslash\mathbb{A}^{1,\times})}{\mathrm{vol}(k'^\times\backslash\mathbb{A}'^{1,\times})} = L(1,\eta).$$

∎

4. The relative trace identity

4.1. **Orbits.** Let $M = \mathrm{Mat}_n \times k_n \times k^n$ be an algebraic variety over k. The group $\mathrm{GL}_{n,k}$ acts on M by

$$[\gamma,x,y].g = [g^{-1}\xi g, xg, g^{-1}y].$$

An element $[\gamma,x,y] \in \mathrm{Mat}_n(k) \times k_n \times k^n$ is called regular semisimple if the orbit of $[\gamma,x,y]$ is closed in M and its stablizer in $\mathrm{GL}_n(k)$ is trivial. We shall call the $2n$-uple

$$\mathrm{Tr} \wedge^i \gamma, \quad x\gamma^j y, \quad i=1,\cdots,n;\ j=0,\cdots,n-1$$

the invariants of $[\gamma,x,y]$.

Let S_n be the algebraic variety over k' whose set of R-points is given by

$$S_n(R) = \{x \in \mathrm{GL}_n(R \otimes_{k'} k) \mid xx^\tau = 1\},$$

where R is an k'-algebra and $\mathrm{Gal}(k/k')$ acts on $R \otimes_{k'} k$ by acting on the second factor. Note that the map

$$\mathrm{GL}_n(k) \to S_n(k'), \quad a \mapsto aa^{\tau,-1}$$

is surjective by the Hilbert's Theorem 90. Let $X_n = S_n \times k'_n \times k^{-,n}$ and the group GL_n acts on X_n from the right by

$$[\gamma, v^\vee, v].g = [g^{-1}\gamma g, v^\vee g, g^{-1}v].$$

The space $X_n(k')$ embeds in $M(k)$ in a natural way. An element in $X_n(k')$ is called regular semisimple if it is regular semisimple as an element in $M(k)$. The invariants of an element in $X_n(k')$ are the invariants of it as an element in $M(k)$. A $\mathrm{GL}_n(k')$-orbit in $X_n(k')$ is called regular semisimple if all elements in the orbit are regular semisimple. We denote by $(X_n(k')//\mathrm{GL}_n(k'))_{\mathrm{rss}}$ the set of regular semisimple orbits.

Let V be a skew-hermitian space over k. Let $Y(V) = \mathrm{U}(V) \times V^\vee$ be an algebraic variety over k'. The group $\mathrm{U}(V)$ acts on $Y(V)$ from the right by

$$[\xi, w^\vee].g = [g^{-1}\xi g, w^\vee g].$$

Choosing a basis of V and the dual basis of $V^\vee$, we then identify V (resp. $V^\vee$) as column (resp. row) vectors. The skew-hermitian form on V is represented by a skew-hermitian matrix β. The space $Y(V)(k')$ embeds in $M(k)$ via

$$[\xi, w^\vee] \mapsto [\xi, w^\vee, \beta^{-1}\, {}^t w^{\vee\tau}].$$

An element in $Y(V)(k')$ is regular semisimple if it is regular semisimple as an element in $M(k)$. The invariants of an element in $Y(V)(k')$ are the invariants of it as an element in $M(k)$. A $\mathrm{U}(V)(k')$-orbit in $Y(V)(k')$ is called regular semisimple if all elements in the orbit are regular semisimple. We denote by $(Y(V)(k')//\mathrm{U}(V)(k'))_{\mathrm{rss}}$ the set of regular semisimple orbits.

We say that two regular semisimple orbits $[\gamma, v^\vee, v] \in (X_n(k')//\mathrm{GL}_n(k'))_{\mathrm{rss}}$ and $[\xi, w^\vee] \in (Y(V)(k')//\mathrm{U}(V)(k'))_{\mathrm{rss}}$ match if they have the same invariants.

Lemma 4.1.1 ([Liu14]). *The matching of orbits defines a natural bijection*

$$(X_n(k')//\mathrm{GL}_n(k'))_{\mathrm{rss}} \simeq \coprod_{V \in \mathsf{Herm}_n(k)} (Y(V)(k')//\mathrm{U}(V)(k'))_{\mathrm{rss}}.$$

We also need the Lie algebra version of Lemma 4.1.1. Let $\mathfrak{s}_n$ be the algebraic variety over k' whose R-points is given by

$$\mathfrak{s}_n(R) = \{x \in \mathfrak{gl}_n(k \otimes_{k'} R) \mid x + x^\tau = 0\},$$

where R is an k'-algebra. Let $\mathfrak{x}_n = \mathfrak{s}_n \times k'_n \times k^{-,n}$ and $\mathfrak{y}(V) = \mathfrak{u}(V) \times V^\vee$ be algebraic varieties over k'. Then, analogous to the case of groups, $\mathrm{GL}_{n,k'}$ (resp. $\mathrm{U}(V)$) acts on $\mathfrak{x}_n$ (resp. $\mathfrak{y}(V)$). The space $\mathfrak{x}_n(k')$ naturally embeds in $M(k)$ and the space $\mathfrak{y}(V)(k')$ embeds in $M(k)$ via

$$[\xi, w^\vee] \mapsto [\xi, w^\vee, \beta^{-1}\, {}^t w^{\vee\tau}].$$

Then analogous to the case of groups, we define the invariants and the notion of regular semisimple orbits. Denote by $(\mathfrak{x}_n(k')//\mathrm{GL}_n(k'))_{\mathrm{rss}}$ (resp. $(\mathfrak{y}(V)(k')//\mathrm{U}(V)(k'))_{\mathrm{rss}}$) the set of regular semisimple orbits.

We say that two orbits $[\gamma, v^\vee, v] \in (\mathfrak{x}_n(k')//\mathrm{GL}_n(k'))_{\mathrm{rss}}$ and $[\xi, w^\vee] \in (\mathfrak{y}(V)(k')//\mathrm{U}(V)(k'))_{\mathrm{rss}}$ match if they have the same invariants.

Lemma 4.1.2 ([Zha14a, Lemma 3.1]). *The matching of orbits defines a natural bijection*

$$(\mathfrak{x}_n(k')//\mathrm{GL}_n(k'))_{\mathrm{rss}} \simeq \coprod_{V \in \mathsf{Herm}_n(k)} (\mathfrak{y}(V)(k')//\mathrm{U}(V)(k'))_{\mathrm{rss}}.$$

4.2. **Orbital integrals.** Let v be a place of k'. We consider only the objects over k'_v in this subsection. To simplify notation, we suppress all the subscripts v from the notation. So $k' = k'_v$ is a local field and $k = k_v$ is a quadratic etale algebra over k'.

Let $F_1, F_2 \in \mathcal{C}_c^\infty(\mathrm{GL}_n(k))$ and $\Phi \in \mathcal{S}(k_n)$. We define a partial Fourier transform $-^\dagger : \mathcal{S}(k_n) \to \mathcal{S}(k'_n \times k^{-,n})$ by

$$\Phi^\dagger(x, y) = \int_{k_n^-} \Phi(x + x^-)\psi(x^- y)\mathrm{d}x^-, \quad \Phi \in \mathcal{S}(k_n),\ x \in k'_n,\ y \in k^{-,n}.$$

We usually write F for the triple (F_1, F_2, Φ) and call F a test function on the general linear group. Define the integral transform

$$\mathcal{C}_c^\infty(\mathrm{GL}_n(k)) \otimes \mathcal{C}_c^\infty(\mathrm{GL}_n(k)) \otimes \mathcal{S}(k_n) \to \mathcal{S}(X_n(k')), \quad F \mapsto F_\natural,$$

where

$$F_\natural(\gamma, v^\vee, v) = \begin{cases} \int\limits_{\mathrm{GL}_n(k')} \int\limits_{\mathrm{GL}_n(k)} F_1(g^{-1})F_2(g^{-1}ah)\overline{(\omega_{\psi',\mu}(g)\Phi)^\dagger(v^\vee, v)}\mathrm{d}g\mathrm{d}h, & n \text{ is odd,} \\ \int\limits_{\mathrm{GL}_n(k')} \int\limits_{\mathrm{GL}_n(k)} F_1(g^{-1})F_2(g^{-1}ah)\mu(\det ah)\overline{(\omega_{\psi',\mu}(g)\Phi)^\dagger(v^\vee, v)}\mathrm{d}g\mathrm{d}h, & n \text{ is even,} \end{cases} \tag{4.2.1}$$

and $\gamma = aa^{\tau,-1} \in S_n(k')$, $a \in \mathrm{GL}_n(k)$. Recall that $\omega_{\psi',\mu}$ is the Weil representation of $\mathrm{GL}_n(k)$ given by $\omega_{\psi',\mu}(g)\Phi(x) = \mu(\det g)|\det g|^{\frac{1}{2}}\Phi(xg)$. The definition of $F_\natural$ does not depend on the choice of a.

Let $[\gamma, x, y] \in X_n(k')$ be a regular semisimple element. We define the orbital integral of $F = (F_1, F_2, \Phi)$ as

$$O([\gamma, x, y], F) = \int\limits_{\mathrm{GL}_n(k')} F_\natural([\gamma, x, y].h)\eta(\det h)\mathrm{d}h. \tag{4.2.2}$$

The orbital integral is absolutely convergent [Xue14, Lemma 3.1.1].

We defines a transfer factor as follows. Let $[\gamma, x, y] \in X_n(k')$ be a regular semisimple element. Define

$$\mathbf{T}_{[\gamma,x,y]} = \det \begin{pmatrix} x \\ x\gamma \\ \vdots \\ x\gamma^{n-1} \end{pmatrix},$$

and define the transfer factor $\mathbf{t}([\gamma, x, y]) = \mu\left(\mathbf{T}_{[\gamma,x,y]}\right)$.

It follows that $\mathbf{t}([\gamma, x, y])O([\gamma, x, y], F)$ depends only on the orbit of $[\gamma, x, y]$.

Let V be a skew-hermitian space over k. Let $\operatorname{Res} V^\vee = \mathbf{L} + \mathbf{L}^\vee$ where $\mathbf{L}$ and $\mathbf{L}^\vee$ are Lagrangian subspaces of $\operatorname{Res} V^\vee$. We define a partial Fourier transform $-^\ddagger : \mathcal{S}(\mathbf{L}(k')) \otimes \mathcal{S}(\mathbf{L}(k')) \to \mathcal{S}(V^\vee)$ by

$$(\phi_1 \otimes \phi_2)^\ddagger(w^\vee) = \int\limits_{\mathbf{L}(k')} \phi_1(x+z)\phi_2(x-z)\psi(\langle z, y\rangle)\mathrm{d}z,$$

where $w^\vee = x + y$, $x \in \mathbf{L}$, $y \in \mathbf{L}^\vee$ and $\langle -, -\rangle$ is the pairing on $\operatorname{Res} V^\vee$.

Let $f_1, f_2 \in \mathcal{C}_c^\infty(\mathrm{U}(V)(k'))$ and $\phi_1, \phi_2 \in \mathcal{S}(\mathbf{L}(k'))$. We usually write f for the quadruple $(f_1, f_2, \phi_1, \phi_2)$ and call f a test function on the unitary group $\mathrm{U}(V)$. If we need to specify the skew-hermitian space V, we write f^V for f. Define an integral transform

$$\mathcal{C}_c^\infty(\mathrm{U}(V)(k')) \otimes \mathcal{C}_c^\infty(\mathrm{U}(V)(k')) \otimes \mathcal{S}(\mathbf{L}(k')) \otimes \mathcal{S}(\mathbf{L}(k')) \to \mathcal{S}(Y(V)(k')), \quad f \mapsto f_\natural,$$

where

$$f_\natural([\xi, w^\vee]) = \int\limits_{\mathrm{U}(V)(k')} f_1(g^{-1})f_2(g^{-1}\xi)(\overline{\omega_{\psi',\mu}(g)\phi_1} \otimes \phi_2)^\ddagger(w^\vee)\mathrm{d}g. \tag{4.2.3}$$

Let $[\xi, w^\vee] \in Y(V)(k')$ be a regular semisimple element. Define the orbital integral of $f = (f_1, f_2, \phi_1, \phi_2)$ by

$$O([\xi, w^\vee], f) = \int\limits_{\mathrm{U}(V)(k')} f_\natural([\xi, w^\vee].h)\mathrm{d}h. \tag{4.2.4}$$

This orbital integral is absolutely convergent [Xue14, Lemma 4.1.1].

Let F be a test function on the general linear group and $\{f^V : V \in \mathsf{Herm}_n(k)\}$ be a collection of test functions on $\mathrm{U}(V)$ for each (isometry class of) skew-hermitian space V. We say that F and $\{f^V : V \in \mathsf{Herm}_n(k)\}$ match, or they are smooth matching of each other, if for all matching regular semisimple orbits $[\gamma, x, y] \in X_n(k')$ and $[\xi, w^\vee] \in Y(V)(k')$, we have

$$\mathbf{t}([\gamma, x, y])O([\gamma, x, y], F) = O([\xi, w^\vee], f^V).$$

Proposition 4.2.1 ([Xue14, Proposition 5.2.1]). *Assume that k' is non-archimedean. Then for any test function F on the general linear group, there is a collection of test functions $\{f^V : V \in \mathsf{Herm}_n(k)\}$ that matches F. Conversely, given a collection of test functions $\{f^V : V \in \mathsf{Herm}_n(k)\}$ on the unitary group, there is a test function F on the general linear group that matches $\{f^V : V \in \mathsf{Herm}_n(k)\}$.*

Remark 4.2.2. To establish the relative trace formulae, we need slightly more than the existence of the smooth matching. More precisely, let $F_1 \in \mathcal{C}_c^\infty(\mathrm{GL}_n(k))$ and $z_1 \in Z_n(k)$. We define $F_1^{z_1} \in \mathcal{C}_c^\infty(\mathrm{GL}_n(k))$ by $F_1^{z_1}(g) = F_1(gz_1)$. Similarly we define $f_1^{z_1}$ for $f_1 \in \mathcal{C}_c^\infty(\mathrm{U}(V)(k'))$ and $z_1 \in Z_{\mathrm{U}(V)}(k')$. We need the following fact. If $F = (F_1, F_2, \Phi)$ and $\{f^V = (f_1^V, f_2^V, \phi_1^V, \phi_2^V) : V \in \mathsf{Herm}_n(k)\}$ match, then $(F_1^{z_1}, F_2^{z_2}, \Phi)$ and $\{(f_1^{V,\sigma(z_1)}, f_2^{V,\sigma(z_2)}, \phi_1^V, \phi_2^V) : V \in \mathsf{Herm}_n(k)\}$ also match, where $\sigma(z_i) = z_i z_i^{\tau,-1}, i = 1, 2$. We have used this fact to fix the central character of the representations that appear on the spectral side of the trace formulae, c.f. [Xue14, Proposition 5.5.2 and Proposition 5.5.5]. We are not able to establish this fact in general, but only under the condition that either $k = k' \times k'$ or we are in the situation of the fundamental lemma (see below), c.f. [Xue14, Lemma 5.5.1].

Assume now that $k = k' \times k'$. We identify $V^\vee$ with $k'_n \times k'_n$. The symplectic form on $\operatorname{Res} V^\vee$ is then given by $\langle (x_1, x_2), (y_1, y_2) \rangle = x_1 {}^t y_2 - x_2 {}^t y_1$. We choose the Lagrangian $\mathbf{L} = k'_n$ (resp. $\mathbf{L}^\vee = k'_n$) that embeds in $\operatorname{Res} V^\vee$ via the first (resp. second) component.

Proposition 4.2.3 ([Xue14, Proposition 5.3.1]). *Assume that $k = k' \times k'$. Then the test functions (F_1, F_2, Φ) and $(f_1, f_2, \phi_1, \phi_2)$ match, where*

$$f_i(g) = \int_{\mathrm{GL}_n(k')} F_i(gh, h) \mathrm{d}h, \quad i = 1, 2, \quad \Phi = \phi_1 \otimes \overline{\phi_2}. \tag{4.2.5}$$

We now recall the fundamental lemma. Suppose that k' is non-archimedean and k is an unramified field extension of k'. Assume that the conductor of ψ' (resp. μ) is $\mathfrak{o}'$ (resp. $\mathfrak{o}^\times$). The set $\mathsf{Herm}_n(k)$ consists of two elements and one of them has a self-dual $\mathfrak{o}$-lattice. We denote this skew-hermitian space by V^+ and the other one by V^-. The skew-hermitian space V^+ is defined over $\mathfrak{o}$. We take a Lagrangian $\mathbf{L} \subset \operatorname{Res} V^{+,\vee}$ that is defined over $\mathfrak{o}'$. The group $\mathrm{U}(V^+)$ has a smooth model over $\mathfrak{o}'$ defined by the self-dual lattice and $K^+ = \mathrm{U}(V^+)(\mathfrak{o}')$ is a hyperspecial maximal compact subgroup.

Theorem 4.2.4 ([Liu14]). *There is constant $c(n)$, depending on n only, such that if the residue characteristic of v is larger than $c(n)$, then in the situation described above, the test functions*

$$F = \left(\frac{1}{\operatorname{vol} \mathrm{GL}_n(\mathfrak{o})} \mathbf{1}_{\mathrm{GL}_n(\mathfrak{o})}, \frac{1}{\operatorname{vol} \mathrm{GL}_n(\mathfrak{o})} \mathbf{1}_{\mathrm{GL}_n(\mathfrak{o})}, \mathbf{1}_{\mathfrak{o}_n} \right),$$

and

$$\left\{ f^{V^+} = \left(\frac{1}{\operatorname{vol} K^+} \mathbf{1}_{K^+}, \frac{1}{\operatorname{vol} K^+} \mathbf{1}_{K^+}, \mathbf{1}_{\mathbf{L}(\mathfrak{o}')}, \mathbf{1}_{\mathbf{L}(\mathfrak{o}')} \right), \quad f^{V^-} = (0,0,0,0) \right\}$$

match.

For later use, we introduce the orbital integrals on the level of Lie algebra. Let $F \in \mathcal{C}_c^\infty(\mathfrak{x}_n(k'))$ and $[X, v^\vee, v] \in \mathfrak{x}_n(k')$ be a regular semisimple element. Define

$$O(X, F) = \int_{\mathrm{GL}_n(k')} F([X, v^\vee, v].h) \eta(\det h) \mathrm{d}h.$$

Let V be a skew-hermitian space over k. Let $f^V \in \mathcal{C}_c^\infty(\mathfrak{y}(V)(k'))$ and $[Y, w^\vee] \in \mathfrak{y}(V)(k')$ a regular semisimple element. Define

$$O([Y, w^\vee], f^V) = \int_{\mathrm{U}(V)(k')} f^V([Y, w^\vee].h) \mathrm{d}h.$$

We define the transfer factor on the level of Lie algebra as follows. Let $[X, v^\vee, v] \in \mathfrak{x}_n(k')$. Let

$$\mathbf{T}_{[X,v^\vee,v]} = \det \begin{pmatrix} v^\vee \\ v^\vee X \\ \vdots \\ v^\vee X^{n-1} \end{pmatrix}.$$

If $[X, v^\vee, v] \in \mathfrak{x}_n(k')$ and $\mathbf{T}_{[X,v^\vee,v]} \neq 0$, we define the transfer factor $\mathfrak{t}([X, v^\vee, v]) = \mu(\mathbf{T}_{[X,v^\vee,v]})$. We say that $F \in \mathcal{C}_c^\infty(\mathfrak{x}_n(k'))$ and a collection of test functions $\{f^V \in \mathcal{C}_c^\infty(\mathfrak{y}(V)(k')) : V \in \mathsf{Herm}_n(k)\}$ match, or they are smooth matching of each other if for all matching regular semisimple orbits $[X, v^\vee, v] \in \mathfrak{x}_n(k')$ and $[Y, w^\vee] \in \mathfrak{y}(V)(k')$, we have

$$\mathfrak{t}([X, v^\vee, v])O([X, v^\vee, v], F) = O([Y, w^\vee], f^V).$$

4.3. **A relative trace identity.** We now come back to the global situation. Thus k' is a number field and k is a quadratic field extension of k'.

Let $F_1 = \otimes F_{1,v}, F_2 = \otimes F_{2,v} \in \mathcal{C}_c^\infty(\mathrm{GL}_n(\mathbb{A}))$ and $\Phi = \otimes \Phi_v \in \mathcal{S}(\mathbb{A}_n)$. Note that the products $\otimes$ are over all the places of k'. We write F for the triple (F_1, F_2, Φ) and call it a test function on the general linear group. For each place v of k', we write F_v for $(F_{1,v}, F_{2,v}, \Phi_v)$.

Let V be a skew-hermitian space over k. Let $f_1 = \otimes f_{1,v}, f_2 = \otimes f_{2,v} \in \mathcal{C}_c^\infty(\mathrm{U}(V)(\mathbb{A}'))$ and $\phi_1 = \otimes \phi_{1,v}, \phi_2 = \otimes \phi_{2,v} \in \mathcal{S}(\mathbf{L}(\mathbb{A}'))$. We write f for the quadruple $(f_1, f_2, \phi_1, \phi_2)$ and call it a test function on the unitary group $\mathrm{U}(V)$. We write f^V for f if we need to specify the skew-hermitian space V. For each place v of k', we write f_v for $(f_{1,v}, f_{2,v}, \phi_{1,v}, \phi_{2,v})$.

Let F be a test function on the general linear group and $\{f^V : V \in \mathsf{Herm}_n(k)\}$ a collection of test functions on the unitary group for each (isometry class of) skew-hermitian space V. We say that F and $\{f^V : V \in \mathsf{Herm}_n(k)\}$ match if for all the places v of k', F_v and $\{f_v^V : V \in \mathsf{Herm}_n(k)\}$ match.

In this paper, we usually need to fix a skew-hermitian space. So we introduce the following terminology. For a fixed skew-hermitian space V, we say that the test functions F and f^V (local or global, on the groups or Lie algebras) match, if there is a test function f^W on $\mathrm{U}(W)$ for each $W \not\simeq V$, such that F and the collection $\{f^V, f^W : W \not\simeq V\}$ match.

For the rest of the subsection, we assume that all the archimedean places of k' split in k.

We say that a test function $F = (F_1, F_2, \Phi)$ on the general linear group is good if the following conditions hold.

(1) There is a non-archimedean split place v_1 of k', such that F_{1,v_1} and F_{2,v_1} are truncated matrix coefficients of supercuspidal representations. This means

$$F_{i,v_1}(g) = \widetilde{F_{i,v_1}}(g)\,\mathbf{1}_{\mathrm{GL}_n(k_{v_1})^*}(g), \quad i = 1, 2$$

where $\widetilde{F_{i,v_1}}$ is a matrix coefficient of a supercudpidal representation of $\mathrm{GL}_n(k_{v_1})$ and

$$\mathrm{GL}_n(k_{v_1})^* = \{g \in \mathrm{GL}_n(k_{v_1}) \mid \det g \in \mathfrak{o}_{v_1}^\times\}.$$

In particular, F_{1,v_1} and F_{2,v_1} are compactly supported.

(2) There is another non-archimedean split place $v_2 \neq v_1$, such that the function $F_{v_2,\natural}$ is supported on the regular semisimple locus of $X_n(k'_{v_2})$.

(3) For all archimedean places v, $k_v = k'_v \times k'_v$, the Schwartz function Φ is a finite linear combination of functions of the form $\Phi_1 \otimes \Phi_2$ where $\Phi_1, \Phi_2 \in \mathcal{S}(k'_{v,n})$. Moreover, F_v is K_v-finite where K_v is a maximal compact subgroup of $\mathrm{GL}_n(k_v) \times \mathrm{GL}_n(k_v)$.

We say that a test function $f = (f_1, f_2, \phi_1, \phi_2)$ on the unitary group $\mathrm{U}(V)$ is good if the following conditions hold.

(1) There is a non-archimedean split place v_1 of k' such that f_{1,v_1}, f_{2,v_1} are truncated matrix coefficients of supercuspidal representations of $\mathrm{U}(V)(k'_{v_1}) \simeq \mathrm{GL}_n(k'_{v_1})$.

(2) There is another split non-archimedean place $v_2 \neq v_1$, such that $f_{v_2,\natural}$ is supported on the regular semisimple locus of $Y(V)(k'_{v_2})$.

(3) For any archimedean place v, the function f_v is K_v-finite where K_v is a maximal compact subgroup of $\mathrm{U}(V)(k'_v) \times \mathrm{U}(V)(k'_v)$.

Proposition 4.3.1. *Suppose π_1 and π_2 are irreducible cuspidal automorphic representations of $\mathrm{U}(V)(\mathbb{A}')$ satisfying*

$$\mathrm{Hom}_{\mathrm{U}(V)(\mathbb{A}')}(\pi_1 \otimes \pi_2 \otimes \overline{\omega_{\psi',\mu}}, \mathbb{C}) \neq 0.$$

Assume that at a non-archimedean split place v of k', both $\pi_{1,v}$ and $\pi_{2,v}$ are supercuspidal. Assume that F and f^V are matching good test functions on the general linear group and the unitary group $\mathrm{U}(V)$ respectively.

Then

$$I_{\mathrm{BC}(\pi_1),\mathrm{BC}(\pi_2)}(F) = 4L(1,\eta)^2 J_{\pi_1,\pi_2}(f^V).$$

Proof. We apply [Xue14, Proposition 5.6.1]. Note that we assume in [Xue14, Proposition 5.6] that both $\pi_{1,v'}$ and $\pi_{2,v'}$ are supercuspidal at another non-archimedean split place v' of k'. This is not needed for the desired identity, but it is needed to show that there is a good test function f^V so that $J_{\pi_{1,v'},\pi_{2,v'}}(f^V_{v'}) \neq 0$. We will relax this condition in Appendix C, showing that the same nonvanishing result holds if $\pi_{1,v'}$ and $\pi_{2,v'}$ are tempered.

We also need to take into account of the different normalizations of the Petersson inner product. Implicitly, the Petersson inner product

$$\langle \varphi_1, \varphi_2 \rangle' = \int_{Z_{\mathrm{U}(V)}(\mathbb{A}')\,\mathrm{U}(V)(k')\backslash\,\mathrm{U}(V)(\mathbb{A}')} \varphi_1(g)\overline{\varphi_2(g)}\mathrm{d}g$$

is used in [Xue14], where $Z_{\mathrm{U}(V)}$ is the center of $\mathrm{U}(V)$. Note that $\mathrm{vol}\, Z_{\mathrm{U}(V)}(k')\backslash Z_{\mathrm{U}(V)}(\mathbb{A}') = 2L(1,\eta)$ by our choices of the measures. Taking into account of this modification, we find that

$$I_{\mathrm{BC}(\pi_1),\mathrm{BC}(\pi_2)}(F) = 4L(1,\eta)^2 \sum_{W\in\mathsf{Herm}_n(k)} \sum_{\sigma_1,\sigma_2} J_{\sigma_1,\sigma_2}(f^W),$$

where in the outer sum, W runs over all the (isometry classes of) skew-hermitian spaces of dimension n over k, and in the inner sum, σ_1 (resp. σ_2) runs over all the irreducible cuspidal automorphic representations that are nearly equivalent to π_1 (resp. π_2). Moreover, σ_{1,v_1} (resp. σ_{2,v_1}) is isomorphic to π_{1,v_1} (resp. π_{2,v_1}), thus it is supercuspidal.

By Hypothesis 1.2.1 (2), the σ_1's (resp. σ_2's) appearing on the right hand side are not isomorphic to each other. Moreover, for any place v, the representations $\sigma_{1,v}$'s (resp. $\sigma_{2,v}$'s) all lie in the same Vogan packet and this packet is generic. Therefore by Hypothesis 1.2.2, there is only one nonzero term on the right hand side of the equality, namely $J_{\pi_1,\pi_2}(f^V)$. We then conclude that

$$I_{\mathrm{BC}(\pi_1),\mathrm{BC}(\pi_2)}(F) = 4L(1,\eta)^2 J_{\pi_1,\pi_2}(f^V).$$

□

4.4. **A local distribution identity.** Recall that we have fixed a purely imaginary element $\delta \in k^-$. For any place v of k', we put

$$\kappa_v = |\delta|_{k_v}^{-d_n} \epsilon\left(\frac{1}{2}, \eta_v, \psi'_v\right)^{-\frac{n(n+1)}{2}} \Omega_{\Pi_{1,v}}(\delta)^{-1}\Omega_{\Pi_{2,v}}(\delta)^{-1}\mu_v(\mathrm{disc}\, V)^{-1}\eta_v(2)^{\frac{n(n-1)}{2}},$$

where $d_n = \binom{n}{3}$ and $\mathrm{disc}\, V \in k^-$ is the discriminant of V.

Note that for any $a \in k_v'^{\times}$, the determinant of the conjugation action of $\underline{a} = \mathrm{diag}[a^{n-1}, \cdots, a]$ on $N_{n-1}(k'_v)$ is a^{d_n}. It follows that if we choose a measure $\mathrm{d}h'$ on $N_{n-1}(k'_v)\backslash \mathrm{GL}_{n-1}(k'_v)$ such that $\mathrm{d}h = \mathrm{d}n\mathrm{d}h'$ where $\mathrm{d}h$ (resp. $\mathrm{d}n$) is the Haar measure on $\mathrm{GL}_{n-1}(k'_v)$ (resp. $N_{n-1}(k'_v)$), then $\mathrm{d}(\underline{a}h') = |a|_{k'_v}^{-d_n}\mathrm{d}h'$.

Theorem 4.4.1. *Let v be a place of k'. Let $\pi_{1,v}$ and $\pi_{2,v}$ be irreducible tempered representations of $\mathrm{U}(V)(k'_v)$ such that $\mathrm{BC}(\pi_{1,v})$ and $\mathrm{BC}(\pi_{2,v})$ are generic and unitary. Assume that one of the following conditions holds.*

1. *The place v is non-archimedean and is unramified in k. The residue characteristic of v is larger than $\max(2, c(n))$, the representations $\pi_{1,v}$ and $\pi_{2,v}$ are unramified, the character μ_v is unramified.*
2. *The place v is split.*
3. *The place v is non-archimedean and $\pi_{1,v}$, $\pi_{2,v}$ are supercuspidal.*

Then there are matching test functions F_v and f_v on the general linear group and the unitary group $\mathrm{U}(V)$ respectively, such that $I_{\Pi_{1,v},\Pi_{2,v},s}(F_v)$ is holomorphic at $s = \frac{1}{2}$ and

$$J_{\pi_{1,v},\pi_{2,v}}(f_v) = L(1,\eta_v)\kappa_v I_{\mathrm{BC}(\pi_{1,v}),\mathrm{BC}(\pi_{2,v})}(F_v) \neq 0. \tag{4.4.1}$$

Proof of Theorem 1.2.3 assuming Theorem 4.4.1. By Lemma 1.3.5, we only have to prove identity (1.3.4) under the hypotheses of Theorem 1.2.3.

By Proposition 4.3.1 and Proposition 3.2.1, for any matcing good (global) test functions $F = \otimes F_v$ and $f = \otimes f_v$, we have

$$J_{\pi_1,\pi_2}(f) = \frac{1}{4}\frac{L\left(\frac{1}{2}, \mathrm{BC}(\pi_1)\times \mathrm{BC}(\pi_2)\otimes\mu^{-1}\right)}{L(1,\pi_1,\mathrm{Ad})L(1,\pi_2,\mathrm{Ad})}\prod_v I^{\natural}_{\mathrm{BC}(\pi_{1,v}),\mathrm{BC}(\pi_{2,v})}(F_v).$$

We choose the test functions as follows.

(1) At the inert unramified places v such that the condition (1) of Theorem 4.4.1 is satisfied, we choose F_v and f_v as in the fundamental lemma, c.f. Theorem 4.2.4.
(2) At almost all split places v, we choose the unit element in the spherical Hecke algebra.
(3) At each $v \in \Sigma$, we choose F_v and f_v as in Theorem 4.4.1.
(4) At finitely many other split places v, including v_0, we choose matching F_v and f_v so that F and f are good test functions and $J_{\pi_{1,v},\pi_{2,v}}(f_v) \neq 0$.

We prove in Appendix C that for each non-archimedean split place v, there is a test function f_v such that $J_{\pi_{1,v},\pi_{2,v}}(f_v) \neq 0$ and $f_{v,\natural}$ is supported in the regular semisimple locus of $Y(V)(k'_v)$. Thus the choice (4) above is possible. We have proved this fact in [Xue14, Proposition 6.1.2] assuming that $\pi_{1,v}$ and $\pi_{2,v}$ are both supercuspidal.

Theorem 4.4.1 is equivalent to the following identity on the normalized distributions

$$I^{\natural}_{\mathrm{BC}(\pi_{1,v}),\mathrm{BC}(\pi_{2,v})}(F_v) = \kappa_v^{-1}L(1,\eta_v)^{-1}\left(\prod_{i=1}^{n} L(i,\eta_v^i)\right)\times J^{\natural}_{\pi_{1,v},\pi_{2,v}}(f_v)\neq 0.$$

Since $\prod_v \kappa_v = 1$, with the above choice of the test functions F and f, we conclude that

$$J_{\pi_1,\pi_2}(f) = \frac{1}{4}L(1,\eta)^{-1}\Delta_{\mathrm{U}(V)}\frac{L\left(\frac{1}{2}, \mathrm{BC}(\pi_1)\times \mathrm{BC}(\pi_2)\otimes\mu^{-1}\right)}{L(1,\pi_1,\mathrm{Ad})L(1,\pi_2,\mathrm{Ad})}\prod_v J^{\natural}_{\pi_{1,v},\pi_{2,v}}(f_v)\neq 0.$$

We have used the Tamagawa measure in Theorem 1.2.3 which is $L(1,\eta)^{-1}\mathrm{d}g$ where $\mathrm{d}g$ is the measure in Theorem 4.4.1. Taking into account this modification, we have proved Theorem 1.2.3. □

4.5. **Convention for the rest of this paper.** The proof of Theorem 4.4.1 occupies the rest of this paper.

We will be working with a place v of k' at a time. So we suppress the subscripts v from all the notation. Then k' is a local field and k is a quadratic etale algebra of k'.

We also change all the measures to the unnormalized ones, i.e. the ones without the normalization factor $\zeta_k(1)$, $L(1,\eta)$, etc..

Lemma 4.5.1. *With this new choice of the measures, the identity* (4.4.1) *becomes*

$$J_{\pi_{1,v},\pi_{2,v}}(f_v) = \kappa_v I_{\mathrm{BC}(\pi_{1,v}),\mathrm{BC}(\pi_{2,v})}(F_v)\neq 0. \tag{4.5.1}$$

Proof. The proof is the same as [Zha14b, Lemma 4.7]. □

5. Local distributions on the general linear groups

5.1. **Preliminaries.** In this section, we compute $I_{\Pi_1,\Pi_2}(F)$ for some carefully chosen test functions F on the general linear group. This computation is an analogue of [Zha14b, Theorem 8.5]. The general line of the argument is the same as [Zha14b, Section 7, 8] while the detailed computations are technically different at various points.

In this section, we assume that k' is a non-archimedean local field and k is a quadratic field extension of k'. We fix a uniformizer ϖ of k'. To make the notation simple, we deviate slightly from our convention here to denote the uniformizer by ϖ instead of ϖ'. To unify notation, we often write k^+ for k' and $\mathfrak{o}^+$ for $\mathfrak{o}'$.

We shall often make use of the Fourier transform. Let $\phi \in \mathcal{C}_c^\infty(k_n)$. Its Fourier transform is a function $\widehat{\phi} \in \mathcal{C}_c^\infty(k^n)$ given by

$$\widehat{\phi}(y) = \int_{k_n}\phi(x)\psi(xy)\mathrm{d}x. \tag{5.1.1}$$

Note that by our choices of the measure, we have

$$\phi(-x) = \int_{k^n} \widehat{\phi}(y)\psi(xy)\mathrm{d}y.$$

Suppose that $\phi = \phi^+ \otimes \phi^-$ where $\phi^\pm \in \mathcal{C}_c^\infty(k_n^\pm)$. Then we define the Fourier transform $\widehat{\phi^\pm}$ using the same formula (5.1.1), except that we integrate over $k_n^\pm$ instead. Then $\widehat{\phi} = \widehat{\phi^+} \otimes \widehat{\phi^-}$.

We also need the Fourier transform on $M_n(k)$. The pairing $(X, Y) \mapsto \mathrm{Tr}\, XY$ on $M_n(k)$ is non-degenerate. Let $f \in \mathcal{C}_c^\infty(M_n(k))$. Define its Fourier transform by

$$\widehat{f}(Y) = \int_{M_n(k)} f(X)\psi(\mathrm{Tr}\, XY)\mathrm{d}X.$$

Recall that the measure $\mathrm{d}X$ is normalized so that $\widehat{\widehat{f}}(X) = f(-X)$.

We identify $\mathfrak{s}_n(k')$ with $M_n(k^-)$. The decomposition $M_n(k) = M_n(k') \oplus \mathfrak{s}_n(k')$ is orthogonal with respect to the pairing $(X, Y) \mapsto \mathrm{Tr}_{k/k'}(\mathrm{Tr}\, XY)$. Thus if $f = f^+ \otimes f^-$ and $f^\pm \in \mathcal{C}_c^\infty(M_n(k^\pm))$, then we have the Fourier transform $\widehat{f^\pm}$.

We recall the definition of convolutions. Let G be a locally compact topological group and P a unimodular closed subgroup of G. Let f be a function on G and ϕ a compactly supported function on P. Choose a Haar measure $\mathrm{d}p$ on P. Then we define the convolution of f and ϕ as

$$f * \phi(g) = \int_P f(gp^{-1})\phi(p)\mathrm{d}p, \quad \phi * f(g) = \int_P \phi(p) f(p^{-1}g)\mathrm{d}p.$$

We define two involutions on $\mathcal{C}^\infty(G)$ as follows. Let $f \in \mathcal{C}^\infty(G)$. We put $f^*(g) = \overline{f(g^{-1})}$ and $f^\star(g) = f(g^{-1})$. Then

$$(f * \phi)^* = \phi^* * f^*, \quad (f * \phi)^\star = \phi^\star * f^\star.$$

If P_1 and P_2 are two closed unimodular subgroups and $f \in \mathcal{C}_c^\infty(G)$ and $\phi_i \in \mathcal{C}_c^\infty(P_i)$, $(i = 1, 2)$, then we can define $f * \phi_1 * \phi_2$, $\phi_1 * f * \phi_2$, etc.

Suppose that π is a representation of G and $\phi \in \mathcal{C}_c^\infty(P)$. Then we define $\pi(\phi) \in \mathrm{End}(\pi)$ as

$$\pi(\phi) = \int_P \phi(p)\pi(p)\mathrm{d}p.$$

Moreover, if π is unitary, i.e. there is a G-invariant hermitian inner product $\langle -, - \rangle$ on π, then

$$\langle \pi(\phi)v, v^\vee \rangle = \langle v, \pi(\phi^*)v^\vee \rangle, \quad v, v^\vee \in \pi.$$

5.2. **The smoothing technique.** We define the admissible test functions following [Zha14b] and introduce the "smoothing technique" [Zha14b, Section 7]. Most of the definitions are the same as in [Zha14b], except that we need to "smoothen" the Rankin–Selberg period in a different way.

We introduce the following notation. In the rest of this paper, if the residue characteristic of k is not two, then there is a decomposition $\mathfrak{o} = \mathfrak{o}' \oplus \mathfrak{o}^-$, where $\mathfrak{o}^-$ consists of elements $x \in \mathfrak{o}$ such that $x^\tau = -x$. If the residue characteristic is two, we define $\mathfrak{o} = \mathfrak{o}' \oplus \mathfrak{o}^-$. Note that this might not be a maximal order of k.

Let $\mathbf{1}_\mathfrak{o}$ be the characteristic function of $\mathfrak{o}$, and its Fourier transform $\widehat{\mathbf{1}_\mathfrak{o}} = \mathbf{1}_{\mathfrak{o}^*}$ for some $\mathfrak{o}$-lattice $\mathfrak{o}^*$ in k. Let $\mathfrak{o}^{\pm *} = \mathfrak{o}^* \cap k^\pm$. We denote by $\mathcal{C}_c^\infty(\varpi^m \mathfrak{o}/\varpi^{2m}\mathfrak{o})$ the subspace of $\mathcal{C}_c^\infty(k)$ spanned by the functions that are $\varpi^{2m}\mathfrak{o}$-invariant and supported $\varpi^m\mathfrak{o}$.

We now make some definitions. Let m be a positive integer.

Definition 5.2.1. We define $\mathcal{C}_c^\infty(\varpi^m \mathfrak{o}/\varpi^{2m}\mathfrak{o})^\heartsuit$ to be the subspace of $\mathcal{C}_c^\infty(\varpi^m \mathfrak{o}/\varpi^{2m}\mathfrak{o})$ spanned by the functions of the form $\phi = \phi^+ \otimes \phi^-$, where $\phi^\pm \in \mathcal{C}_c^\infty(\varpi^m \mathfrak{o}^\pm)$ and

(1) ϕ^+ is a multiple of $\mathbf{1}_{\varpi^m \mathfrak{o}^+}$,
(2) $\widehat{\phi^-}$ is supported in $\varpi^{-2m}\mathfrak{o}^{-*} \backslash \varpi^{-2m+1}\mathfrak{o}^{-*}$.

Definition 5.2.2. We define $\mathcal{C}_c^\infty(k^{n-1})_m^\heartsuit$ to be the subspace of $\mathcal{C}_c^\infty(k^{n-1})$ spanned by the functions of the form $\phi = \otimes \phi_i$ where ϕ_i is a function on the i-th entry satisfying

(1) $\phi_i = \mathbf{1}_{\varpi^m \mathfrak{o}}$, $1 \le i \le n-2$,
(2) $\phi_{n-1} \in \mathcal{C}_c^\infty(\varpi^m \mathfrak{o}/\varpi^{2m}\mathfrak{o})^\heartsuit$.

Definition 5.2.3. We define $\mathcal{C}_c^\infty(\mathrm{GL}_{n-1}(k))_m^\heartsuit$ to be the subspace of $\mathcal{C}_c^\infty(\mathrm{GL}_{n-1}(k))$ spanned by the functions of the form $\varphi = (\varphi_{ij})$ where φ_{ij} is a function on the (i,j) entry satisfying

(1) $\varphi_{ij} = \mathbf{1}_{\varpi^m \mathfrak{o}}$ if $i \ne j$ or $j-1$,
(2) $\varphi_{ii} = \mathbf{1}_{1+\varpi^m \mathfrak{o}}$, $i = 1, \cdots, n-1$,
(3) $\varphi_{i,i+1} \in \mathcal{C}_c^\infty(\varpi^m \mathfrak{o}/\varpi^{2m}\mathfrak{o})^\heartsuit$.

Definition 5.2.4. A pair of functions (φ, ϕ) with $\varphi \in \mathcal{C}_c^\infty(\mathrm{GL}_{n-1}(k))$ and $\phi \in \mathcal{C}_c^\infty(k^{n-1})$ is said to be m-admissible if $\varphi \in \mathcal{C}_c^\infty(\mathrm{GL}_{n-1}(k))_m^\heartsuit$ and $\phi \in \mathcal{C}_c^\infty(k^{n-1})_m^\heartsuit$.

Definition 5.2.5. Let (φ, ϕ) be a pair of functions that is m-admissible for some $m > 0$. Put $\varphi_{n-1} = \varphi$ and $\phi_{n-1} = \phi$. We recursively define $\varphi_i \in \mathcal{C}_c^\infty(M_i(k))$, $\phi_i \in \mathcal{C}_c^\infty(k^i)$ and $\phi_i^\flat \in \mathcal{C}_c^\infty(k_i)$ as follows. Note that φ_i is viewed as a function on $M_i(k)$ even though it is supported in $\mathrm{GL}_i(k)$. Suppose for some i, the function φ_i is defined. For $X_i = (X_{st})_{1\le s,t\le i} \in M_i(k)$, write

$$X_{i-1} = (X_{st})_{1\le s,t\le i-1} \in M_{i-1}(k),\ u_{i-1} = {}^t(X_{1i}, \cdots, X_{i-1,i}) \in k^{i-1},\ v_i = (X_{i1}, \cdots, X_{i,i}) \in k_i.$$

Define $\varphi_{i-1} \in \mathcal{C}_c^\infty(M_{i-1}(k))$, $\phi_i^\flat \in \mathcal{C}_c^\infty(k_i)$ and $\phi_{i-1} \in \mathcal{C}_c^\infty(k^{i-1})$ so that

$$\varphi_i(X_i) = \varphi_{i-1}(X_{i-1})\phi_{i-1}(u_{i-1})\phi_i^\flat(v_i).$$

Finally we set $\phi_1^\flat = \varphi_1$. Define $\underline{\phi^\flat} = \otimes \phi_i^\flat$ as a function on $B_{n-1,-}(k)$.

Lemma 5.2.6 ([Zha14b, Proposition 7.5])**.** *Suppose (φ, ϕ) is m-admissible for some $m > 0$.*

(1) *The functions $\phi_i^\flat$ is the characteristic function of $e_i + \varpi^m \mathfrak{o}_i$ and $\phi_i \in \mathcal{C}_c^\infty(k^i)_m^\heartsuit$.*
(2) *The function φ is invariant under the right and left multiplication of elements in $1 + \varpi^m \mathfrak{b}_{n-1,-}(\mathfrak{o})$ and $1 + \varpi^m M_{n-1}(\mathfrak{o}')$.*
(3) *Decompose φ as $\varphi^+ \otimes \varphi^-$ where $\varphi^\pm \in \mathcal{C}_c^\infty(M_n(k^\pm))$. Then φ^+ is a multiple of $\mathbf{1}_{1+\varpi^m M_{n-1}(\mathfrak{o}')}$.*

For any functions $\varphi \in \mathcal{C}_c^\infty(\mathrm{GL}_{n-1}(k))$ and $\phi \in \mathcal{C}_c^\infty(k^{n-1})$, define a function $\widetilde{W_{\varphi,\phi}}$ on $\mathrm{GL}_{n-1}(k)$ by

$$\widetilde{W_{\varphi,\phi}}(g) = \widehat{\phi}(-e_{n-1}g) \int\limits_{N_{n-1}(k')\backslash \mathrm{GL}_{n-1}(k')} \int\limits_{N_{n-1}(k)} \varphi(g^{-1}u\epsilon_{n-1}h)\overline{\psi(u)}\eta^{n+1}(\det h)\mathrm{d}u\mathrm{d}h. \tag{5.2.1}$$

The integral is absolutely convergent. Recall that $\epsilon_{n-1} = \mathrm{diag}[\delta^{n-1}, \cdots, \delta] \in \mathrm{GL}_{n-1}(k)$. It is clear that

$$\widetilde{W_{\varphi,\phi}}(ug) = \overline{\psi}(u)\widetilde{W_{\varphi,\phi}}(g), \quad u \in N_{n-1}(k).$$

Lemma 5.2.7 ([Zha14b, Lemma 7.6])**.** *Assume that (φ, ϕ) is m-admissible for some $m > 0$. Define the functions $\varphi_i, \phi_i, \phi_i^\flat, \underline{\phi^\flat}$ as in Definition 5.2.5.*

(1) *The function $\widetilde{W_{\varphi,\phi}}$ is supported in $N_{n-1}(k)B_{n-1,-}(k)$ and $\widetilde{W_{\varphi,\phi}} \in \mathcal{C}_c^\infty(N_{n-1}(k)\backslash \mathrm{GL}_{n-1}(k), \overline{\psi})$.*
(2) *Suppose $g = yv \in \mathrm{GL}_{n-1}(k')$, where $y = \mathrm{diag}[y_1 \cdots y_{n-1}, y_1 \cdots y_{n-2}, \cdots, y_1]$, and*

$$v = \prod_{i=1}^{n-2} \begin{pmatrix} 1_i & \\ v_i & 1 \end{pmatrix} \in N_{n-1,-}(k'), \quad v_i \in k_i'.$$

Then

$$\widetilde{W_{\varphi,\phi}}(\epsilon_{n-1}g) = |\delta|_k^{d_n} \int\limits_{B_{n-1,-}(k')} \underline{\phi^\flat}(b)\mathrm{d}b \times \eta^{n+1}(\det y)|\delta_{n-1}(y)|_{k'} \prod_{i=1}^{n-1} \widehat{\phi_{n-i}}(-y_i(v_{n-i-1}, 1)\delta),$$

where $d_n = \binom{n}{3}$ and δ_{n-1} is the modulus character of B_{n-1}. The measure $\mathrm{d}b$ is either the right or the left invariant measure on $B_{n-1,-}(k')$. They give the same results since $\underline{\phi^\flat}$ is supported in $1 + \varpi \mathfrak{b}_{n-1}(\mathfrak{o})$.

Let Π_1 and Π_2 be two irreducible admissible generic unitary representations of $\mathrm{GL}_n(k)$ with central characters Ω_{Π_1} and Ω_{Π_2} respectively. Suppose (φ, ϕ) is m-admissible for some $m > 0$. By Lemma 5.2.7, there is a Whittaker function $W_{\varphi,\phi} \in \mathcal{W}(\Pi_2, \overline{\psi})$, such that $W|_{\mathrm{GL}_{n-1}(k)} = \widetilde{W_{\varphi,\phi}}$. The "smoothing technique" alluded in the title of this subsection refers to the following proposition.

Proposition 5.2.8 ([Zha14b, Proposition 7.7])**.** *Suppose (φ, ϕ) is m-admissible for some $m > 0$. Then for all $W \in \mathcal{W}(\Pi_2, \overline{\psi})$, we have*

$$\beta_n(\Pi_2(\varphi^*)\Pi_2(\phi^*)W) = \langle W, W_{\varphi,\phi}\rangle,$$

where $\langle -, - \rangle$ on the right hand side is the inner product on $\mathcal{W}(\Pi_2, \overline{\psi})$, c.f. (3.1.1).

We also need to "smoothen" the Rankin–Selberg period λ. Our situation is slightly different from [Zha14b, Proposition 7.9].

Definition 5.2.9. Let $m_2 > m_1$ be two integers and $\Phi \in \mathcal{C}_c^\infty(k_n)$. We say that Φ is (m_1, m_2)-admissible if $\Phi = \otimes \Phi_i$ where Φ_i is the function on the i-th entry satisfying

(1) $\Phi_i = \mathbf{1}_{\varpi^{m_2}\mathfrak{o}}$, $1 < i < n-1$,
(2) $\Phi_n = \Phi_n^+ \otimes \Phi_n^-$, where Φ_n^+ (resp. Φ_n^-) is a multiple of $\mathbf{1}_{\varpi^{m_2}\mathfrak{o}'}$ (resp. $\mathbf{1}_{\delta(1+\varpi^{m_1}\mathfrak{o}')}$).

We choose m_2 so that $\operatorname{supp}\Phi_n \subset \delta(1 + \varpi^{m_1}\mathfrak{o})$.

Proposition 5.2.10. *Let $W_1 \in \mathcal{W}(\Pi_1, \psi)$ and $W \in \mathcal{W}(\Pi_2, \overline{\psi})$. Suppose that $W|_{\mathrm{GL}_{n-1}}$ is compactly supported modulo $N_{n-1}(k)$. Let $W^\sharp \in \mathcal{W}(\overline{\Pi_1}, \overline{\psi})$ so that*

$$W^\sharp(\iota(a_{n-1})) = W(\iota(a_{n-1}))|\det a_{n-1}|^{-\frac{1}{2}}\overline{\mu}(\det a_{n-1}), \quad a_{n-1} \in \mathrm{GL}_{n-1}(k).$$

Let $m_1 < m_2$ be two integers such

(1) *$\delta^{-1}\varpi^{m_2}\mathfrak{o} \subset \varpi^{m_1}\mathfrak{o}$,*
(2) *W and $W^\sharp$ are invariant under the right multiplication by elements in $1 + \varpi^{m_1}M_n(\mathfrak{o})$,*
(3) *Ω_{Π_1}, Ω_{Π_2}, μ are trivial when restricted to $1 + \varpi^{m_1}\mathfrak{o}$.*

If $\Phi \in \mathcal{C}_c^\infty(k_n)$ is (m_1, m_2)-admissible, then $Z(s, W_1, W, \mu, \Phi)$ is holomorphic at $s = \frac{1}{2}$ and

$$\lambda(W_1, W, \mu, \Phi) = |\delta|^{-\frac{n}{2}}\Omega_{\Pi_1}(\delta)\Omega_{\Pi_2}(\delta)\overline{\mu}(\delta)^n \times \langle W_1, \overline{W^\sharp}\rangle \times \int_{k_n} \overline{\Phi(v_n)}\mathrm{d}v_n,$$

where $\langle -, - \rangle$ is the inner product on $\mathcal{W}(\Pi_1, \psi)$.

Proof. We first show that $Z(s, W_1, W, \mu, \Phi)$ is holomorphic at $s = \frac{1}{2}$. Recall that

$$Z(s, W_1, W, \mu, \Phi) = \int_{N_n(k)\backslash \mathrm{GL}_n(k)} W_1(g)W(g)\overline{\mu(\det g)\Phi(e_n g)}|\det g|^s \mathrm{d}g.$$

By the support condition on Φ, if $e_n g \in \operatorname{supp}\Phi$, then we can write

$$g = y_1 \begin{pmatrix} a_{n-1} & \\ & 1 \end{pmatrix} \begin{pmatrix} 1_{n-1} & \\ v_{n-1} & 1 \end{pmatrix},$$

where $y_1 \in Z_n(k) \simeq k^\times$, $a_{n-1} \in N_{n-1}(k)\backslash \mathrm{GL}_{n-1}(k)$ and $v_{n-1} \in k_{n-1}$. The measure $\mathrm{d}g$ decomposes as

$$\mathrm{d}g = \frac{1}{|\det a_{n-1}|}\mathrm{d}a_{n-1}\mathrm{d}v_{n-1}\mathrm{d}y_1.$$

By the admissibility condition, the function W is invariant under the right multiplication of $\begin{pmatrix} 1_{n-1} & \\ v_{n-1} & 1 \end{pmatrix}$ if $y_1(v_{n-1}, 1) \in \operatorname{supp}\Phi$. Since $W|_{\mathrm{GL}_{n-1}(k)} \in \mathcal{C}_c^\infty(N_{n-1}(k)\backslash \mathrm{GL}_{n-1}(k), \overline{\psi})$, we conclude that the integration of g is in fact over some compact region. Therefore it is convergent for any s. In particular, $Z(s, W_1, W, \mu, \Phi)$ is holomorphic at $s = \frac{1}{2}$.

We conclude that

$$\begin{aligned}&\lambda(W_1,W,\mu,\Phi)\\=&\int\limits_{k^\times}\int\limits_{k_{n-1}}\int\limits_{N_{n-1}(k)\backslash\mathrm{GL}_{n-1}(k)} W_1\left(\begin{pmatrix}a_{n-1}&\\&1\end{pmatrix}\begin{pmatrix}1_{n-1}&\\v_{n-1}&1\end{pmatrix}\right)W\left(\begin{pmatrix}a_{n-1}&\\&1\end{pmatrix}\begin{pmatrix}1_{n-1}&\\v_{n-1}&1\end{pmatrix}\right)\\&\overline{\mu(\det a_{n-1})\mu(y_1)^n\Phi(y_1(v_{n-1},1))}\Omega_{\Pi_1}(y_1)\Omega_{\Pi_2}(y_1)|y_1|^{\frac{n}{2}}|\det a_{n-1}|^{-\frac12}\mathrm{d}a_{n-1}\mathrm{d}v_{n-1}\mathrm{d}y_1.\end{aligned}$$

Since the functions W and $W^\sharp$ are invariant under the right multiplication of $\begin{pmatrix}1_{n-1}&\\v_{n-1}&1\end{pmatrix}$ if $y_1(v_{n-1},1)\in\operatorname{supp}\Phi$, we have

$$\begin{aligned}&\lambda(W,W_2,\mu,\Phi)\\=&\int\limits_{k^\times}\int\limits_{k_{n-1}}\int\limits_{N_{n-1}(k)\backslash\mathrm{GL}_{n-1}(k)} W_1\left(\begin{pmatrix}a_{n-1}&\\&1\end{pmatrix}\begin{pmatrix}1_{n-1}&\\v_{n-1}&1\end{pmatrix}\right)W^\sharp\left(\begin{pmatrix}a_{n-1}&\\&1\end{pmatrix}\begin{pmatrix}1_{n-1}&\\v_{n-1}&1\end{pmatrix}\right)\\&\overline{\Phi(y_1(v_{n-1},1))}\Omega_{\Pi_1}(y_1)\Omega_{\Pi_2}(y_1)\overline{\mu}(y_1)^n|y_1|^{\frac{n}{2}}\mathrm{d}a_{n-1}\mathrm{d}v_{n-1}\mathrm{d}y_1.\end{aligned}$$

Since the integration over $a_{n-1}\in N_{n-1}(k)\backslash\mathrm{GL}_{n-1}(k)$ gives the $\mathrm{GL}_n(k)$-invariant inner product on $\mathcal{W}(\Pi_1,\psi)$, the expression simplifies to

$$\lambda(W,W_2,\mu,\Phi)=\langle W_1,\overline{W^\sharp}\rangle\int\limits_{k^\times}\int\limits_{k_{n-1}}\overline{\Phi(y_1(v_{n-1},1))}\Omega_{\Pi_1}(y_1)\Omega_{\Pi_2}(y_1)\overline{\mu}(y_1)^n|y_1|^{\frac{n}{2}}\mathrm{d}v_{n-1}\mathrm{d}y_1,$$

where $\langle -,-\rangle$ is the inner product on $\mathcal{W}(\Pi_1,\psi)$. Finally we make a change of variables $v_{n-1}\mapsto y_1^{-1}v_{n-1}$ in the last integral and replace the multiplicative measure $\mathrm{d}y_1$ by the additive one. Note that $y_1\in\delta(1+\varpi^{m_1}\mathfrak{o})$. Since Ω_{Π_1}, Ω_{Π_2} and μ are trivial when restricted to $1+\varpi^{m_1}\mathfrak{o}$, the proposition follows. □

5.3. **A formula for I_{Π_1,Π_2}.** Let Π_1 and Π_2 be irreducible admissible generic representations of $\mathrm{GL}_n(k)$ with central characters Ω_{Π_1} and Ω_{Π_2} respectively. Let $\varphi\in\mathcal{C}_c^\infty(\mathrm{GL}_{n-1}(k))$ and $\phi\in\mathcal{C}_c^\infty(k^{n-1})$. Define the function $\widetilde{W_{\varphi,\phi}}$ as (5.2.1). If $\widetilde{W_{\varphi,\phi}}\in\mathcal{C}_c^\infty(N_{n-1}(k)\backslash\mathrm{GL}_{n-1}(k),\overline{\psi})$, then we define $W_{\varphi,\phi}\in\mathcal{W}(\Pi_2,\overline{\psi})$ and $W^\sharp_{\varphi,\phi}\in\mathcal{W}(\overline{\Pi_1},\overline{\psi})$ such that

$$\begin{aligned}W_{\varphi,\phi}(\iota(a_{n-1}))&=\widetilde{W_{\varphi,\phi}}(a_{n-1}),\\W^\sharp_{\varphi,\phi}(\iota(a_{n-1}))&=\widetilde{W_{\varphi,\phi}}(a_{n-1})|\det a_{n-1}|_k^{-\frac12}\overline{\mu}(\det a_{n-1}).\end{aligned}\tag{5.3.1}$$

Definition 5.3.1. Let $r>m_2>m_1>m>0$ be four integers. Let (F_0,φ,ϕ,Φ) be a quadruple of functions with $F_0\in\mathcal{C}_c^\infty(\mathrm{GL}_n(k))$, $\varphi\in\mathcal{C}_c^\infty(\mathrm{GL}_{n-1}(k))$, $\phi\in\mathcal{C}_c^\infty(k^{n-1})$ and $\Phi\in\mathcal{C}_c^\infty(k_n)$.

We say that (F_0,φ,ϕ,Φ) is (m,m_1,m_2,r)-admissible if the following conditions are satisfied.

(1) The pair (φ,ϕ) is m-admissible.
(2) The function Φ is (m_1,m_2)-admissible.
(3) The function F_0 is a multiple of $\mathbf{1}_{1+\varpi^rM_n(\mathfrak{o})}$ so that $\int_{\mathrm{GL}_n(k)}F_0(g)\mathrm{d}g=1$.

We require that r, m_1 and m_2 are sufficiently large so that the conditions in Proposition 5.2.10 are satisfied, and that the functions $W_{\varphi,\phi}$, $W^\sharp_{\varphi,\phi}$ and Φ are invariant under the right multiplication of elements in $1+\varpi^rM_n(\mathfrak{o})$.

We say that (F_0,φ,ϕ,Φ) is sufficiently admissible if it is (m,m_1,m_2,r)-admissible for sufficiently large m,m_1,m_2 and r.

To shorten notation, we write $F_0^{\varphi,\phi}=F_0*\varphi*\phi$. We say that the test function F on the general linear group is (m,m_1,m_2,r)-admissible if it is of the form $F=(F_0,F_0^{\varphi,\phi},\Phi)$ and the quadruple (F_0,φ,ϕ,Φ) is admissible.

Lemma 5.3.2. *If the test function F is (m,m_1,m_2,r)-admissible. Then $I_{\Pi_1,\Pi_2,s}$ is holomorphic at $s=\frac12$ and we have*

$$I_{\Pi_1,\Pi_2}(F)=|\delta|^{-\frac{n}{2}}(\Omega_{\Pi_1}\Omega_{\Pi_2}\overline{\mu}^n)(\delta)\times\int\limits_{k_n}\overline{\Phi(v_n)}\mathrm{d}v_n\times\beta_n(W^\sharp_{\varphi,\phi}).$$

Proof. The proof of this lemma is identical to [Zha14b, Proposition 8.2]. To avoid repetition, we only sketch the argument here. We have

$$\begin{aligned}
I_{\Pi_1,\Pi_2,s}(F) &= \sum_{W_1,W_2} Z(s,\Pi_1(F_0)W_1,\Pi_2(F_0^{\varphi,\phi})W_2,\mu,\Phi)\overline{\beta_n(W_1)\beta_n(W_2)} \\
&= \sum_{W_1,W_2} Z(s,\Pi_1(F_0)W_1,W_2,\mu,\Phi)\overline{\beta_n(W_1)\beta_n(\Pi_2(\phi^*)\Pi_2(\varphi^*)\Pi_2(F_0^*)W_2)} \\
&= \sum_{W_1,W_2} Z(s,\Pi_1(F_0)W_1,W_2,\mu,\Phi)\overline{\beta_n(W_1)\langle W_2,W_{\varphi,\phi}\rangle} \qquad \text{Proposition 5.2.8} \\
&= \sum_{W_1} Z(s,\Pi_1(F_0)W_1,W_{\varphi,\phi},\mu,\Phi)\overline{\beta_n(W_1)}
\end{aligned}$$

It follows from Proposition 5.2.10 that $Z(s,\Pi_1(F_0)W_1,W_{\varphi,\phi},\mu,\Phi)$ is holomorphic at $s=\frac{1}{2}$, so is $I_{\Pi_1,\Pi_2,s}(F)$. Then by Proposition 5.2.10, we have

$$\begin{aligned}
I_{\Pi_1,\Pi_2}(F) &= \sum_{W_1}|\delta|^{-\frac{n}{2}}(\Omega_{\Pi_1}\Omega_{\Pi_2}\overline{\mu}^n)(\delta)\langle W_1,\overline{W^\sharp_{\varphi,\phi}}\rangle \times \int_{k_n}\overline{\Phi(v_n)}\mathrm{d}v_n \times \overline{\beta_n(W_1)} \\
&= |\delta|^{-\frac{n}{2}}(\Omega_{\Pi_1}\Omega_{\Pi_2}\overline{\mu}^n)(\delta) \times \int_{k_n}\overline{\Phi(v_n)}\mathrm{d}v_n \times \beta_n(W^\sharp_{\varphi,\phi}).
\end{aligned}$$

□

Lemma 5.3.3. *Suppose the test function $F=(F_0,F_0^{\varphi,\phi},\Phi)$ is (m,m_1,m_2,r)-admissible. Define the functions $\phi_i,\underline{\phi}^\flat$ for the pair (φ,ϕ) as in Definition 5.2.5. Then*

$$\begin{aligned}
I_{\Pi_1,\Pi_2}(F) =&|\delta|^{d_n-\frac{n}{2}-\frac{n(n-1)}{4}}\Omega_{\Pi_1}(\delta)\Omega_{\Pi_2}(\delta)\overline{\mu}(\delta)^{\frac{n(n+1)}{2}} \times \int_{k_n}\overline{\Phi(v_n)}\mathrm{d}v_n \times \int_{B_{n-1,-}(k')}\underline{\phi}^\flat(b)\mathrm{d}b \\
&\times \int |\det y|_{k'}^{-1}\eta(\det y)\prod_{i=1}^{n-1}\widehat{\phi_{n-i}}(-y_i(v_{n-i-1},1)\delta)\mathrm{d}y_1\cdots\mathrm{d}y_{n-1}\mathrm{d}v_1\cdots\mathrm{d}v_{n-2},
\end{aligned}$$

where the domain of the last integral is $y_i\in k'^\times$ and $v_i\in k'_i$. Recall that $y=\begin{pmatrix} y_1\cdots y_{n-1} & & \\ & \ddots & \\ & & y_1\end{pmatrix}\in \mathrm{GL}_{n-1}(k')$.

Proof. By Lemma 5.3.2, we only have to compute $\beta_n(W^\sharp_{\varphi,\phi})$. By definition,

$$\begin{aligned}
\beta_n(W^\sharp_{\varphi,\phi}) &= \int_{N_{n-1}(k')\backslash \mathrm{GL}_{n-1}(k')} W^\sharp_{\varphi,\phi}(\iota(\epsilon_{n-1}g))\eta^{n+1}(\det g)\mathrm{d}g \\
&= \int_{N_{n-1}(k')\backslash \mathrm{GL}_{n-1}(k')} \widetilde{W_{\varphi,\phi}}(\epsilon_{n-1}g)|\det\epsilon_{n-1}g|_k^{-\frac{1}{2}}\overline{\mu}(\det\epsilon_{n-1}g)\eta^{n+1}(\det g)\mathrm{d}g.
\end{aligned}$$

Then Lemma 5.2.7 (2) yields the desired result. We only have to note that the measure $\mathrm{d}g$ decomposes according to $g=yv$ as

$$\mathrm{d}g = |\delta_{n-1}(y)|_{k'}^{-1}\mathrm{d}y_1\cdots\mathrm{d}y_{n-1}\mathrm{d}v_1\cdots\mathrm{d}v_{n-2}.$$

□

5.4. **Computing $F_\natural$.** Let $F=(F_0,F_0^{\varphi,\phi},\Phi)$ be an (m,m_1,m_2,r)-admissible test function on the general linear group. Note that $F_0=F_0^\star * F_0$. Let $F_\natural$ be the integral transform defined by (4.2.1). Let P_n be the mirabolic subgroup of GL_n. For later use, we define $\Psi\in\mathcal{C}_c^\infty(P_n(k))$ by

$$\Psi\left(\begin{pmatrix} a & \\ & 1\end{pmatrix}\begin{pmatrix} 1_{n-1} & u \\ & 1\end{pmatrix}\right)=\varphi(a)\phi(u), \quad a\in\mathrm{GL}_{n-1}(k),\ u\in k_{n-1}.$$

Let $\mathfrak{p}$ be the Lie algebra of P_n. Then $\mathfrak{p}\simeq M_{n-1,n}$. We always view $M_{n-1,n}$ as a subspace of M_n consisting of matrices whose last row is zero. By admissibility, the support of Ψ (resp. φ, resp. ϕ) is contained in

$1+\varpi^m\mathfrak{p}(\mathfrak{o})$ (resp. $1+\varpi^m M_{n-1}(\mathfrak{o})$, resp. $\varpi^m\mathfrak{o}^{n-1}$). We have the decomposition $\mathfrak{p}(k)=\mathfrak{p}^+ + \mathfrak{p}^-$ where $\mathfrak{p}^\pm = \mathfrak{p}(k)\cap M_n(k^\pm)$. Note that any element in $P_n(k)$ can be written as $p=p_+(1+p_-)$ where $p_+\in P_n(k')$ and $p_-\in\mathfrak{p}^-$.

We decompose Ψ as $\Psi=\Psi^+\otimes\Psi^-$, where $\Psi^+\in\mathcal{C}_c^\infty(1+\varpi\mathfrak{p}^+(\mathfrak{o}'))$, $\Psi^-\in\mathcal{C}_c^\infty(\mathfrak{p}^-)$ and

$$\Psi(p_+(1+p_-)) = \Psi^+(p_+)\Psi^-(p_+p_-), \quad p_+\in P_n(k'),\ p_-\in\mathfrak{p}^-.$$

Note that Ψ^+ is a multiple of $\mathbf{1}_{1+\varpi^m M_{n-1,n}(\mathfrak{o}')}$. It follows from the admissibility condition that if $p_+\in\operatorname{supp}\Psi^+$ and $p_-\in\mathfrak{p}^-$, then

$$\Psi^-(p_+p_-)=\Psi^-(p_-). \tag{5.4.1}$$

We have $\Psi^-=\varphi^-\otimes\phi^-$ where $\varphi^-\in\mathcal{C}_c^\infty(M_{n-1}(k^-))$ and $\phi^-\in\mathcal{C}_c^\infty(k^{-,n-1})$.

Lemma 5.4.1. *The support of $F_\natural$ is contained in $(1+\varpi^m M_n(\mathfrak{o}^-))\times k_n'\times k^{-,n}$.*

Proof. This follows from the definition of admissible test functions. □

By this lemma, we may identify $F_\natural$ as a function on $\mathfrak{x}_n(k')$ supported in $\varpi^m M_n(\mathfrak{o}^-)\times k_n'\times k^{-,n}$ via the Cayley transform. We shall always make this identification in the following. Recall that the Caylay transform is a map

$$\mathfrak{c}:\mathfrak{s}_n(k')\to S_n(k'),\quad X\to(1+X)(1-X)^{-1}, \tag{5.4.2}$$

which gives a homeomorphism from a neighborhood of $0\in\mathfrak{s}_n(k')$ to a neighborhood of $1\in S_n(k')$. We extend the map $\mathfrak{c}$ to a map $\mathfrak{x}_n(k')\to X_n(k')$ which is the identity on $k_n'\times k^{-,n}$. We also call this the Caylay transform. Similarly, the support of F_0 is $1+\varpi^r M_n(\mathfrak{o})$ for some sufficiently large r. We define a function $F_{0,\natural}\in\mathcal{C}_c^\infty(\mathfrak{s}_n(k'))$ by

$$F_{0,\natural}(X)=\begin{cases}\displaystyle\int\limits_{\mathrm{GL}_n(k')} F_0((1+X)h)\mathrm{d}h & n\text{ odd},\\ \displaystyle\int\limits_{\mathrm{GL}_n(k')} F_0((1+X)h)\mu((1+X)h)\mathrm{d}h & n\text{ even}.\end{cases}$$

Lemma 5.4.2. *Put*

$$c(\Psi^+)=\int\limits_{1+\varpi\mathfrak{p}^+(\mathfrak{o}')}\Psi^+(p)\mathrm{d}p.$$

Then for $(X,v^\vee,v)\in\mathfrak{s}_n(k')\times k_n'\times k^{-,n}$, we have

$$F_\natural(X,v^\vee,v)=c(\Psi^+)\times\int\limits_{\mathfrak{p}^-}F_{0,\natural}(X+p_-)\Psi^-(p_-)\mathrm{d}p_-\times\overline{\Phi^\dagger(v^\vee,v)}.$$

Proof. This computation is very similar to [Zha14b, Lemma 8.7]. To avoid repetition, we only sketch the argument for the case n being odd. The case n being even only requires the change of notation at several places.

By the admissibility conditions, we have

$$F_\natural(X,v^\vee,v)=\int\limits_{\mathrm{GL}_n(k')}\int\limits_{P(k)}\Psi(p)F_0(p(1+X)h)\mathrm{d}p\mathrm{d}h\times\overline{\Phi^\dagger(v^\vee,v)}.$$

We write p as $p_+(1+p_-)$. We make use of (5.4.1) and the fact that F_0 is invariant under the conjugation of elements in $1+\varpi M_n(\mathfrak{o})$. We get

$$F_\natural(X,v^\vee,v)=c(\Psi^+)\times\int\limits_{\mathfrak{p}^-}\int\limits_{\mathrm{GL}_n(k')}F_0((1+p_-)(1+X)h)\Psi^-(p_-)\mathrm{d}h\mathrm{d}p_-\times\overline{\Phi^\dagger(v^\vee,v)}.$$

It follows that

$$\begin{aligned}
F_\natural(X,v^\vee,v) &= c(\Psi^+)\times\int_{\mathfrak{p}^-}\int_{\mathrm{GL}_n(k')} F_0((1+p_-X)(1+(1+p_-X)^{-1}(p_-+X)h))\Psi^-(p_-)\mathrm{d}h\mathrm{d}p_-\times\overline{\Phi^\dagger(v^\vee,v)}\\
&= c(\Psi^+)\times\int_{\mathfrak{p}^-}\int_{\mathrm{GL}_n(k')} F_0(1+(1+p_-X)^{-1}(p_-+X)h)\Psi^-(p_-)\mathrm{d}h\mathrm{d}p_-\times\overline{\Phi^\dagger(v^\vee,v)}\\
&= c(\Psi^+)\times\int_{\mathfrak{p}^-} F_{0,\natural}((1+p_-X)^{-1}(p_-+X))\Psi^-(p_-)\mathrm{d}p_-\times\overline{\Phi^\dagger(v^\vee,v)}\\
&= c(\Psi^+)\times\int_{\mathfrak{p}^-} F_{0,\natural}(X+p_-)\Psi^-(p_-)\mathrm{d}p_-\times\overline{\Phi^\dagger(v^\vee,v)},
\end{aligned}$$

where in the last equality, we have used the fact that $F_{0,\natural}$ is a multiple of $\varpi^r\mathfrak{s}_n(\mathfrak{o}')$ and is thus invariant under the multiplication of elements in $1+\varpi M_n(\mathfrak{o})$. □

5.5. **Some lemmas from** [Zha14b]**.** We recall some lemmas from [Zha14b]. We write them in the form that we are going to use.

Let $\widehat{\mathfrak{r}_n}=\mathfrak{s}_n\times k'^n\times k_n^-$. The group GL_n acts on $\widehat{\mathfrak{r}_n}$ by

$$[\gamma,v,v^\vee].h=[h^{-1}\gamma h,h^{-1}v,v^\vee h].$$

We say that any element $[\gamma,v,v^\vee]$ is regular semisimple if its stabilizer in GL_n is trivial and its GL_n-orbit is closed.

Let $x,y\in k'^n$. Set

$$A(x)=\delta\begin{pmatrix} 0 & & & & x_n\\ 1 & 0 & & & x_{n-1}\\ & \ddots & \ddots & & \vdots\\ & & \ddots & 0 & x_2\\ & & & 1 & x_1\end{pmatrix}\in M_n(k'),\quad x=\begin{pmatrix}x_n\\ \vdots\\ x_1\end{pmatrix},$$

and $A(x,y)=[\delta A(x),y,\delta e_n]\in\mathfrak{s}_n(k')\times k'^n\times k_n^-$. Note that for any x,y, the stabilizer of $A(x,y)$ in GL_n is trivial.

Lemma 5.5.1 ([Zha14b, Corollary 6.5])**.** *The map $\beta:\mathrm{GL}_n(k')\times(k'^n\times k'^n)\to\mathfrak{s}_n(k')\times k'^n\times k_n^-$ given by*

$$(h,(x,y))\mapsto A(x,y).h$$

is an open immersion. The image consists of all elements $[\gamma,v,v^\vee]$ with $\det\begin{pmatrix}v^\vee\\ \vdots\\ v^\vee\gamma^{n-1}\end{pmatrix}\neq 0.$

Let $[\gamma,v,v^\vee]\in\widehat{\mathfrak{r}_n}$. Let W be the subvariety of $\widehat{\mathfrak{r}_n}$ defined by the following equations, where we write $\gamma=(\gamma_{ij})_{1\le i,j\le n}$.

$$\gamma_{i,i-1}=\delta,\ (i=2,\cdots,n);\quad \gamma_{ij}=0,\ (i>j+1);\quad v^\vee=\delta e_n.$$

Let V be the subvariety of $\widehat{\mathfrak{r}_n}$ defined by the same equations as W, plus the equation $v=0$. There is a natural morphism $p:W\to V$, $[\gamma,v,v^\vee]\mapsto[\gamma,0,v^\vee]$. For $x,y\in k'^n$, define a map $\alpha_{x,y}:N_n(k')\to W(k')$ by $u\mapsto A(x,y).u$.

Lemma 5.5.2 ([Zha14b, Lemma 6.6])**.** *For any $x,y\in k'^n$, the composition $p\circ\alpha_{x,y}:N_n(k')\to V(k')$ is an isomorphism whose Jacobian is ±1.*

Let $X\in\widehat{\mathfrak{r}_n}(k')$ and $F\in\mathcal{C}_c^\infty(\widehat{\mathfrak{r}_n}(k'))$. We define the orbital integral

$$O(X,F,s)=\int_{\mathrm{GL}_n(k')}F(X.h)\eta(\det h)|\det h|^s\mathrm{d}h \tag{5.5.1}$$

If X is regular semisimple, then this integral is convergent for any s. We define $O(X,F)=O(X,F,0)$ if X is regular semisimple.

Let $\xi_- = A(0,0) \in \widehat{\mathfrak{x}_n}(k')$, i.e.

$$\xi_- = \left[\delta \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, (0, \cdots, 0, \delta)\right].$$

Lemma 5.5.3 ([Zha14b, Lemma 6.9])**.** *As a function of s, $O(\xi_-, F, s)$ is absolutely convergent if $\Re s > 1 - \frac{1}{n}$. It extends to a meromorphic function for all $s \in \mathbb{C}$ which is holomorphic at $s = 0$.*

By this lemma, we define the regular nilpotent orbital integral $O(\xi_-, F) = O(\xi_-, F, 0)$. If for some F, the integral (5.5.1) is convergent when $s = 0$, then it computes $O(\xi_-, F)$.

5.6. **Orbital integrals of $\widehat{F_\natural}$.**

Lemma 5.6.1. *Let $\Phi = \Phi^+ \otimes \Phi^- \in \mathcal{S}(k_n)$, $\Phi^\pm \in \mathcal{S}(k_n^\pm)$. Then*

$$\widehat{\widehat{\Phi^\dagger}}(v, v^\vee) = \widehat{\widehat{\Phi^+}}(v)\overline{\Phi^-}(v^\vee), \quad v \in k'^n, \ v^\vee \in k_n^-.$$

Proof. This is the Fourier inversion formula. □

Put

$$F_{0,\natural} * \Psi^-(X) = \int_{\mathfrak{p}^-} F_{0,\natural}(X + p_-)\Psi^-(p_-)\mathrm{d}p_-.$$

Then $F_{0,\natural} * \Psi^- \in \mathcal{C}_c^\infty(M_n(k^-))$. Similarly to Definition 5.2.5, we define the functions φ_i^-, ϕ_i^-, $\phi_i^{\flat,-}$ and $\underline{\phi^{\flat,-}}$. More precisely, we put $\varphi_n^- = F_{0,\natural} * \Psi^-$ and define inductively the functions φ_i^-, ϕ_i^-, $\phi_i^{\flat,-}$ so that $\varphi_i^-(X_i) = \varphi_{i-1}^-(X_{i-1})\phi_i^{\flat,-}(v_i)\phi_{i-1}^-(u_{i-1})$, where

$$X_i = \begin{pmatrix} X_{i-1} & u_{i-1} \\ & v_i \end{pmatrix}, \quad X_{i-1} \in \mathrm{GL}_{i-1}(k'), \quad u_{i-1} \in k^{i-1}, \quad v_i \in k_i.$$

Note that $\phi_n^{\flat,-}$ is a multiple of $\mathbf{1}_{\varpi^r \mathfrak{o}_n^-}$. We define $\underline{\phi^{\flat,-}} = \otimes \phi_i^{\flat,-}$ and view it as a function on $\mathfrak{b}_n(k^-)$.

Lemma 5.6.2. *Suppose $\delta h^{-1}A(x)h \in \operatorname{supp} \widehat{F_{0,\natural} * \Psi^-}$. Then the last column of $h^{-1}A(x)h$ is bounded by a polynomial of the norms of x_i ($1 \leq i \leq n$) and ϖ^{-m}, independent of m_2 and r.*

Proof. This is the "*claim*" in the Step 1 of the proof of [Zha14b, Lemma 8.8, p. 603]. For the convenience of the reader, we point out that our $A(x)$ (resp. $F_{0,\natural} * \Psi^-$) is the $\varrho(x)$ (resp. φ_n^-) in [Zha14b]. □

If $[\gamma, v, v^\vee] \in \widehat{\mathfrak{x}_n}(k')$, then we put

$$\widehat{\mathbf{T}}([\gamma, v, v^\vee]) = \det \begin{pmatrix} v^\vee \\ \vdots \\ v^\vee \gamma^{n-1} \end{pmatrix},$$

and $\widehat{\mathbf{t}}([\gamma, v, v^\vee]) = \mu(\widehat{\mathbf{T}}([\gamma, v, v^\vee]))$.

Lemma 5.6.3. *Let $\mathcal{Y}$ be an open compact neighborhood of $[0,0,0] \in \widehat{\mathfrak{x}_n}(k')$. Then there are sufficiently large integers (m, m_1, m_2, r) and an (m, m_1, m_2, r)-admissible test function $F = (F_0, F_0^{\varphi,\phi}, \Phi)$, such that $\widehat{\mathbf{t}}(X)O(X, \widehat{F_\natural})$ is a constant if $X \in \mathcal{Y}$. Moreover, this constant equals $\widehat{\mathbf{t}}(\xi_-)O(\xi_-, \widehat{F_\natural})$.*

Proof. By Lemma 5.5.1, it is enough to prove that for any compact subset $\mathcal{X} \subset k'^n \times k'^n$, we can find a sufficiently admissible test function F, such that the orbital integral $O(A(x,y), \widehat{F_\natural})$ is a constant for regular semisimple $A(x,y)$ with $(x,y) \in \mathcal{X}$ and this constant equals $O(\xi_-, \widehat{F_\natural})$. Note that $\widehat{\mathbf{t}}(A(x,y)) = \mu\left((-1)^{\left[\frac{n}{2}\right]}\delta^{\frac{n(n+1)}{2}}\right)$ for any x, y.

By Lemma 5.4.2, if $A(x,y)$ is regular semisimple, then the orbital integral of $\widehat{F_\natural}$ takes the form

$$c(\Psi^+) \times \int_{\mathrm{GL}_n(k')} \widehat{F_{0,\natural} * \Psi^-}(\delta h^{-1}A(x)h)\widehat{\widehat{\Phi^\dagger}}(h^{-1}y, \delta e_n h)\eta(\det h)\mathrm{d}h.$$

Without loss of generality, we may assume that $c(\Psi^+) = 1$.

It follows from Lemma 5.6.1 that if $\widehat{\Phi^\dagger}(h^{-1}y, \delta e_n h) \neq 0$, then the lower right entry of h is not zero. We therefore decompose h as

$$h = z_1 \begin{pmatrix} 1_{n-1} & w_{n-1} \\ & 1 \end{pmatrix} \begin{pmatrix} a_{n-1} & \\ & 1 \end{pmatrix} \begin{pmatrix} 1_{n-1} & \\ v_{n-1} & 1 \end{pmatrix},$$

where $z_1 \in Z_n(k') \simeq k'^\times$, $a_{n-1} \in \mathrm{GL}_{n-1}(k')$, $w_{n-1} \in k'^{n-1}$, $v_{n-1} \in k'_{n-1}$. The measure $\mathrm{d}h$ decomposes as

$$\mathrm{d}h = \frac{1}{|\det a_{n-1}|} \mathrm{d}z_1 \mathrm{d}a_{n-1} \mathrm{d}w_{n-1} \mathrm{d}v_{n-1}.$$

It follows from the admissibility conditions on Φ that $z_1 \in 1 + \varpi^{m_1}\mathfrak{o}'$ and $v_{n-1} \in \delta^{-1}\varpi^{m_2}\mathfrak{o}^-_{n-1}$. Therefore by Lemma 5.6.2, we can take a suitably large m_2 (and r), such that $\widehat{F_{0,\natural} * \Psi^-}$ (resp. $\widehat{\Phi^+}$) is invariant under the conjugation (resp. left multiplication) by $\begin{pmatrix} 1_{n-1} & \\ v_{n-1} & 1 \end{pmatrix}$. Then

$$O(A(x,y), \widehat{F_\natural}) = \int \widehat{F_{0,\natural} * \Psi^-}(\delta h_{n-1}^{-1} A(x) h_{n-1}) \widehat{\overline{\Phi^+}}(z_1^{-1} h_{n-1}^{-1} y)$$
$$\overline{\Phi^-}(\delta z_1(v_{n-1}, 1)) \eta(\det z_1 a_{n-1}) \frac{1}{|\det a_{n-1}|_{k'}} \mathrm{d}a_{n-1} \mathrm{d}w_{n-1} \mathrm{d}z_1 \mathrm{d}v_{n-1},$$

where the domain of integration is $z_1 \in Z_n(k') \simeq k'^\times$, $v_{n-1} \in k'_{n-1}$, $h_{n-1} = \begin{pmatrix} 1_{n-1} & w_{n-1} \\ & 1 \end{pmatrix} \begin{pmatrix} a_{n-1} & \\ & 1 \end{pmatrix}$, $w_{n-1} \in k'^{n-1}$, $a_{n-1} \in \mathrm{GL}_{n-1}(k')$.

We now argue as in [Zha14b, Proof of Lemma 8.8, Step 2 and 3]. We may inductively show, using the admissibility conditions of $F_{0,\natural} * \Psi^-$, that we can write a_{n-1} as uav, where

$$u \in N_{n-1}(k'), \quad a = \begin{pmatrix} z_2 \cdots z_n & & \\ & \ddots & \\ & & z_2 \end{pmatrix}, \quad v = \prod_{i=1}^{n-2} \begin{pmatrix} 1_i & \\ v_i & 1 \end{pmatrix} \in \mathrm{GL}_{i+1}(k'), \quad v_i \in k'_i.$$

Moreover, $v_i \in \varpi^m \mathfrak{o}'_i$ for $1 \leq i \leq n-2$. Note that the measure decomposes as

$$\mathrm{d}a_{n-1} = |\delta_{n-1}^{-1}(a)| \mathrm{d}z_2 \cdots \mathrm{d}z_n \mathrm{d}u \mathrm{d}v_1 \cdots \mathrm{d}v_{n-2}.$$

Recall that we have defined the functions ϕ_i^-, $\phi_i^{\flat,-}$ and $\underline{\phi^{\flat,-}}$. Using the fact that $\widehat{F_{0,\natural} * \Psi^-}$ is invariant under the left and right multiplication by $1 + \varpi^m M_n(\mathfrak{o}')$, we conclude that

$$O(A(x,y), \widehat{F_\natural}) = \int \widehat{\underline{\phi^{\flat,-}}}(\delta(ua)^{-1} A(x)(ua)) \left(\prod_{i=1}^{n-1} \widehat{\phi_i^-}(-\delta z_{n-i+1}(v_{i-1}, 1)) \right) \widehat{\overline{\Phi^+}}(z_1^{-1}(ua)^{-1} y) \tag{5.6.1}$$
$$\overline{\Phi^-}(\delta z_1(v_{n-1}, 1)) \eta(\det z_1 a) |\delta_{n-1}^{-1}(a)| \frac{1}{|\det a|} \mathrm{d}z_1 \cdots \mathrm{d}z_n \mathrm{d}u \mathrm{d}v_1 \cdots \mathrm{d}v_{n-1},$$

where $z_i \in k'^\times$, $u \in N_{n-1}(k')$, $v_i \in k'_i$. Furthermore, by the admissibility conditions, $z_i \in \delta^{-1}\varpi^{-m}\mathfrak{o}'$ $(i \geq 2)$, $v_i \in \varpi^m \mathfrak{o}'_i$ $(1 \leq i \leq n-2)$.

We now make use of Lemma 5.5.2 and make a change of variable $u \mapsto u_1 = p \circ \alpha_{x,y}(u)$. The integral in (5.6.1) over u then takes the form

$$\int_{\mathfrak{b}_{n-1}(k')} \widehat{\underline{\phi^{\flat,-}}}(\delta a^{-1} X(u_1) a) \widehat{\overline{\Phi^+}}(z_1^{-1} a^{-1} *) \mathrm{d}u_1,$$

where $X(u_1) = X_{ij}$ is a matrix of the following form. The triangle (X_{ij}) with $1 \leq i \leq j \leq n-1$ is the matrix u_1, $X_{i,i-1} = 1$ for $i = 2, \cdots, n$, the last column is a polynomial in the entries of u_1, u, x, y. The other entries of $X(u_1)$ are zero. The $*$ is also a polynomial in the entries of u_1, u, x, y. Note that the last colum of $\widehat{\underline{\phi^{\flat,-}}}$ (resp. $\widehat{\overline{\Phi^+}}$) is a multiple of $\mathbf{1}_{\varpi^{-r}\mathfrak{o}^{-*,n}}$ (resp. $\mathbf{1}_{\varpi^{-m_2}\mathfrak{o}'^*_n}$). Since x, y lie inside some compact region, z_i's are bounded by some bound which is independent of m_2 and r, we may increase m_2 and r suitably so that the restrictions on the last column of $\delta a^{-1}X(u_1)a$ and $z_1^{-1}a^{-1}*$ are superfluous. We conclude that

$$
\begin{aligned}
O(A(x,y),\widehat{F_\natural}) = \int \widehat{\underline{\phi^{\flat,-}}}(\delta a^{-1}X_1(u_1)a)\widehat{\overline{\Phi^+}}(0) \times \left(\prod_{i=1}^{n-1}\widehat{\phi_i^-}(-\delta z_{n-i+1}(v_{i-1},1))\right) \\
\overline{\Phi^-}(\delta z_1(v_{n-1},1))\eta(\det z_1 a)|\delta_{n-1}^{-1}(a)|\frac{1}{|\det a|}\mathrm{d}z_1\cdots\mathrm{d}z_n\mathrm{d}u_1\mathrm{d}v_1\cdots\mathrm{d}v_{n-1},
\end{aligned}
\tag{5.6.2}
$$

where $X_1(u_1) = (X_{ij})$ is a matrix of the following form. The triangle X_{ij} with $1 \le i \le j \le n-1$ is the matrix u_1, $X_{i,i-1} = 1$ for $i = 2,\cdots,n$, the other entries of $X(u_1)$ are zero. The domain of integration is $u_1 \in \mathfrak{b}_{n-1}(k')$, $z_i \in k'^{\times}$ and $v_i \in k'_i$. The integral (5.6.2) is then independent of x, y. It also follows that the integral equals $O(A(0,0),\widehat{F_\natural})$, which is precisely the nilpotent orbital integral of $\widehat{F_\natural}$. □

Lemma 5.6.4. *The nilpotent orbital integral of $\widehat{F_\natural}$ equals*

$$
\begin{aligned}
O(\xi_-,\widehat{F_\natural}) =& c(\Psi^+) \times |\delta|_k^{-\frac{n}{2}-\frac{n(n-1)}{4}} \prod_{i=1}^{n-1}\phi_i^{\flat,-}(0) \times \int_{k_n}\overline{\Phi(x)}\mathrm{d}x \\
&\times \int \prod_{i=1}^{n-1}\widehat{\phi_i^-}(-\delta z_{n-i+1}(v_{i-1},1))\eta(\det a)\frac{1}{|\det a|}\mathrm{d}z_2\cdots\mathrm{d}z_n\mathrm{d}v_1\cdots\mathrm{d}v_{n-1},
\end{aligned}
\tag{5.6.3}
$$

where the domain of the last integration is $z_i \in k'^{\times}$ and $v_i \in k'_i$. Recall also that $a = \begin{pmatrix} z_2\cdots z_n & & \\ & \ddots & \\ & & z_2 \end{pmatrix} \in \mathrm{GL}_{n-1}(k')$.

Proof. All we need to compute is the integral (5.6.2). Since $\phi_n^{\flat,-}$ is normalized so that $\widehat{\phi_n^{\flat,-}}(0) = 1$, it follows that

$$
\widehat{\underline{\phi^{\flat,-}}}(\delta a^{-1}X_1(u_1)a) = \left(\bigotimes_{i=1}^{n-1}\widehat{\phi_i^{\flat,-}}\right)(\delta a^{-1}u_1 a),
$$

where $u_1 \in \mathfrak{b}_{n-1}(k')$. We make a change of variable $u_1 \mapsto a u_1 a^{-1}$, then

$$
\begin{aligned}
\int_{N_{n-1}(k')}\left(\bigotimes_{i=1}^{n-1}\widehat{\phi_i^{\flat,-}}\right)(\delta a^{-1}u_1 a)\mathrm{d}u_1 &= |\delta_{n-1}(a)|\int_{N_{n-1}(k')}\left(\bigotimes_{i=1}^{n-1}\widehat{\phi_i^{\flat,-}}\right)(\delta u_1)\mathrm{d}u_1 \\
&= |\delta_{n-1}(a)||\delta|_k^{-\frac{n(n-1)}{4}}\prod_{i=1}^{n-1}\phi_i^{\flat,-}(0).
\end{aligned}
$$

Note that the extra factor $|\delta|_k^{-\frac{n(n-1)}{4}}$ comes from the difference between the measures on k^- and $\delta k'$. In the integral over z_1 and v_{n-1}, we make a change of variable $v_{n-1} \mapsto v_{n-1}z_1^{-1}$ and replace $\mathrm{d}z_1$ by the additive measure. By the support condition of Φ^-, z_1 lies in a small neighborhood of 1. It follows that

$$
\int_{k'^{\times}}\int_{k'_{n-1}}\overline{\Phi^-}(\delta z_1(v_{n-1},1))\mathrm{d}v_{n-1}\mathrm{d}z_1 = \int_{k'_n}\overline{\Phi^-}(\delta v_n)\mathrm{d}v_n = \frac{1}{|\delta|_k^{\frac{n}{2}}}\int_{k_n^-}\overline{\Phi^-}(v_n)\mathrm{d}v_n.
$$

Thus

$$
\begin{aligned}
O(\xi_-,F_\natural) =& |\delta|_k^{-\frac{n}{2}-\frac{n(n-1)}{4}}\prod_{i=1}^{n-1}\phi_i^{\flat,-}(0)\widehat{\overline{\Phi^+}}(0)\int_{k_n^-}\overline{\Phi^-}(v_n)\mathrm{d}v_n \\
&\int\prod_{i=1}^{n-1}\widehat{\phi_i^-}(-\delta z_{n-i+1}(v_{i-1},1))\eta(\det a)\frac{1}{|\det a|}\mathrm{d}z_2\cdots\mathrm{d}z_n\mathrm{d}v_1\cdots\mathrm{d}v_{n-1},
\end{aligned}
$$

Finally, since

$$
\widehat{\overline{\Phi^+}}(0) = \int_{k'_n}\overline{\Phi^+}(w_n)\mathrm{d}w_n,
$$

we have

$$\widehat{\Phi^+}(0)\int_{k_n^-}\overline{\Phi^-}(v_n)\mathrm{d}v_n=\int_{k_n}\overline{\Phi}(x)\mathrm{d}x.$$

We then get the desired result. ☐

5.7. A truncated germ expansion of I_{Π_1,Π_2}.

Proposition 5.7.1. *Let Π_1 and Π_2 be two irreducible admissible generic representations of $\mathrm{GL}_n(k)$ with central characters Ω_{Π_1} and Ω_{Π_2} respectively. Then there is a test function F on the general linear group, such that $I_{\Pi_1,\Pi_2,s}(F)$ is holomorphic at $s=\frac{1}{2}$ and $F_\natural$ is supported in $A\times k'_n\times k^{-,n}$ where A is a sufficiently small neighborhood of $1_n\in S_n(k')$. Moreover, if we view $F_\natural$ as a function on $\mathfrak{x}_n(k')$ via the Caylay transform (5.4.2), then*

$$I_{\Pi_1,\Pi_2}(F)=|\delta|_k^{d_n}\Omega_{\Pi_1}(\delta)\Omega_{\Pi_2}(\delta)\overline{\mu}(\delta)^{\frac{n(n+1)}{2}}O(\xi_-,\widehat{F_\natural})\neq 0.$$

Proof. We choose integers (m,m_1,m_2,r) as in Lemma 5.6.3 and take an (m,m_1,m_2,r)-admissible test function $F=(F_0,F_0^{\varphi,\phi},\Phi)$. It follows from Lemma 5.3.3 and Lemma 5.6.4 that

$$I_{\Pi_1,\Pi_2}(F)=|\delta|_k^{d_n}\Omega_{\Pi_1}(\delta)\Omega_{\Pi_2}(\delta)\overline{\mu}(\delta)^{\frac{n(n+1)}{2}}O(\xi_-,\widehat{F_\natural}).$$

It is clear from the formula (5.6.3) that we can choose a test function so that $O(\xi_-,\widehat{F_\natural})\neq 0$. ☐

6. Proof of Theorem 4.4.1

6.1. The unramified case.

Proof of Theorem 4.4.1 in case (1). First we note that if we replace δ by $a\delta$ where $a\in k'^\times$, then the distribution I_{Π_1,Π_2} is multiplied by $|a|_k^{d_n}$. Moreover, replacing ψ' by ψ'_a amounts to replacing δ by $a\delta$, where $\psi'_a(x)=\psi'(ax)$. Therefore we may assume in addition that $\delta\in\mathfrak{o}'^\times$ and the conductor of ψ' is $\mathfrak{o}'$.

It follows from Theorem 4.2.4 that we have a matching pair of test functions

$$F=\left(\frac{1}{\mathrm{vol}\,\mathrm{GL}_n(\mathfrak{o})}\mathbf{1}_{\mathrm{GL}_n(\mathfrak{o})},\frac{1}{\mathrm{vol}\,\mathrm{GL}_n(\mathfrak{o})}\mathbf{1}_{\mathrm{GL}_n(\mathfrak{o})},\mathbf{1}_{\mathfrak{o}_n}\right),$$

and

$$\left\{f^{V^+}=\left(\frac{1}{\mathrm{vol}\,K^+}\mathbf{1}_{K^+},\frac{1}{\mathrm{vol}\,K^+}\mathbf{1}_{K^+},\mathbf{1}_{\mathbf{L}(\mathfrak{o}')},\mathbf{1}_{\mathbf{L}(\mathfrak{o}')}\right),\quad f^{V^-}=(0,0,0,0)\right\}.$$

Let W_1^0 (resp. W_2^0) be the unique normalized spherical Whittaker function in $\mathcal{W}(\Pi_1,\psi)$ (resp. $\mathcal{W}(\Pi_2,\overline{\psi})$), i.e. $W_1^0(1)=W_2^0(1)=1$. Then

$$\mathrm{BC}(\pi_i)(F_i)(W_i^0)=\frac{1}{\mathrm{vol}\,\mathrm{GL}_n(\mathfrak{o})}W_i^0,\quad i=1,2.$$

It then follows from (3.1.2), (3.1.5) and (3.1.9) that

$$I_{\mathrm{BC}(\pi_1),\mathrm{BC}(\pi_2)}(F)=\frac{L(\frac{1}{2},\mathrm{BC}(\pi_1)\otimes\mathrm{BC}(\pi_2)\otimes\mu^{-1})}{L(1,\pi_1,\mathrm{Ad})L(1,\pi_2,\mathrm{Ad})}.$$

The distribution J_{π_1,π_2} is computed in a similar way where we use Proposition 1.1.1 instead. We finally conclude that

$$I_{\mathrm{BC}(\pi_1),\mathrm{BC}(\pi_2)}(F)=J_{\pi_1,\pi_2}(f^{V^+}).$$

The Theorem 4.4.1 in case (1) is thus proved. ☐

6.2. **The split case.** We assume that $k = k' \times k'$ in this subsection. We make an identification $\mathrm{U}(V) \simeq \mathrm{GL}_n$ once and for all in this subsection.

Let π_1 and π_2 be two irreducible admissible unitary generic tempered representations of $\mathrm{GL}_n(k')$. Let $\Pi_1 = \pi_1 \boxtimes \overline{\pi_1}$, $\Pi_2 = \pi_2 \boxtimes \overline{\pi_2}$ be the base change of π_1 and π_2 to $\mathrm{GL}_n(k)$ respectively. Let $\mathcal{W}(\pi_1, \psi)$ and $\mathcal{W}(\pi_2, \overline{\psi})$ be their Whittaker models respectively. Let $f = (f_1, f_2, \phi_1, \phi_2)$ be a test function on the unitary group. We define a distribution

$$J'_{\pi_1,\pi_2}(f) = \sum_{W_1, W_2} \lambda(\pi_1(f_1)W_1, \pi_2(f_2)W_2, \mu, \phi_1)\overline{\lambda(W_1, W_2, \mu, \phi_2)},$$

where W_1 (resp. W_2) runs over an orthonormal basis of $\mathcal{W}(\pi_1, \psi)$ (resp. $\mathcal{W}(\pi_2, \overline{\psi})$).

Lemma 6.2.1. *Let $F = (F_1, F_2, \Phi)$ be a test function on the general linear group. Suppose F and f match as described in Proposition 4.2.3. Then $I_{\Pi_1,\Pi_2}(F) = \kappa^{-1} J'_{\pi_1,\pi_2}(f)$.*

Proof. By definition

$$I(F_1, F_2, \Phi) = \sum \lambda(\Pi_1(F_1)(W_1 \otimes \overline{W_1^\vee}), \Pi_2(F_2)(W_2 \otimes \overline{W_2^\vee}), \mu, \Phi) \overline{\beta_n(W_1 \otimes \overline{W_1^\vee})\beta_n(W_2 \otimes \overline{W_2^\vee})},$$

where W_1 and $W_1^\vee$ (resp. W_2 and $W_2^\vee$) independently run through an orthonormal basis of $\mathcal{W}(\pi_1, \psi')$ (resp. $\mathcal{W}(\pi_2, \overline{\psi'})$).

Recall that we have fixed a purely imaginary element $\delta \in k^-$. Then by the definition of β_n, we have

$$\beta_n(W_i \otimes \overline{W_i^\vee}) = |\delta|_k^{d_n/2} \langle W_i, W_i^\vee \rangle, \quad i = 1, 2,$$

where the pairing on the right hand side is the inner product on the Whittaker model of π_i.

It follows that

$$I(F_1, F_2, \Phi) = |\delta|_k^{d_n} \sum_{W_1, W_2} \lambda(\Pi_1(F_1)(W_1 \otimes \overline{W_1}), \Pi_2(F_2)(W_2 \otimes \overline{W_2}), \mu, \Phi),$$

where W_1 (resp. W_2) runs through an orthonormal basis of $\mathcal{W}(\pi_1, \psi')$ (resp. $\mathcal{W}(\pi_2, \overline{\psi'})$). The desired equality then follows from the definitions of I and J'. $\square$

We fix an isomorphism

$$\pi_1 \to \mathcal{W}(\pi_1, \psi'), \quad \varphi_1 \to W_1,$$

so that $\langle \varphi_1, \varphi_1 \rangle_{\pi_1} = \langle W_1, W_1 \rangle_{\mathcal{W}(\pi_1, \psi')}$ where $\langle -, - \rangle_{\pi_1}$ is the inner product on π_1 while $\langle -, - \rangle_{\mathcal{W}(\pi_1, \psi')}$ is the inner product on the Whittaker model. We fix an isomorphism $\pi_2 \to \mathcal{W}(\pi_2, \overline{\psi'})$ with the same property. Under these isomorphisms, we write the linear form α as a linear functional on the Whittaker models

$$\alpha(W_1, W_2, \phi) = \int_{\mathrm{GL}_n(k')} \langle \pi_1(g)W_1, W_1 \rangle \langle \pi_2(g)W_2, W_2 \rangle \overline{\langle \omega_{\psi',\mu}(g)\phi, \phi \rangle} \mathrm{d}g.$$

Recall that $\omega_{\psi',\mu}(g)\phi(x) = \mu(\det g)|\det g|^{\frac{1}{2}}\phi(xg)$.

Proposition 6.2.2. *For all $W_1 \in \mathcal{W}(\pi_1, \psi')$, $W_2 \in \mathcal{W}(\pi_2, \overline{\psi'})$ and $\phi \in \mathcal{S}(k'_n)$, we have*

$$\alpha(W_1, W_2, \phi) = \lambda(W_1, W_2, \mu, \phi)\overline{\lambda(W_1, W_2, \mu, \phi)}.$$

Proof. Since both α and $\lambda \otimes \overline{\lambda}$ belong to the one dimensional space $\mathrm{Hom}_{\mathrm{GL}_n(k') \times \mathrm{GL}_n(k')}(\pi_1 \otimes \overline{\pi_1} \otimes \pi_2 \otimes \overline{\pi_2} \otimes \overline{\omega_{\psi',\mu}} \otimes \omega_{\psi',\mu}, \mathbb{C})$, it is enough to prove that for some W_1, W_2 and ϕ, we have

$$\alpha(W_1, W_2, \phi) = \lambda(W_1, W_2, \mu, \phi)\overline{\lambda(W_1, W_2, \mu, \phi)} \neq 0.$$

We will denote by ι the embedding $\mathrm{GL}_{n-1} \to \mathrm{GL}_n, g \mapsto \mathrm{diag}[g, 1]$, or the embedding $\mathrm{GL}_n \to \mathrm{GL}_{n+1}$ defined in a similar way. We always consider GL_{n-1} as a subgroup of GL_n via this embedding. Let $f \in \mathcal{C}_c^\infty(B_{n-1,-}(k'))$. Then there is a Whittaker function $W_f \in \mathcal{W}(\pi_1, \psi')$, such that

$$\mathrm{supp}\, W_f|_{\mathrm{GL}_{n-1}} \subset N_{n-1}(k')B_{n-1,-}(k'), \ W_f|_{B_{n-1,-}} = f.$$

Let $\phi \in \mathcal{C}_c^\infty(k'_{n-1} \times k'^\times)$ and $F(g) = W_f(g)\overline{\phi(e_n g)\mu(\det g)}|\det g|^{\frac{1}{2}} \in \mathcal{C}^\infty(\mathrm{GL}_n(k'))$. If $F(g) \neq 0$, then the lower right entry of g is nonzero. It follows that g takes the form

$$g = zn \begin{pmatrix} a_{n-1} & \\ & 1 \end{pmatrix} \begin{pmatrix} 1_{n-1} & \\ v_{n-1} & 1 \end{pmatrix},$$

where $z \in Z_n(k')$, $n \in N_n(k')$, $a_{n-1} \in \mathrm{GL}_{n-1}(k')$ and $v_{n-1} \in k'_{n-1}$. Moreover, a_{n-1} lies in a compact subset of $B_{n-1,-}(k')$. It follows that $F(g) \in \mathcal{C}_c^\infty(N_n(k') \backslash \mathrm{GL}_n(k'), \psi')$.

Let σ be any irreducible generic tempered representation of $\mathrm{GL}_{n+1}(k')$. Then $\mathrm{Hom}_{\mathrm{GL}_n(k')}(\sigma \otimes \pi_2, \mathbb{C}) \neq 0$. There is a Whittaker function $W_F \in \mathcal{W}(\sigma, \psi')$, such that $W_F|_{\mathrm{GL}_n(k')} = F$.

We now apply [Zha14b, Proposition 4.10]. It follows that

$$\int\limits_{\mathrm{GL}_n(k')} \langle \sigma(\iota(g))W_F, W_F\rangle \langle \pi_2(g)W_2, W_2\rangle \mathrm{d}g = \int\limits_{\mathrm{GL}_n(k')} W_F(\iota(g))W_2(g)\mathrm{d}g \overline{\int\limits_{\mathrm{GL}_n(k')} W_F(\iota(g))W_2(g)\mathrm{d}g}, \tag{6.2.1}$$

where $\langle -, - \rangle$ stands for the inner products on the corresponding Whittaker models.

We have

$$\begin{aligned} \langle \sigma(\iota(g))W_F, W_F\rangle &= \int\limits_{N_n(k')\backslash \mathrm{GL}_n(k')} W_F(hg)\overline{W_F(h)}\mathrm{d}h \\ &= \int\limits_{N_n(k')\backslash \mathrm{GL}_n(k')} \pi_1(g)W_f(h)\overline{W_f(h)\omega_{\psi',\mu}(g)\phi(e_n h)}\phi(e_n h)|\det h|\mathrm{d}h. \end{aligned}$$

Note that if $\phi(e_n h) \neq 0$, then the lower right entry of h is nonzero. We then decompose

$$h = z \begin{pmatrix} a_{n-1} & \\ & 1 \end{pmatrix} \begin{pmatrix} 1_{n-1} & \\ v_{n-1} & 1 \end{pmatrix},$$

where $z \in Z_n(k') \simeq k'^\times$, $a_{n-1} \in N_{n-1}(k')\backslash \mathrm{GL}_{n-1}(k')$, $v_{n-1} \in k'_{n-1}$. The measure $\mathrm{d}h$ decomposes as

$$\mathrm{d}h = \frac{1}{|\det a_{n-1}|}\mathrm{d}z\mathrm{d}a_{n-1}\mathrm{d}v_{n-1}.$$

We conclude that

$$\begin{aligned} \langle \sigma(\iota(g))W_F, W_F\rangle = &\int\limits_{N_{n-1}(k')\backslash \mathrm{GL}_{n-1}(k')} \int\limits_{k'^\times} \int\limits_{k'_{n-1}} \pi_1(g)W_f(\iota(a_{n-1}))\overline{W_f(\iota(a_{n-1}))} \\ &\overline{\omega_{\psi',\mu}(g)\phi(z(v_{n-1},1))}\phi(z(v_{n-1},1))|z|^n \mathrm{d}v_{n-1}\mathrm{d}z\mathrm{d}a_{n-1}. \end{aligned}$$

Note that we have used the fact that the pairing on $\mathcal{W}(\pi_1, \psi')$

$$\langle W_1, W_1^\vee\rangle = \int\limits_{N_{n-1}(k')\backslash \mathrm{GL}_{n-1}(k')} W_1(\iota(a_{n-1}))\overline{W_1^\vee(\iota(a_{n-1}))}\mathrm{d}a_{n-1}$$

is indeed $\mathrm{GL}_n(k')$-invariant.

Recall that the inner product on $\mathcal{S}(k'_n)$ is given by

$$\langle \phi, \phi^\vee\rangle = \int\limits_{k'_n} \phi(x)\overline{\phi^\vee(x)}\mathrm{d}x,$$

where $\mathrm{d}x$ is the self-dual additive measure on k'_n. Thus

$$\langle \sigma(\iota(g))W_F, W_F\rangle = \langle \pi_1(g)W_f, W_f\rangle \overline{\langle \omega_{\psi',\mu}(g)\phi, \phi\rangle}.$$

Therefore identity (6.2.1) gives

$$\alpha(W_f, W_2, \phi) = \lambda(W_f, W_2, \mu, \phi)\overline{\lambda(W_f, W_2, \mu, \phi)}.$$

We are left to show that we can choose f, W_2 and ϕ, such that $\lambda(W_f, W_2, \mu, \phi) \neq 0$. For this, we choose an $\widetilde{f} \in \mathcal{C}_c^\infty(B_{n-1,-}(k'))$ and let $W_2 \in \mathcal{W}(\pi_2, \overline{\psi'})$ be the Whittaker function such that $\operatorname{supp} W_2|_{\mathrm{GL}_{n-1}(k')} \subset N_{n-1}(k')B_{n-1,-}(k')$ and $W_2|_{B_{n-1,-}(k')} = \widetilde{f}$. By definition,

$$\lambda(W_f, W_2, \mu, \phi) = \int\limits_{N_n(k)\backslash \mathrm{GL}_n(k)} W_f(g)W_2(g)\overline{\mu(\det g)\phi(e_n g)}|\det g|^{\frac{1}{2}} \mathrm{d}g.$$

In order that $\phi(e_n g) \neq 0$, the lower right entry of g is nonzero. Then by a similar argument as above, we have

$$\lambda(W_f, W_2, \mu, \phi) = \int\limits_{B_{n-1,-}} f(b)\widetilde{f}(b)\overline{\mu}(\det b)|\det b|^{\frac{1}{2}} \mathrm{d}b \times \int\limits_{k_{n-1}} \int\limits_{k^\times} \overline{\phi(z(v_{n-1}, 1))}|z|^{\frac{n}{2}} \mathrm{d}z \mathrm{d}v_{n-1}.$$

It is clear that we can choose f, $\widetilde{f}$ and ϕ such that $\lambda(W_f, W_2, \mu, \phi) \neq 0$. ☐

Proof of Theorem 4.4.1 in case (2). Theorem 4.4.1 (2) now follows from Lemma 6.2.1 and Proposition 6.2.2. ☐

6.3. **Some preparations for the proof in the supercuspidal case.** From now on, we always assume that k' is a non-archimedean local field and k is a quadratic field extension of k'. We need some preparations for the proof of Theorem 4.4.1 in the supercuspidal case.

Recall that we have defined the Caylay transform $\mathfrak{c} : \mathfrak{u}(V)(k') \to U(V)(k')$, c.f. (2.2.1). We extend it to a map $\mathfrak{y}(V)(k') \to Y(V)(k')$ which is the identity on $V^\vee$. We also call this map the Caylay transform.

Lemma 6.3.1. *Suppose that F is a test function on the general linear group and $\{f^V : V \in \mathsf{Herm}_n(k)\}$ is a collection of test functions on the unitary group $\mathrm{U}(V)$ for each V. Suppose that the support of the integral transform $F_\natural$ (resp. $f^V_\natural$) is contained in a neighborhood of $[1_n, 0, 0] \in X_n(k')$ (resp. $[1_n, 0] \in \mathrm{U}(V)(k')$) such that the Caylay transform is well-defined and is a homeomorphism. We then identify $F_\natural$ (resp. $f^V_\natural$ for each V) with a function on $\mathfrak{x}_n(k')$ (resp. $\mathfrak{y}(V)(k')$ for each V) via the Caylay transform. If F and $\{f^V : V \in \mathsf{Herm}_n(k)\}$ match, then so do $\eta(2)^{\frac{n(n-1)}{2}}\eta(-1)^{[\frac{n}{2}]}F_\natural$ and $\{f^V_\natural : V \in \mathsf{Herm}_n(k)\}$.*

Proof. We only need to compare the transfer factors on $X_n(k')$ and $\mathfrak{x}_n(k')$. Let $X \in \mathfrak{s}_n(k')$ be an element that lies in a sufficiently small neighborhood of $0 \in \mathfrak{s}_n(k')$. Then $\mathfrak{c}(X) = (1-X)^{-1}(1+X) = 2(1-X)^{-1} - 1$. It follows that

$$\begin{aligned} \det \begin{pmatrix} v^\vee \\ \vdots \\ v^\vee \mathfrak{c}(X)^{n-1} \end{pmatrix} &= 2^{\frac{n(n-1)}{2}} \det \begin{pmatrix} v^\vee \\ \vdots \\ v^\vee (1-X)^{-(n-1)} \end{pmatrix} \\ &= 2^{\frac{n(n-1)}{2}} \det(1-X)^{-(n-1)} (-1)^{[\frac{n}{2}]} \det \begin{pmatrix} v^\vee \\ \vdots \\ v^\vee X^{n-1} \end{pmatrix}. \end{aligned}$$

We temporarily denote by $\mathbf{t}_{\mathrm{gp}}$ and $\mathbf{t}_{\mathrm{alg}}$ the transfer factors on $X_n(k')$ and on $\mathfrak{x}_n(k')$ respectively. It follows that

$$\mathbf{t}_{\mathrm{gp}}([\mathfrak{c}(X), v^\vee, v]) = \eta(2)^{\frac{n(n-1)}{2}} \eta(-1)^{[\frac{n}{2}]} \mathbf{t}_{\mathrm{alg}}([X, v^\vee, v]).$$

☐

Lemma 6.3.2 ([Zha14a, Theorem 2.6], [Xue14, Theorem 5.2.4])**.** *For any $F \in \mathcal{C}_c^\infty(\mathfrak{x}_n(k'))$, there is a collection of test functions $\{f^V \in \mathcal{C}_c^\infty(\mathfrak{y}(V)(k')) : V \in \mathsf{Herm}_n(k)\}$ which matches F. The converse also holds.*

We make an identification $\mathfrak{x}_n \simeq \widehat{\mathfrak{x}_n}$ by $x = [\gamma, v^\vee, v] \mapsto {}^t x = [{}^t\gamma, {}^t v, {}^t v^\vee]$. A test function F on $\widehat{\mathfrak{x}_n}(k')$ is thus identified with a function on $\mathfrak{x}_n(k')$. We say that two regular semisimple orbits $x \in \widehat{\mathfrak{x}_n}(k')$ and $y \in \mathfrak{y}(V)(k')$ match if ${}^t x \in \mathfrak{x}_n(k')$ and $y \in \mathfrak{y}(V)(k')$ match. We say that the test functions F on $\widehat{\mathfrak{x}_n}(k')$ and f on $\mathfrak{y}(V)(k')$ match if they match when F is viewed as a function on $\mathfrak{x}_n(k')$ via the above identification.

Lemma 6.3.3 ([Zha14a, Theorem 4.17])**.** *If the functions $F \in \mathcal{C}_c^\infty(\mathfrak{x}_n(k'))$ and $f \in \mathcal{C}_c^\infty(\mathfrak{y}(V)(k'))$ match, the so do $\widehat{F}$ and $\epsilon(\frac{1}{2}, \eta, \psi')^{n(n+1)/2}\widehat{f}$, where $\widehat{\cdot}$ stands for the Fourier transform on the corresponding spaces.*

Lemma 6.3.4 ([Zha14a, Theorem 4.6])**.** *If $f_1, f_2 \in \mathcal{C}_c^\infty(\mathfrak{y}(V)(k'))$, then*

$$\int_{\mathfrak{y}(V)(k')} f_1(Y)O(Y,\widehat{f_2})\mathrm{d}Y = \int_{\mathfrak{y}(V)(k')} \widehat{f_1}(Y)O(Y,f_2)\mathrm{d}Y.$$

Remark 6.3.5. Despite the name "Lemma" that we put on the last three theorems, they are in fact very deep and hard and are the main ingredients in Zhang's proof of the global Gan–Gross–Prasad conjecture for $\mathrm{U}(n)\times\mathrm{U}(n+1)$. We proved Proposition 4.2.1 in [Xue14] by reducing it to Lemma 6.3.2.

Definition 6.3.6. Let f be a test function on the unitary group $\mathrm{U}(V)$. Let $r > m_2 > m_1 > m$ be four positive integers. We say that f is (m, m_1, m_2, r)-admissible (resp. sufficiently admissible) if there is a test function F on the general linear group that matches f and F is (m, m_1, m_2, r)-admissible (resp. sufficiently admissible).

We briefly recall some facts about the categorical quotients. We refer the readers to [Zha14a, Section 3.1] for a discussion of this notion.

Let $\mathfrak{u}(V) \to \mathfrak{u}(V)/\!/\,\mathrm{U}(V)$ and $\mathfrak{s}_n \to \mathfrak{s}_n/\!/\,\mathrm{GL}_n$ be categorical quotients. In this case, the quotients $\mathfrak{u}(V)/\!/\,\mathrm{U}(V) \simeq \mathfrak{s}_n/\!/\,\mathrm{GL}_n \simeq \mathrm{Spec}\,k'[X_1,\cdots,X_n]$. The morphism $\mathfrak{u}(V) \to \mathfrak{u}(V)/\!/\,\mathrm{U}(V) \simeq \mathrm{Spec}\,k'[X_1,\cdots,X_n]$ is given by $X_i = \mathrm{Tr}\wedge^i g, i = 1,\cdots,n,\ g \in \mathfrak{u}(V)$ and similarly for the morphism $\mathfrak{s}_n \to \mathfrak{s}_n/\!/\,\mathrm{GL}_n \simeq \mathrm{Spec}\,k'[X_1,\cdots,X_n]$. The induced maps $\mathfrak{u}(V)(k') \to (\mathfrak{u}(V)/\!/\,\mathrm{U}(V))(k')$ and $\mathfrak{s}_n(k') \to (\mathfrak{s}_n/\!/\,\mathrm{GL}_n)(k')$ are continuous.

We also need to consider the categorical quotients $\mathfrak{y}(V) \to \mathfrak{y}(V)/\!/\,\mathrm{U}(V)$ and $\mathfrak{x}_n \to \mathfrak{x}_n/\!/\,\mathrm{GL}_n$. In this case, the quotients $\mathfrak{y}(V)/\!/\,\mathrm{U}(V) \simeq \mathfrak{x}_n/\!/\,\mathrm{GL}_n \simeq \mathrm{Spec}\,k'[X_1,\cdots,X_n,Y_1,\cdots,Y_n]$. The morphisms $\mathfrak{y}(V) \to \mathfrak{y}(V)/\!/\,\mathrm{U}(V) \simeq \mathrm{Spec}\,k'[X_1,\cdots,X_n,Y_1,\cdots,Y_n]$ and $\mathfrak{y}(V)/\!/\,\mathrm{U}(V) \simeq \mathfrak{x}_n/\!/\,\mathrm{GL}_n \simeq \mathrm{Spec}\,k'[X_1,\cdots,X_n,Y_1,\cdots,Y_n]$ are given respectively by

$$X_i = \mathrm{Tr}\wedge^i g,\ Y_i = w^\tau g^{i-1}w, \quad [g,w] \in \mathfrak{y}(V) = \mathfrak{u}(V)\times V^\vee,$$

and

$$X_i = \mathrm{Tr}\wedge^i \gamma,\ Y_i = v^\vee\gamma^{i-1}v, \quad [\gamma, v^\vee, v] \in \mathfrak{x}_n = \mathfrak{s}_n \times k'_n \times k^{-,n}.$$

The induced maps $\mathfrak{y}(V)(k') \to (\mathfrak{y}(V)/\!/\,\mathrm{U}(V))(k')$ and $\mathfrak{x}_n(k') \to (\mathfrak{x}_n/\!/\,\mathrm{GL}_n)(k')$ are continuous.

We have $(\mathfrak{x}_n/\!/\,\mathrm{GL}_n)(k') = \mathfrak{x}_n(k')_{\mathrm{ss}}/\!/\,\mathrm{GL}_n(k')$ and

$$(\mathfrak{y}(V)/\!/\,\mathrm{U}(V))(k') = \coprod_{W\in\mathsf{Herm}_n(k)} \mathfrak{y}(W)(k')_{\mathrm{ss}}/\!/\,\mathrm{U}(W)(k'),$$

where the right hand side of each identity stands for the set of semisimple orbits (i.e. closed orbits) in $\mathfrak{x}_n(k')$ and $\mathfrak{y}(W)(k')$ respectively.

Lemma 6.3.7. *There is an open and closed neighborhood of $0 \in \mathfrak{s}_n(k')$ (resp. $0 \in \mathfrak{u}(V)(k')$) such that*

1. *it is invariant under the conjugation action of $\mathrm{GL}_n(k')$ (resp. $\mathrm{U}(V)(k')$);*
2. *the Caylay transform is well-defined and is a homeomorphism.*

Proof. We prove the case of $\mathfrak{u}(V)(k')$. The case of $\mathfrak{s}_n(k')$ is similar.

Let $\mathfrak{a}_0$ be a neighborhood of $0 \in (\mathfrak{u}(V)/\!/\,\mathrm{U}(V))(k')$. We can take this neighborhood to be small enough so that the Caylay transform is well-defined on the inverse image of this neighborhood in $\mathfrak{u}(V)(k')$. The inverse image of $\mathfrak{a}_0$ is then the desired neighborhood. □

6.4. **The supercuspidal case.**

Lemma 6.4.1. *Let π_1 and π_2 be two irreducible supercuspidal representations of $\mathrm{U}(V)(k')$ which satisfy*

$$\mathrm{Hom}_{\mathrm{U}(V)(k')}(\pi_1\otimes\pi_2\otimes\overline{\omega_{\psi',\mu}},\mathbb{C}) \neq 0.$$

Then there is a $\mathrm{U}(V)(k')$ invariant neighborhood $\mathfrak{a}$ of $[0,0] \in \mathfrak{y}(V)(k') = \mathfrak{u}(V)(k')\times V^\vee$ and a locally constant locally integrable function $\Theta \in \mathcal{C}_c^\infty(\mathfrak{y}(V)(k'))$ supported in $\mathfrak{a}$, such that

1. *the Caylay transform $\mathfrak{c}$ is well-defined and is a homeomorphism on $\mathfrak{a}$;*
2. *$\Theta(0) = 1$;*

(3) *if f is a test function on the unitary group such that* $\operatorname{supp} f_\natural \subset A = \mathfrak{c}(\mathfrak{a})$, *then*

$$J_{\pi_1,\pi_2}(f) = \int\limits_{\mathfrak{y}(V)(k')} f_\natural(y) O(y,\Theta)\mathrm{d}y,$$

where we have identified $f_\natural$ with a function on $\mathfrak{y}(V)(k')$ via the Caylay transform.

Proof. Let $\varphi_1^\circ \in \pi_1$, $\varphi_2^\circ \in \pi_2$ and $\phi^\circ \in \mathcal{S}(\mathbf{L}(k'))$ where $\mathbf{L} \subset \operatorname{Res} V^\vee$ is a Lagrangian subspace and $\omega_{\psi',\mu}$ is realized on $\mathcal{S}(\mathbf{L}(k'))$. Assume that $\alpha(\varphi_1^\circ, \varphi_2^\circ, \phi^\circ) = 1$. Let $\widetilde{\Theta} \in \mathcal{C}_c^\infty(Y(V)(k')) = \mathcal{C}_c^\infty(\mathrm{U}(V)(k') \times V^\vee)$ defined by

$$\widetilde{\Theta}(g,w) = \int\limits_{\mathrm{U}(V)(k')} \langle \pi_1(h^{-1})\varphi_1^\circ, \varphi_1^\circ\rangle \langle \pi_2(h^{-1}g)\varphi_2^\circ, \varphi_2^\circ\rangle \overline{\left(\overline{\omega_{\psi',\mu}(h)\phi^\circ} \otimes \phi^\circ\right)^\ddagger (w)}\mathrm{d}h, \quad g \in \mathrm{U}(V)(k'),\ w \in V^\vee.$$

This function is compactly supported because π_1 and π_2 are supercuspidal.

We prove in Appendix C that the function on $Y(V)(k')$ defined by

$$y \mapsto O(y, \widetilde{\Theta})$$

is locally integrable as π_1 and π_2 are both supercuspidal. Moreover, $\widetilde{\Theta}(1,0) = 1$ and

$$J_{\pi_1,\pi_2}(f) = \int\limits_{Y(V)(k')} f_\natural(y) O(y,\widetilde{\Theta})\mathrm{d}y.$$

Let $\mathfrak{a}_0$ be a neighborhood of $0 \in (\mathfrak{u}(V)/\!/\,\mathrm{U}(V))(k')$ as in Lemma 6.3.7 and $\mathfrak{a}$ the inverse image of $\mathfrak{a}_0 \times k'_n \subset (\mathfrak{y}(V)/\!/\,\mathrm{U}(V))(k')$ in $\mathfrak{y}(V)(k')$. Put $A = \mathfrak{c}(\mathfrak{a}) \subset Y(V)(k')$. Let $\mathbf{1}_A$ be the characteristic function of A and $\Theta = \widetilde{\Theta} \cdot \mathbf{1}_A$. Then $\Theta \in \mathcal{C}_c^\infty(Y(V)(k'))$ since $\widetilde{\Theta}$ is compactly supported. Note that we still have $\Theta(1,0) = 1$. We identify Θ with a function on $\mathfrak{y}(V)(k')$ via the Caylay transform. Then Θ is the desired function in the lemma. □

Lemma 6.4.2. *There is a sufficiently admissible test function F on the general linear group and a matching test function f on the unitary group* $\mathrm{U}(V)$, *such that $O(\xi_-, \widehat{F_\natural}) \neq 0$ and $f_\natural$ is supported in A.*

Proof. Let $\mathfrak{a}_0$ be a neighborhood of $0 \in (\mathfrak{u}(V)/\!/\,\mathrm{U}(V))(k') = (\mathfrak{s}_n/\!/\,\mathrm{GL}_n)(k')$ as in Lemma 6.3.7 and $\mathfrak{a} \subset \mathfrak{y}(V)(k')$ the inverse image of $\mathfrak{a}_0 \times k'_n \subset (\mathfrak{y}(V)/\!/\,\mathrm{U}(V))(k')$. Then $A = \mathfrak{c}(\mathfrak{a}) \subset Y(V)(k')$. Let F be an (m, m_1, m_2, r)-admissible test function on the general linear group. We can take (m, m_1, m_2, r) to be sufficiently large such that the support of $F_\natural$ is contained in a $B \times k'_n \times k^{-,n}$ where B is a small neighborhood of $1 \in S_n(k')$ and thus we can identify $F_\natural$ with a function on $\mathfrak{x}_n$. We require that the image of $\operatorname{supp} F_\natural$ in $(\mathfrak{x}_n/\!/\,\mathrm{GL}_n)(k')$ is contained in $\mathfrak{a}_0 \times k'_n$. We also require that $O(\xi_-, \widehat{F_\natural}) \neq 0$. Such an F exists.

Let f_0^V be a test function on $\mathrm{U}(V)$ that matches F. Let $f_\natural^V = f_{0,\natural}^V \cdot \mathbf{1}_{\mathfrak{a}} \in \mathcal{C}_c^\infty(\mathfrak{y}(V)(k'))$. Then $f_\natural^V$ has the same regular semisimple orbital integrals as f_0^V since the integral of $f_{0,\natural}^V$ on any regular semisimple orbit which is not contained in $\mathfrak{a}$ is zero by the support condition of F. We identify $f_\natural^V$ with a function on $Y(V)(k')$ supported in A via the Caylay transform. Let $f^V \in \mathcal{C}_c^\infty(\mathrm{U}(V)(k') \times \mathrm{U}(V)(k') \times \mathbf{L}(k') \times \mathbf{L}(k'))$ be a test function on the unitary group so that its image under the integral transform (4.2.3) is $f_\natural^V$. We can find such an f^V since the integral transform (4.2.3) is surjective (c.f. [Xue14, Lemma 4.1.2]). The function f^V is the desired test function on the unitary group. □

Proof of Theorem 4.4.1 in case (3). Let F and f be sufficiently admissible matching test functions as in Lemma 6.4.2. It follows from Lemma 6.3.4 that

$$J_{\pi_1,\pi_2}(f) = \int\limits_{\mathfrak{y}(V)(k')} \check{\Theta}(y) O(y, \widehat{f_\natural})\mathrm{d}y,$$

where $\check{\Theta}$ is the inverse Fourier transform of Θ. Therefore by Lemma 6.3.3, for regular semisimple matching orbits $y \in \mathfrak{y}(V)(k')$ and $x \in \widehat{\mathfrak{x}_n}(k')$, we have

$$O(y, \widehat{f_\natural}) = \eta(2)^{\frac{n(n-1)}{2}} \eta(-1)^{[\frac{n}{2}]} \epsilon\left(\frac{1}{2}, \eta, \psi'\right)^{-\frac{n(n+1)}{2}} \mathfrak{t}({}^t x) O(x, \widehat{F_\natural}).$$

We need to compare $\mathbf{t}({}^t x)$ and $\widehat{\mathbf{t}}(x)$. Suppose $x = [\gamma, v, v^\vee] \in \widehat{\mathfrak{x}_n}(k')$ and $y = [\xi, w^\vee] \in \mathfrak{y}(V)(k')$ match. We first note that

$$\det(v^\vee \gamma^{i+j} v)_{0 \le i,j \le n-1} = \det(\langle w^\vee \xi^i, w^\vee \xi^j \rangle_{V^\vee})_{0 \le i,j \le n-1} \in k^-.$$

The right hand side and $\operatorname{disc} V^\vee$ differ by an element in $N_{k/k'} k^\times$. Note that both of them are purely imaginary elements in k^-. It follows from the definitions of $\mathbf{t}$ and $\widehat{\mathbf{t}}$ that

$$\begin{aligned} \frac{\mathbf{t}({}^t x)}{\widehat{\mathbf{t}}(x)} &= \mu(\det(v, v\gamma, \cdots, v\gamma^{n-1})) \cdot \mu\left(\det\begin{pmatrix} v^\vee \\ \vdots \\ v^\vee \gamma^{n-1} \end{pmatrix}\right) \cdot \mu(\delta)^{-n(n+1)} \\ &= \mu(\operatorname{disc} V)^{-1} \mu(\delta)^{-n(n+1)}. \end{aligned}$$

For any subset $\mathcal{X} \subset \widehat{\mathfrak{x}_n}(k')$, we put ${}^t\mathcal{X} = \{ {}^t x \mid x \in \widehat{\mathfrak{x}_n}(k') \} \subset \mathfrak{x}_n(k')$. Since $\check{\Theta}$ is compactly supported, we may choose a sufficiently large compact neighborhood $\mathcal{X} \subset \widehat{\mathfrak{x}_n}(k')$ of 0, such that the image of ${}^t\mathcal{X}$ in $(\mathfrak{x}_n // \mathrm{GL}_n)(k') = (\mathfrak{y}(V) // \mathrm{U}(V))(k')$ contains the image of $\operatorname{supp} \check{\Theta}$ in $(\mathfrak{y}(V) // \mathrm{U}(V))(k')$. Thus we may take F (and hence f) sufficiently admissible so that $\widehat{\mathbf{t}}(x) O(x, \widehat{F_\natural})$ is a constant and equals $\eta(-1)^{[\frac{n}{2}]} \mu(\delta)^{\frac{n(n+1)}{2}} O(\xi_-, \widehat{F_\natural})$ if $x \in \mathcal{X}$. Thus if $y \in \operatorname{supp} \check{\Theta}$, then $O(y, \widehat{f_\natural})$ is a constant. Therefore

$$\begin{aligned} J_{\pi_1,\pi_2}(f) &= |\delta|^{-\frac{n}{2}} \eta(2)^{\frac{n(n-1)}{2}} \epsilon\left(\frac{1}{2}, \eta, \psi'\right)^{-\frac{n(n+1)}{2}} \mu(\operatorname{disc} V)^{-1} \mu(\delta)^{-\frac{n(n+1)}{2}} O(\xi_-, \widehat{F_\natural}) \int_{y \in \mathfrak{y}(V)(k')} \check{\Theta}(y) \mathrm{d}y \\ &= |\delta|^{-\frac{n}{2}} \eta(2)^{\frac{n(n-1)}{2}} \epsilon\left(\frac{1}{2}, \eta, \psi'\right)^{-\frac{n(n+1)}{2}} \mu(\operatorname{disc} V)^{-1} \mu(\delta)^{-\frac{n(n+1)}{2}} O(\xi_-, \widehat{F_\natural}) \neq 0, \end{aligned}$$

where in the second equality, we have used the fact that $\Theta(0) = 1$.

It then follows from Proposition 5.7.1 that

$$J_{\pi_1,\pi_2}(f) = |\delta|^{-d_n} \epsilon\left(\frac{1}{2}, \eta, \psi'\right)^{-\frac{n(n+1)}{2}} \Omega_{\Pi_1}(\delta)^{-1} \Omega_{\Pi_2}(\delta)^{-1} \mu(\operatorname{disc} V)^{-1} \eta(2)^{\frac{n(n-1)}{2}} I_{\Pi_1,\Pi_2}(F) \neq 0.$$

□

Appendix A. Theta correspondences for unitary groups

We recall briefly the relevant facts on theta correspondences for unitary groups. The readers may refer to [MVW87, How89, Wal90, GI, Yam14, GQT, GTa, GTb] and the references therein for more details.

A.1. **Local theory.** Let k' be a local field and k a quadratic etale algebra over k', i.e. either $k = k' \times k'$ or k is a quadratic field extension of k'. Let τ be the nontrivial element in $\mathrm{Gal}(k/k')$. Let η be the quadratic character of $k'^\times$ associated to the extension k/k' by the local class field theory. Let $\mu : k^\times \to \mathbb{C}^\times$ be a character such that $\mu|_{k'^\times} = \eta$. Let $\psi' : k' \to \mathbb{C}^\times$ be a non-trivial additive character.

Let $(W, \langle -, - \rangle_W)$ (resp. $(V, \langle -, - \rangle_V)$) be a skew-hermitian (resp. hermitian) space over k of dimension m (resp. n). We consider the symplectic space $\operatorname{Res} W \otimes V$ of dimension $2mn$ over k' whose underline vector space is $W \otimes V$, viewed as a space over k' and whose symplectic form is given by

$$\langle\langle -, - \rangle\rangle = \frac{1}{2} \operatorname{Tr}_{k/k'} \langle -, - \rangle_W \otimes \langle -, - \rangle_V.$$

We then have an embedding $\mathrm{U}(W) \times \mathrm{U}(V) \to \mathrm{Sp}(W \otimes V)$. Let $\mathrm{Mp}(W \otimes V)$ be the metaplectic group which is an extension

$$1 \to \mathbb{C}^\times \to \mathrm{Mp}(W \otimes V)(k') \to \mathrm{Sp}(W \otimes V)(k') \to 1.$$

We choose the pair of characters $(\mu^{\dim V}, \mu^{\dim W})$ and split the map $\mathrm{Mp}(W \otimes V)(k') \to \mathrm{Sp}(W \otimes V)(k')$ over $\mathrm{U}(W)(k') \times \mathrm{U}(V)(k')$, This means that we fix a homomorphism $\mathrm{U}(W)(k') \times \mathrm{U}(V)(k') \to \mathrm{Mp}(W \otimes V)(k')$ over $\mathrm{Sp}(W \otimes V)(k')$ using $(\mu^{\dim V}, \mu^{\dim W})$ as in [Kud94]. Then we get a Weil representation $\omega_{W,V,\psi',\mu}$ of $\mathrm{U}(W)(k') \times \mathrm{U}(V)(k')$. As the notation suggests, this representation depends on the character ψ', μ and the spaces W, V. To simply notation, we usually omit some (or all) subscripts when there is no confusion. If $V = k$, the one dimensional hermitian space whose hermitian form is given by $\langle x, y \rangle = x^\tau y$, then we speak of the Weil representation of $\mathrm{U}(W)(k')$.

Let π be an irreducible admissible representation of $\mathrm{U}(W)(k')$. The π-isotypic part of $\omega_{W,V,\psi',\mu}$ is of the form $\pi \boxtimes \Theta_{W,V,\psi',\mu}(\pi)$ where $\Theta_{W,V,\psi',\mu}(\pi)$ is a smooth representation of $\mathrm{U}(V)(k')$ of finite length. Let $\theta_{W,V,\psi',\mu}(\pi)$ be the maximal semisimple quotient of $\Theta_{W,V,\psi',\mu}(\pi)$. We remark here that even though we talk about $\theta_{W,V,\psi',\mu}(\pi)$, the following discussion is valid for the similarly defined representation $\theta_{V,W,\psi',\mu}(\sigma)$ where σ is an irreducible admissible representation of $\mathrm{U}(V)(k')$.

Theorem A.1.1. (1) *The representation $\theta_{W,V,\psi',\mu}(\pi)$ is irreducible.*

(2) *$\pi \simeq \pi'$ if and only if $\theta_{W,V,\psi',\mu}(\pi) \simeq \theta_{W,V,\psi',\mu}(\pi')$.*

This theorem is known as the Howe duality conjecture. If k' is archimedean, then the Howe duality conjecture is proved by Howe [How89]. If k' is non-archimedean and the residue characteristic of k' is not two, it is proved by Waldspurger [Wal90]. It is recently proved for non-archimedean local fields of characteristic not two and arbitrary residue characteristic and for all unitary (and symplectic-orthogonal) dual pairs by Gan–Takeda [GTa, GTb].

A.2. **Explicit local theta liftings.** We keep the notation from the previous subsection. Let $\mathbf{L} \subset \mathrm{Res}(W \otimes V)^\vee$ be a Lagrangian subspace of $\mathrm{Res}(W \otimes V)^\vee$, where $\mathrm{Res}(W \otimes V)^\vee$ is the dual space of $\mathrm{Res}\, W \otimes V$. Then we can realize the Weil representation $\omega_{W,V,\psi',\mu}$ on $\mathcal{S}(\mathbf{L}(k'))$, the space of Schwartz functions on $\mathbf{L}(k')$. There is a natural pairing on $\mathcal{S}(\mathbf{L}(k'))$

$$\langle \phi_1, \phi_2 \rangle = \int_{\mathbf{L}(k')} \phi_1(x)\overline{\phi_2(x)}\mathrm{d}x.$$

Let π be an irreducible admissible unitary representation of $\mathrm{U}(W)(k')$. We assume that π is a local component of an irreducible unitary cuspidal automorphic representation. This assumption is innocuous to the applications in this paper. Let $\langle -, - \rangle$ be the hermitian form on π. Let $\varphi_1, \varphi_2 \in \pi$, $\phi_1, \phi_2 \in \mathcal{S}(\mathbf{L}(k'))$. Define

$$Z(\varphi_1, \varphi_2, \phi_1, \phi_2) = \int_{\mathrm{U}(V)(k')} \overline{\langle \pi(g)\varphi_1, \varphi_2 \rangle} \langle \omega_{\psi',\mu}(g, h)\phi_1, \phi_2 \rangle \mathrm{d}g. \tag{A.2.1}$$

Lemma A.2.1 ([GI14, Theorem 9.1; Yam14, Lemma 7.2])**.** *Assume that $n - m = 0$ or 1 and that π is tempered.*

(1) *The integral (A.2.1) is convergent.*

(2) *If $\theta_{W,V,\psi',\mu}(\pi) \neq 0$, then the integral (A.2.1) is not identically zero.*

Lemma A.2.2. *Assume that $n - m = 0$ or 1 and that π is tempered. The integral (A.2.1) gives a matrix coefficient of $\theta_{W,V,\psi',\mu}(\pi)$. More precisely, fix a nonzero $\mathrm{U}(W)(k') \times \mathrm{U}(V)(k')$ equivariant map*

$$\ell : \overline{\pi} \otimes \omega_{W,V,\psi',\mu} \to \theta_{W,V,\psi',\mu}(\pi).$$

Then there is a pairing on $\theta_{W,V,\psi',\mu}(\pi)$ such that

$$Z(\varphi_1, \varphi_2, \phi_1, \phi_2) = \langle \sigma(h)\ell(\varphi_1, \phi_1), \ell(\varphi_2, \phi_2) \rangle.$$

Proof. One argues as in [GI11, Lemma 5.6, Lemma 5.7]. Even though the authors studied only the case of theta lifting from GO(4) to GSp(4), the method is completely general. Note that we make use of Theorem A.1.1 here instead of the results in [GI11, Appendix A]. □

A.3. **A local seesaw identity.** In this subsection, we assume that $n = m + 1$. Suppose that V admits an orthogonal decomposition $V = V_0 + k$ where V_0 is an m-dimensional hermitian space. Let $\iota : \mathrm{U}(V_0) \to \mathrm{U}(V)$ be the embedding corresponding to the inclusion $V_0 \subset V$. Let π (resp. σ) be an irreducible admissible tempered representation of $\mathrm{U}(W)(k')$ (resp. $\mathrm{U}(V_0)(k')$). Let $\varphi_1, \varphi_2 \in \pi$, $f_1, f_2 \in \sigma$ and $\phi_1, \phi_2 \in \mathcal{S}(\mathbf{L}(k'))$ where $\mathbf{L} \subset \mathrm{Res}(W \otimes V)^\vee$ is a Lagrangian subspace. Consider the integral

$$\int_{\mathrm{U}(W)(k')} \int_{\mathrm{U}(V_0)(k')} \langle \pi(g)\varphi_1, \varphi_2 \rangle \langle \sigma(h) f_1, f_2 \rangle \langle \omega_{W,V,\psi',\mu}(g, \iota(h))\phi_1, \phi_2 \rangle \mathrm{d}h\mathrm{d}g, \tag{A.3.1}$$

where $\langle -, - \rangle$ stands for various hermitian inner products on the corresponding spaces.

Lemma A.3.1. *The integral (A.3.1) is absolutely convergent.*

Proof. The lemma is proved in the same way as [GI11, Lemma 9.1]. We omit the details. Note that we assume that π and σ are both tempered while [GI11, Lemma 9.1] does not. The Kim–Shahidi estimate is used in [GI11, Lemma 9.1] instead of the temperedness. $\square$

A.4. **Global theory.** Now let k' be a number field and k a quadratic etale algebra over k'. Let τ be the non-trivial element in $\mathrm{Gal}(k/k')$. We fix a non-trivial additive character $\psi' : k'\backslash\mathbb{A}' \to \mathbb{C}$ and multiplicative character $\mu : k^\times\backslash\mathbb{A}^\times \to \mathbb{C}^\times$ such that $\mu|_{\mathbb{A}'^\times} = \eta$, the quadratic character associated to the extension k/k' by the global class field theory.

Let $(W, \langle -,-\rangle_W)$ (resp. $(V, \langle -,-\rangle_V)$) be a skew-hermitian (resp. hermitian) space over k of dimension m (resp. n). Analogue to the local case, we define the sympletic space $\operatorname{Res} W \otimes V$ of dimension $2mn$ and a Lagrangian subspace $\mathbf{L} \subset \operatorname{Res}(W \otimes V)^\vee$ over k'. By taking the restricted tensor product of the local Weil representations, we get a global Weil representation $\omega_{W,V,\psi',\mu}$, realized on the space $\mathcal{S}(\mathbf{L}(\mathbb{A}'))$ of Schwartz functions on $\mathbf{L}(\mathbb{A}')$. We form the theta series on $\mathrm{U}(W)(\mathbb{A}') \times \mathrm{U}(V)(\mathbb{A}')$ by

$$\theta_{W,V,\psi',\mu}(g,h,\phi) = \sum_{x\in\mathbf{L}(k')} \omega_{W,V,\psi',\mu}(g,h)\phi(x), \quad g \in \mathrm{U}(W)(\mathbb{A}'),\ h \in \mathrm{U}(V)(\mathbb{A}'),\ \phi \in \mathcal{S}(\mathbf{L}(k')).$$

Let π be an irreducible cuspidal automorphic form on $\mathrm{U}(W)(\mathbb{A}')$. By the theta lifting of π, denoted by $\theta_{W,V,\psi',\mu}(\pi)$, we mean the automorphic representation of $\mathrm{U}(V)(\mathbb{A}')$ generated by the functions of the form

$$h \mapsto \theta_{W,V,\psi',\mu}(\varphi,\phi)(h) = \int_{\mathrm{U}(W)(k')\backslash \mathrm{U}(W)(\mathbb{A}')} \overline{\varphi(g)}\theta_{W,V,\psi',\mu}(g,h,\phi)\mathrm{d}g, \quad \varphi \in \pi$$

Proposition A.4.1 ([GRS93, Proposition 1.2])**.** *If $\theta_{W,V,\psi',\mu}(\pi)$ is a cuspidal automorphic representation of $\mathrm{U}(V)(\mathbb{A}')$, then it is irreducible and is isomorphic to the restricted tensor product $\otimes'\theta_{W_v,V_v,\psi'_v,\mu_v}(\pi_v)$. Moreover, the theta lifting of $\theta_{W,V,\psi'^{-1},\mu^{-1}}(\pi)$ back to $\mathrm{U}(W)(\mathbb{A}')$ is isomorphic to π.*

If $V = k$, the one dimensional hermitian space whose hermitian form is given by $\langle x, y\rangle = x^\tau y$, then we speak of the Weil representation of $\mathrm{U}(W)(\mathbb{A}')$. It is realized on $\mathcal{S}(\mathbf{L}(\mathbb{A}'))$ where $\mathbf{L} \subset \operatorname{Res} W^\vee$ is a Lagrangian subspace. In this case, we suppress the subscript V from the notation.

Suppose that $V = V_0 + k$, where V_0 is an $(n-1)$-dimension hermitian space. Let $\mathbf{L}_0$ be a Lagrangian subspace of $\operatorname{Res}(W \otimes V_0)^\vee$ and $\mathbf{L}_W$ a Lagrangian subspace of $\operatorname{Res} W^\vee$. Then $\mathbf{L} = \mathbf{L}_0 + \mathbf{L}_W$ is a Lagrangian subspace of $\operatorname{Res}(W \otimes V)^\vee$. The Weil representation $\omega_{W,V,\psi',\mu}$ is realized on $\mathcal{S}(\mathbf{L}(\mathbb{A}'))$. Let $\iota : \mathrm{U}(V_0) \to \mathrm{U}(V)$ be the embedding corresponding to the inclusion $V_0 \subset V$. Then we have a $\mathrm{U}(W)(\mathbb{A}') \times \iota(\mathrm{U}(V_0)(\mathbb{A}'))$ equivariant isomorphism

$$\mathcal{S}(\mathbf{L}(\mathbb{A}')) \simeq \mathcal{S}(\mathbf{L}_0(\mathbb{A}')) \otimes \mathcal{S}(\mathbf{L}_W(\mathbb{A}')),$$

where $\iota(\mathrm{U}(V_0)(\mathbb{A}'))$ acts on $\mathcal{S}(\mathbf{L}_W(\mathbb{A}'))$ trivially. If $\phi = \phi_0 \otimes \phi_W$ with $\phi_0 \in \mathcal{S}(\mathbf{L}_0(\mathbb{A}'))$ and $\phi_W \in \mathcal{S}(\mathbf{L}_W(\mathbb{A}'))$, then for $g \in \mathrm{U}(W)(\mathbb{A}')$ and $h \in \mathrm{U}(V_0)(\mathbb{A}')$, we have

$$\theta_{W,V,\psi',\mu}(g,\iota(h),\phi_0\otimes\phi_W) = \theta_{W,V_0,\psi',\mu}(g,h,\phi_0)\theta_{W,\psi',\mu}(g,\phi_W).$$

Suppose $n-m = 0$ or 1 and π is an irreducible cuspidal tempered automorphic representation of $\mathrm{U}(W)(\mathbb{A}')$. We record the Rallis inner product formulae in this setting.

We use the Tamagawa measure $\mathrm{d}g$ (resp. $\mathrm{d}h$) of $\mathrm{U}(W)(\mathbb{A}')$ (resp. $\mathrm{U}(V)(\mathbb{A}')$). We choose a measure $\mathrm{d}g_v$ (resp. $\mathrm{d}h_v$) on $\mathrm{U}(W)(k'_v)$ (resp. $\mathrm{U}(V)(k'_v)$) for each place v so that $\mathrm{d}g = \prod \mathrm{d}g_v$ (resp. $\mathrm{d}h = \prod \mathrm{d}h_v$).

For any multiplicative character $\chi = \otimes\chi_v : k^\times\backslash\mathbb{A}^\times \to \mathbb{C}^\times$ and any place v of k', put

$$b_v(s,\chi_v) = \prod_{i=1}^m L(2s+i, \chi_v|_{k'_v}\eta_v^{n-i}), \quad b(s,\chi) = \prod_v b_v(s,\chi_v).$$

Suppose $\varphi = \otimes\varphi_v \in \pi$ and $\phi = \otimes\phi_v \in \mathcal{S}(\mathbf{L}(\mathbb{A}'))$ are factorizable. Recall that we have defined the integral $Z_v(\varphi_v,\varphi_v,\phi_v,\phi_v)$, c.f. (A.2.1). Put $s_0 = \frac{n-m}{2}$ and

$$Z_v^\natural(\varphi_v,\varphi_v,\phi_v,\phi_v) = \frac{b_v(s_0,\mu_v^n)}{L(s_0+\frac{1}{2},\pi_v\times\mu_v^n)} Z_v(\varphi_v,\varphi_v,\phi_v,\phi_v),$$

where $L(s,\pi_v\times\mu_v^n)$ is the local L-factor defined by the doubling method [LR05].

The following Rallis inner product formula is a culmination of the work of many people. It is due to [Yam14, GQT] in this case. See [GQT, Introduction] for some history of this formula.

Theorem A.4.2. *Assume that $\theta_{W,V,\psi',\mu}(\pi)$ is cuspidal. Then*

$$\int\limits_{\mathrm{U}(V)(k')\backslash \mathrm{U}(V)(\mathbb{A}')} |\theta_{W,V,\psi',\mu}(\varphi,\phi)(h)|^2 \mathrm{d}h = c \times \frac{L(s_0+\frac{1}{2},\pi\times\mu^n)}{b(s_0,\mu^n)} \prod_v Z_v^{\natural}(\varphi_v,\varphi_v,\phi_v,\phi_v),$$

where $L(s,\pi\times\mu^n)$ is the L-function defined by the doubling method [LR05] *and*

$$c = \begin{cases} 1, & n-m=0; \\ 2, & n-m=1. \end{cases}$$

Appendix B. Basic estimates

We recall some basic estimates in this appendix. We follow [II10, Section 4] rather closely. In this appendix, k' is always a local field. For any algebraic group G over k', we denote by G instead of $G(k')$ for its group of k'-points.

Let G be a reductive group over k'. Let A_0 be a maximal split subtorus of G, M_0 the centralizer of A_0 in G. We fix a minimal parabolic subgroup P_0 of G with the Levi decomposition $P_0 = M_0N_0$. Let Δ be the set of simple roots of (P_0, A_0). Let δ_{P_0} be the modulus character of P_0. Let

$$A_0^+ = \{a \in A_0 \mid |\alpha(a)| \leq 1 \text{ for all } \alpha \in \Delta\}.$$

We fix a special maximal compact subgroup K of G. Then we have a Cartan decomposition $G = KA_0^+K$. We also have the Iwasawa decomposition

$$G = M_0N_0K, \quad g = m_0(g)n_0(g)k_0(g).$$

For any function $f \in L^1(G)$,

$$\int\limits_G f(g)\mathrm{d}g = \int_{A_0^+} \nu(m) \iint\limits_{K\times K} f(k_1mk_2)\mathrm{d}k_1\mathrm{d}k_2\mathrm{d}m, \tag{B.0.1}$$

where $\nu(m)$ is a positive function on A_0^+ such that there is a constant $A > 0$ with

$$A^{-1}\delta_{P_0}(m)^{-1} \leq \nu(m) \leq A\delta_{P_0}(m)^{-1}. \tag{B.0.2}$$

Let $\mathbf{1}$ be the trivial representation of M_0 and let $e(g) = \delta_{P_0}(m_0(g))^{\frac{1}{2}}$ be an element in $\mathrm{Ind}_{P_0}^G \mathbf{1}$. Let $\mathrm{d}k$ be the measure on K such that $\mathrm{vol}\, K = 1$. We define the Harish–Chandra function

$$\Xi(g) = \int_K e(kg)\mathrm{d}k = \int_K \delta_{P_0}(m_0(kg))^{\frac{1}{2}}\mathrm{d}k.$$

This function is bi-K-invariant.

The Harish-Chandra function Ξ has the following property. For any $g_1, g_2 \in G$, we have

$$\int\limits_K \Xi(g_1kg_2)\mathrm{d}k = \Xi(g_1)\Xi(g_2). \tag{B.0.3}$$

We define a height function on G. We fix an embedding $\tau : G \to \mathrm{GL}_n$. Write $g \in G$ as $g = (a_{ij})$ and $g^{-1} = (b_{ij})$. Define

$$\|g\| = \sup\{|a_{ij}|, |b_{ij}| \mid 1 \leq i, j \leq n\}, \tag{B.0.4}$$

and $\sigma(g) = \log\|g\|$.

There are constants C_1, C_2 and a positive real number d such that

$$C_1\delta_0(m)^{\frac{1}{2}} \leq \Xi(m) \leq C_2\delta_0(m)^{\frac{1}{2}}(1+\sigma(g))^d. \tag{B.0.5}$$

Now let π be an irreducible admissible tempered representation of G. Let Φ be a matrix coefficient of G. Then there are constants A and B such that

$$|\Phi(g)| \leq A\Xi(g)(1+\sigma(g))^B. \tag{B.0.6}$$

This is called the weak inequality.

Appendix C. Complements on [Xue14]

C.1. **Statement of the results.** In this appendix, we strengthen [Xue14, Proposition 6.1.2]. This will enable us to weaken some local conditions in [Xue14, Theorem 1.1.1]. We keep the notation from the main body of this paper. Fix a non-archimedean place v of k'. We are always in the local situation and will work with objects over k'_v. So we suppress all the subscripts v from the notation in this subsection. Thus k' is a non-archimedean local field and k is a quadratic etale k' algebra. We let A_0 be the maximum split torus in $\mathrm{U}(V)(k')$, A_0^+ as in Appendix B and K a special maximum compact subgroup of $\mathrm{U}(V)(k')$. Then the Cartan decomposition $\mathrm{U}(V)(k') = KA_0^+K$ holds.

Let π_1 and π_2 be irreducible tempered representations of $\mathrm{U}(V)(k')$. Let $\varphi_1 \in \pi_1$, $\varphi_2 \in \pi_2$ and $\phi \in \mathcal{S}(\mathbf{L}(k'))$. Let $\widetilde{\Theta}$ be a function on $Y(V)(k') = \mathrm{U}(V)(k') \times V^\vee$ defined by

$$\widetilde{\Theta}(g, w) = \int\limits_{\mathrm{U}(V)(k')} \langle \pi_1(h^{-1})\varphi_1, \varphi_1\rangle \langle \pi_2(h^{-1}g)\varphi_2, \varphi_2\rangle \overline{\left(\overline{\omega_{\psi',\mu}(h)\phi} \otimes \phi\right)^{\ddagger}(w)} \mathrm{d}h, \quad g \in \mathrm{U}(V)(k'),\ w \in V^\vee.$$

This integral is convergent since π_1 and π_2 are both tempered (c.f. Proposition 1.1.1 and its proof in Appendix D.1).

Define the orbital integral of $\widetilde{\Theta}$ defined by

$$O([g, w], \widetilde{\Theta}) = \int\limits_{\mathrm{U}(V)(k')} \widetilde{\Theta}([g, w].h)\mathrm{d}h.$$

If π_1 and π_2 are both supercuspidal, then $\widetilde{\Theta}$ is compactly supported and $O([g, w], \widetilde{\Theta})$ is defined for all regular semisimple $[g, w]$. The following proposition shows that if $k = k' \times k'$, then the orbital integral is absolutely convergent except for a measure zero subset of $Y(V)(k')$.

Proposition C.1.1. *Assume that either $k = k' \times k'$ or π_1 and π_2 are both supercuspidal. Then as a function on $Y(V)(k')$, $O([g, w], \widetilde{\Theta})$ is locally integrable. This means that for any $f \in \mathcal{C}_c^\infty(Y(V)(k'))$, the integral*

$$\int\limits_{Y(V)(k')} f(g, w)|O([g, w], \widetilde{\Theta})|\mathrm{d}g\mathrm{d}w$$

is convergent.

Corollary C.1.2. *Assume that either $k = k' \times k'$ or π_1 and π_2 are both supercuspidal. Suppose that* $\mathrm{Hom}_{\mathrm{U}(V)(k')}(\pi_1 \otimes \pi_2 \otimes \overline{\omega_{\psi',\mu}}, \mathbb{C}) \neq 0$.

(1) *The distribution J_{π_1,π_2} is represented by $O([g, w], \widetilde{\Theta})$. More precisely, for any test function f on the unitary group, we have*

$$J_{\pi_1,\pi_2}(f) = \int\limits_{Y(V)(k')} f_\natural(g, w)O([g, w], \widetilde{\Theta})\mathrm{d}g\mathrm{d}w.$$

(2) *For any Zariski open subset $\Omega \subset Y(V)(k')$, there is a test function f on the unitary group such that* $\mathrm{supp}\, f_\natural \subset \Omega$ *and $J_{\pi_1,\pi_2}(f) \neq 0$.*

Proof. We have proved this corollary in [Xue14, Proposition 6.1.2, Theorem 7.1.1, Theorem 7.3.3] assuming that both π_1 and π_2 are supercuspidal and $k = k' \times k'$. These conditions are imposed because

(1) we have only proved that the integral of matrix coefficients (1.1.2) is not identically zero under the hypothesis that π_1 and π_2 are both supercuspidal;
(2) in the case $k = k' \times k'$, we have only proved Proposition C.1.1 under the hypothesis that π_1 and π_2 are both supercuspidal.

The representability of the distribution J in the case $k = k' \times k'$ and the nonvanishing of (1.1.2) are now proved under the weakened hypothesis that π_1 and π_2 are both tempered. Then we can repeat the argument in [Xue14, Section 7] to deduce the corollary. ∎

Corollary C.1.3. *In this corollary, we temporarily switch to the notation of* [Xue14]*. Theorem 1.1.1 of* [Xue14] *holds if we replace the first condition by "At a non-archimedean split place v_1 of k', the representations π_{1,v_1} and π_{2,v_1} are both supercuspidal and at another non-archimedean split place v_2 of k', the representations π_{1,v_1} and π_{2,v_1} are both tempered".*

Proof. The supercuspidality condition at v_2 was only used in [Xue14, Proposition 6.1.2]. We now get [Xue14, Theorem 1.1.1] under the weakened hypotheses using Corollary C.1.2 instead of [Xue14, Proposition 6.1.2]. □

C.2. **Proof of Proposition C.1.1.**

Proof of Proposition C.1.1 assuming π_1 and π_2 are supercuspidal. We first assume that k is a field and π_1, π_2 are both supercuspidal. The case $k = k' \times k'$ and π_1, π_2 being supercuspidal has been proved in [Xue14]. It follows that $\widetilde{\Theta}(g, w)$, as a function on $Y(V)(k')$, is compactly supported. We are then reduced to show that for any compactly supported function f on $Y(V)(k')$, the orbital integral

$$O([g,w], f) = \int\limits_{\mathrm{U}(V)(k')} f([g,w].h)\mathrm{d}h,$$

as a function on $Y(V)(k')$, is locally integrable.

Let Ξ be the Harish-Chandra function on $\mathrm{U}(V)(k')$ defined as in Appendix B. Since $\Xi(g) > 0$ for any g, to prove Proposition C.1.1, it is enough to prove that the integral

$$\int\limits_{\mathrm{U}(V)(k')} \Xi(h^{-1}gh)\phi(wh)\mathrm{d}h,$$

as a function on $Y(V)(k')$, is locally integrable, where ϕ is a nonnegative compactly supported locally constant function on $V^\vee$. We choose a basis of V and a dual basis of $V^\vee$. We write elements $g \in \mathrm{U}(V)(k')$ as matrices and elements in $V^\vee$ as row vectors. By the Cartan decomposition for $\mathrm{U}(V)(k')$, it is enough to show that for any $\gamma \in A_0^+$, the integral

$$\int\limits_{K^2} \int\limits_{\mathrm{U}(V)(k')} \int\limits_{k_n} \Xi(h^{-1}k_1\gamma k_2 h)\phi(wh)\phi'(w)\mathrm{d}w\mathrm{d}h\mathrm{d}k_1\mathrm{d}k_2$$

is absolutely convergent for any nonnegative compactly supported locally constant function ϕ' on $V^\vee$. Using identity (B.0.3), we have

$$\int\limits_{K^2} \int\limits_{\mathrm{U}(V)(k')} \int\limits_{k_n} \Xi(h^{-1}k_1\gamma k_2 h)\phi(wh)\phi'(w)\mathrm{d}h\mathrm{d}w\mathrm{d}k_1\mathrm{d}k_2 = \Xi(\gamma) \int\limits_{\mathrm{U}(V)(k')} \int\limits_{k_n} \Xi(h)^2\phi(wh)\phi(w)\mathrm{d}w\mathrm{d}h.$$

Let

$$h = k_1 \operatorname{diag}[a_1, \cdots, a_r, 1, \cdots, 1, a_r^{\tau,-1}, \cdots, a_1^{\tau,-1}]k_2, \quad |a_1| \leq \cdots \leq |a_r| \leq 1$$

be the Cartan decomposition of h. Then

$$\int\limits_{k_n} \phi(wh)\phi'(w)\mathrm{d}w \leq |a_1 \cdots a_r|.$$

By the estimates of ν and Ξ (c.f. (B.0.2), (B.0.5)), it is enough to prove that the integral

$$\int\limits_{|a_1| \leq \cdots \leq |a_r| \leq 1} |a_1 \cdots a_r| \sum_{i=1}^{r} (1 - \log|a_i|)^B \mathrm{d}a$$

is convergent for any real number B, where $\mathrm{d}a$ is the multiplicative measure of $(k'^\times)^r$. But this is clear. Proposition C.1.1 is then proved under the assumption that π_1 and π_2 are both supercuspidal. □

Proof of Proposition C.1.1 assuming $k = k' \times k'$. From now on we assume that $k = k' \times k'$. We fix an isomorphism $\mathrm{U}(V) \simeq \mathrm{GL}_n(k')$ and take $K = \mathrm{GL}_n(\mathfrak{o}')$. The Weil representation of $\mathrm{GL}_n(k')$ is realized on $\mathcal{C}_c^\infty(k'_n)$ by $\omega_{\psi',\mu}(g)\phi(x) = \mu(\det g)|\det g|^{\frac{1}{2}}\phi(xg)$. In particular, it is independent of ψ'. For simplicity, we assume that μ is trivial. The general case requires nothing more than some additional notation. We denote the Weil representation by ω instead of $\omega_{\psi',\mu}$ for the rest of this appendix. Recall that for any $\phi_1, \phi_2 \in \mathcal{C}_c^\infty(k'_n)$, the partial Fourier transform is defined by

$$(\phi_1 \otimes \phi_2)^\ddagger(x,y) = \int_{k'_n} \phi_1(x+z)\phi_2(x-z)\psi'(z\,{}^t y)\mathrm{d}z.$$

Moreover, for any $h \in \mathrm{GL}_n(k')$, we have

$$(\omega(h)\phi_1 \otimes \omega(h)\phi_2)^\ddagger(x,y) = (\phi_1 \otimes \phi_2)^\ddagger(xh, y\,{}^t h^{-1}).$$

By the weak inequality (B.0.6), to prove Proposition C.1.1, we only have to prove that for any $f \in \mathcal{C}_c^\infty(Y(V)(k'))$, the integral

$$\int_{(k'_n)^2} \int_{(\mathrm{GL}_n(k'))^2} \int_{\mathrm{GL}_n(k')} f(g,x,y)\Xi(h_1^{-1})\Xi(h_1^{-1}h_2^{-1}gh_2) \left| \left(\overline{\omega(h_1)\phi_1} \otimes \phi_2\right)^\ddagger (xh_2, y\,{}^t h_2^{-1}) \right| \mathrm{d}g\mathrm{d}h_2\mathrm{d}h_1\mathrm{d}x\mathrm{d}y$$

is convergent for any $\phi_1, \phi_2 \in \mathcal{C}_c^\infty(k'_n)$. It is enough to prove the convergence for $\phi_1 = \mathbf{1}_{A+\varpi'^r\mathfrak{o}'_n}$ and $\phi_2 = \mathbf{1}_{B+\varpi'^r\mathfrak{o}'_n}$, $A, B \in k'_n$. Then by changing variables, we may further assume that $r = 0$. We may also assume that $f(g,x,y)$ is of the form $f(g)\,\mathbf{1}_{\varpi^M\mathfrak{o}'_n}(x)\,\mathbf{1}_{\varpi^N\mathfrak{o}'_n}(y)$ for some compactly supported function f on $\mathrm{GL}_n(k')$ and integers M, N.

Lemma C.2.1. *For any $h_1, h_2 \in \mathrm{GL}_n(k')$, we have*

$$\begin{aligned}&\int_{(k'_n)^2} \mathbf{1}_{\varpi'^M\mathfrak{o}'_n}(x)\,\mathbf{1}_{\varpi'^N\mathfrak{o}'_n}(y) \left| \left(\omega(h_1)\,\mathbf{1}_{A+\mathfrak{o}'_n} \otimes \mathbf{1}_{B+\mathfrak{o}'_n}\right)^\ddagger (xh_2, y\,{}^t h_2^{-1}) \right| \mathrm{d}x\mathrm{d}y \\ \leq &\int_{(k'_n)^2} \mathbf{1}_{\varpi'^M\mathfrak{o}'_n}(x)\,\mathbf{1}_{\varpi'^N\mathfrak{o}'_n}(y) \left| \left(\omega(h_1)\,\mathbf{1}_{\mathfrak{o}'_n} \otimes \mathbf{1}_{\mathfrak{o}'_n}\right)^\ddagger (xh_2, y\,{}^t h_2^{-1}) \right| \mathrm{d}x\mathrm{d}y.\end{aligned}$$

Proof. First using the Cartan decomposition for h_1, we see that we may assume that $h_1 = t_1 \in A_0^+$. Straightforward but tedious computation gives that

$$\left| \left(\omega(t_1)\,\mathbf{1}_{A+\mathfrak{o}'_n} \otimes \mathbf{1}_{B+\mathfrak{o}'_n}\right)^\ddagger (x,y) \right| = \left| \left(\omega(t_1)\,\mathbf{1}_{\mathfrak{o}'_n} \otimes \mathbf{1}_{\mathfrak{o}'_n}\right)^\ddagger \left(x - \frac{At_1^{-1}+B}{2}, y\right) \right|,$$

and

$$\left(\omega(t_1)\,\mathbf{1}_{\mathfrak{o}'_n} \otimes \mathbf{1}_{\mathfrak{o}'_n}\right)^\ddagger (x,y) = \mathbf{1}_{\mathfrak{o}'_n}(xt_1^+)\,\mathbf{1}_{\mathfrak{o}'_n}(yt_1^{-,-1})|\det t_1^+|^{\frac{1}{2}}|\det t_1^-|^{-\frac{1}{2}}\psi'(x\,{}^t y), \tag{C.2.1}$$

where $t_1 = \mathrm{diag}[a_1, \cdots, a_n]$, $|a_1| \leq \cdots \leq |a_r| \leq 1 < |a_{r+1}| \leq \cdots \leq |a_n|$,

$$t_1^+ = \mathrm{diag}[a_1, \cdots, a_r, 1, \cdots, 1], \quad t_1^- = \mathrm{diag}[1, \cdots, 1, a_{r+1}, \cdots, a_n].$$

Let $R = (At_1^{-1}+B)/2$ and $h_2t_1^+ = k_1tk_2$ be the Cartan decomposition. Then we have

$$\begin{aligned}&\int_{k'_n} \mathbf{1}_{\mathfrak{o}'_n}((xh_2 - R)t_1^+)\,\mathbf{1}_{\varpi'^M\mathfrak{o}'_n}(x)\mathrm{d}x = \int_{k'_n} \mathbf{1}_{\mathfrak{o}'_n}(xk_1tk_2 - Rt_1^+)\,\mathbf{1}_{\varpi'^M\mathfrak{o}'_n}(x)\mathrm{d}x \\ =&\int_{k'_n} \mathbf{1}_{\mathfrak{o}'_n t^{-1}}(x - Rt_1^+k_2^{-1}t^{-1})\,\mathbf{1}_{\varpi'^M\mathfrak{o}'_n}(x)\mathrm{d}x \leq \int_{k'_n} \mathbf{1}_{\mathfrak{o}'_n t^{-1}}(x)\,\mathbf{1}_{\varpi'^M\mathfrak{o}'_n}(x)\mathrm{d}x = \int_{k'_n} \mathbf{1}_{\mathfrak{o}'_n}(xht_1^+)\,\mathbf{1}_{\varpi'^M\mathfrak{o}'_n}(x)\mathrm{d}x.\end{aligned}$$

The lemma then follows. ∎

By this lemma, to prove Proposition C.1.1, we only need to prove that the integral

$$\int_{(k'_n)^2}\int_{(\mathrm{GL}_n(k'))^2}\int_{\mathrm{GL}_n(k')} f(g)\,\mathbf{1}_{\varpi^M\mathfrak{o}'_n}(x)\,\mathbf{1}_{\varpi^N\mathfrak{o}'_n}(y)\Xi(h_1^{-1})\Xi(h_1^{-1}h_2^{-1}gh_2)$$

$$\left|\left(\omega(h_1)\,\mathbf{1}_{\mathfrak{o}'_n}\otimes\mathbf{1}_{\mathfrak{o}'_n}\right)^{\ddagger}(xh_2, y\,{}^th_2^{-1})\right|\mathrm{d}g\mathrm{d}h_2\mathrm{d}h_1\mathrm{d}x\mathrm{d}y$$

is convergent. It follows from (C.2.1) that we may write $\mathbf{1}_{\varpi^M\mathfrak{o}'_n}$ and $\mathbf{1}_{\varpi^N\mathfrak{o}'_n}$ as a sum of functions of the form $\mathbf{1}_{A+\varpi^r\mathfrak{o}'_n}$ for some sufficiently large integer r such that if $x\in A+\varpi^r\mathfrak{o}'_n$ and $y\in B+\varpi^r\mathfrak{o}'_n$, then

$$\psi'(-A\,{}^tB)\left(\omega(h_1)\,\mathbf{1}_{\mathfrak{o}'_n}\otimes\mathbf{1}_{\mathfrak{o}'_n}\right)^{\ddagger}(xh_2, y\,{}^th_2^{-1})\geq 0.$$

for any $h_1,h_2\in\mathrm{GL}_n(k')$. So we are reduced to show that

$$\int_{(\mathrm{GL}_n(k'))^2}\int_{\mathrm{GL}_n(k')}\int_{(k'_n)^2} f(g)\,\mathbf{1}_{A+\varpi^r\mathfrak{o}'_n}(x)\,\mathbf{1}_{B+\varpi^r\mathfrak{o}'_n}(y)\Xi(h_1^{-1})\Xi(h_1^{-1}h_2^{-1}gh_2)$$

$$\left(\omega(h_1)\,\mathbf{1}_{\mathfrak{o}'_n}\otimes\mathbf{1}_{\mathfrak{o}'_n}\right)^{\ddagger}(xh_2, y\,{}^th_2^{-1})\mathrm{d}x\mathrm{d}y\mathrm{d}g\mathrm{d}h_2\mathrm{d}h_1$$

is convergent. Since the partial Fourier transform $-^{\ddagger}$ preserves the L^2-inner product, we are reduced to show that

$$\int_{(\mathrm{GL}_n(k'))^2}\int_{\mathrm{GL}_n(k')}\int_{(k'_n)^2} f(g)|\Phi(x,y)|\Xi(h_1^{-1})\Xi(h_1^{-1}h_2^{-1}gh_2)\omega(h_2h_1)\,\mathbf{1}_{\mathfrak{o}'_n}(x)\omega(h_2)\,\mathbf{1}_{\mathfrak{o}'_n}(y)\mathrm{d}x\mathrm{d}y\mathrm{d}g\mathrm{d}h_2\mathrm{d}h_1$$

is convergent, where $\Phi(x,y)$ is the inverse partial Fourier transform of $\mathbf{1}_{A+\varpi^r\mathfrak{o}'_n}\otimes\mathbf{1}_{B+\varpi^r\mathfrak{o}'_n}$. As Φ is a finite sum of the functions of form $\Phi_1(x)\Phi_2(y)$, we are reduced to show the convergence of

(C.2.2)
$$\int_{(\mathrm{GL}_n(k'))^2}\int_{\mathrm{GL}_n(k')}\int_{(k'_n)^2} f(g)\Phi_1(x)\Phi_2(y)\Xi(h_1^{-1})\Xi(h_1^{-1}h_2^{-1}gh_2)\omega(h_2h_1)\,\mathbf{1}_{\mathfrak{o}'_n}(x)\omega(h_2)\,\mathbf{1}_{\mathfrak{o}'_n}(y)\mathrm{d}x\mathrm{d}y\mathrm{d}g\mathrm{d}h_2\mathrm{d}h_1.$$

We may assume that $f=\mathbf{1}_{K\gamma K}$ for some $\gamma\in\mathrm{GL}_n(k')$ and $\Phi_1=\Phi_2=\mathbf{1}_{\varpi'^s\mathfrak{o}'_n}$ for some sufficiently large negative integer s. Then

$$\text{(C.2.2)}=\int_{(\mathrm{GL}_n(k'))^2}\int_{K^2}\int_{(k'_n)^2}\mathbf{1}_{\varpi'^s\mathfrak{o}'_n}(x)\,\mathbf{1}_{\varpi'^s\mathfrak{o}'_n}(y)\Xi(h_1^{-1})\Xi(h_1^{-1}h_2^{-1}k_1\gamma k_2h_2)$$

$$\omega(h_2h_1)\,\mathbf{1}_{\mathfrak{o}'_n}(x)\omega(h_2)\,\mathbf{1}_{\mathfrak{o}'_n}(y)\mathrm{d}x\mathrm{d}y\mathrm{d}k_1\mathrm{d}k_2\mathrm{d}h_2\mathrm{d}h_1.$$

We make a change of variables $h_1\mapsto h_2^{-1}h_1$. Then

$$\text{(C.2.2)}=\int_{(\mathrm{GL}_n(k'))^2}\int_{K^2}\int_{(k'_n)^2}\mathbf{1}_{\varpi'^s\mathfrak{o}'_n}(x)\,\mathbf{1}_{\varpi'^s\mathfrak{o}'_n}(y)\Xi(h_1^{-1}h_2)\Xi(h_1^{-1}k_1\gamma k_2h_2)$$

$$\omega(h_1)\,\mathbf{1}_{\mathfrak{o}'_n}(x)\omega(h_2)\,\mathbf{1}_{\mathfrak{o}'_n}(y)\mathrm{d}x\mathrm{d}y\mathrm{d}k_1\mathrm{d}k_2\mathrm{d}h_2\mathrm{d}h_1.$$

Now using identity (B.0.3) and the Cartan decomposition of h_1 and h_2, we are reduced to prove the convergence of

$$\int_{A_0^+}\int_{k'_n}\Xi(a)^2\nu(a)\omega(a)\,\mathbf{1}_{\mathfrak{o}'_n}(x)\,\mathbf{1}_{\varpi'^s\mathfrak{o}'_n}(x)\mathrm{d}x\mathrm{d}a.$$

Using the estimates of Ξ and ν (c.f. (B.0.2), (B.0.5)), we only need to prove the convergence of

$$\int_{A_0^+}\int_{k'_n}\mathbf{1}_{\mathfrak{o}'_n}(xa)\,\mathbf{1}_{\varpi'^s\mathfrak{o}'_n}(x)|\det a|^{\frac{1}{2}}(1+\sigma(a))^B\mathrm{d}x\mathrm{d}a.$$

Decompose A_0^+ into regions of the form

$$\Omega_r=\{\mathrm{diag}[a_1,\cdots,a_n]\mid|a_1|\leq\cdots\leq|a_r|\leq 1<|a_{r+1}|\leq\cdots\leq|a_n|\},$$

We have

$$\int_{k'_n} \mathbf{1}_{\mathfrak{o}'_n}(xa)\,\mathbf{1}_{\varpi'^s\mathfrak{o}'_n}(x)\mathrm{d}x \le C \times |a_{r+1}\cdots a_n|^{-1}$$

for some constant C. Then the desired convergence follows from convergence of

$$\int_{\Omega_r} |a_1\cdots a_r|^{\frac{1}{2}}|a_{r+1}\cdots a_n|^{-\frac{1}{2}}(1+\sigma(a))^B\mathrm{d}a.$$

Proposition C.1.1 is then proved. □

Appendix D. Some local theory

In this appendix, we prove Proposition 1.1.1. We keep the notation from Proposition 1.1.1. All the subscripts v are suppressed from the notation. We warn the readers that the notation is slightly different from Appendix A.

D.1. **Convergence.**

Proof of Proposition 1.1.1 (1). By the Cartan decomposition (B.0.1), the weak inequality (B.0.6) and the estimate of ν (B.0.2), we only have to prove that

$$\int_{A_0^+} \langle \omega_{\psi',\mu}(a)\phi,\phi\rangle(1+\sigma(a))^B\mathrm{d}a$$

is absolutely convergent, where A_0^+ is defined as in Appendix B,

$$a = \mathrm{diag}[a_1,\cdots,a_r,1,\cdots,1,a_r^{\tau,-1},\cdots,a_1^{\tau,-1}] \in A_0^+, \quad |a_1|\le\cdots\le|a_r|\le 1,$$

and r is the split rank of $\mathrm{U}(V)$. We have

$$|\langle \omega_{\psi',\mu}(a)\phi,\phi\rangle| \le |a_1\cdots a_r|^{\frac{1}{2}}.$$

Proposition 1.1.1 then follows from the fact that for any real number B, the integral

$$\int_{|x_1|\le\cdots\le|x_r|\le 1} |x_1\cdots x_r|^{\frac{1}{2}}\sum_{i=1}^{r}(1-\log|x_i|)^B\mathrm{d}x$$

is convergent, where $\mathrm{d}x$ is the multiplicative measure of $(k'^{\times})^r$. □

D.2. **Nonvanishing and positivity.**

Proof of Proposition 1.1.1 (2). By the local theta dichotomy [HKS96, Corollary 4.4, GI14, Theorem 11.1], there is a unique n-dimensional hermitian space W_n and an irreducible admissible tempered representation σ_n of $\mathrm{U}(W_n)$, such that $\theta_{W_n,V,\psi',\mu}(\overline{\sigma_n}) = \overline{\pi_2}$. Then it follows that

$$\mathrm{Hom}_{\mathrm{U}(W_n)\times\mathrm{U}(V)}(\pi_1\otimes\overline{\sigma_n}\otimes\overline{\omega_{W_n,V,\psi',\mu}}\otimes\overline{\omega_{\psi',\mu}}) \ne 0. \tag{D.2.1}$$

Let $W_{n+1} = W_n + k$ and $\iota : \mathrm{U}(W_n) \to \mathrm{U}(W_{n+1})$ be the corresponding embedding of groups. Let $\mathbf{L}_n \subset \mathrm{Res}(W_n\otimes V)^\vee$ (resp. $\mathbf{L}_1 \subset \mathrm{Res}\,V^\vee$) be a Lagrangian subspace. Then $\mathbf{L}_{n+1} = \mathbf{L}_n + \mathbf{L}_1 \subset \mathrm{Res}(W_{n+1}\otimes V)^\vee$ is a Lagrangian subspace.

It follows from (D.2.1) that

$$\mathrm{Hom}_{\mathrm{U}(W_n)}(\overline{\theta_{W_{n+1},V,\psi',\mu}(\pi_1)}\otimes\overline{\sigma_n},\mathbb{C}) \ne 0.$$

Let $\sigma_{n+1} = \theta_{W_{n+1},V,\psi',\mu}(\pi_1)$. Apply [SV, § 6.4], we see that there are $f_{n+1}, f_{n+1}^\vee \in \sigma_{n+1}$ and $f_n, f_n^\vee \in \sigma_n$, such that

$$\int_{\mathrm{U}(W_n)(k')} \langle \sigma_{n+1}(\iota(h))f_{n+1}, f_{n+1}^\vee\rangle\langle\sigma_n(h)f_n,f_n^\vee\rangle\mathrm{d}h \ne 0.$$

We now apply the explicit local theta lifting from Appendix A. We can then find $\varphi_1, \varphi_1^\vee \in \pi_1$, $\phi_n, \phi_n^\vee \in \mathcal{S}(\mathbf{L}_n)$ and $\phi_1, \phi_1^\vee \in \mathcal{S}(\mathbf{L}_1)$ so that

$$\int_{\mathrm{U}(W_n)(k')} \left(\int_{\mathrm{U}(V)(k')} \overline{\langle \pi_1(g)\varphi_1, \varphi_1^\vee \rangle} \langle \omega_{W_{n+1},V,\psi',\mu}(g, \iota(h))(\phi_n \otimes \phi_1), \phi_n^\vee \otimes \phi_1^\vee \rangle \mathrm{d}g \right) \langle \sigma_n(h) f_n, f_n^\vee \rangle \mathrm{d}h \neq 0.$$

By Lemma A.3.1, we can change the order of integration. Since

$$\omega_{W_{n+1},V,\psi',\mu}(g, \iota(h))(\phi_n \otimes \phi_1) = \omega_{W_n,V,\psi',\mu}(g,h)\phi_n \otimes \omega_{\psi',\mu}(g)\phi_1,$$

we conclude that

$$\int_{\mathrm{U}(V)(k')} \langle \pi_1(g)\varphi_1, \varphi_1^\vee \rangle \langle \pi_2(g)\varphi_2, \varphi_2^\vee \rangle \langle \omega_{\psi',\mu}(g)\phi_1, \phi_1^\vee \rangle \neq 0,$$

where as in Proposition A.2.2,

$$\langle \pi_2(g)\varphi_2, \varphi_2^\vee \rangle = \int_{\mathrm{U}(W_n)(k')} \langle \sigma_n(h) f_n, f_n^\vee \rangle \langle \omega_{W_n,V,\psi',\mu}(g,h)\phi_n, \phi_n^\vee \rangle \mathrm{d}h.$$

We prove $\alpha(\varphi_1, \varphi_2, \phi) \geq 0$ in a similar way. We can find $f_n \in \sigma_n$ and $\phi_n \in \mathcal{S}(\mathbf{L}_n)$ such that

$$\langle \pi_2(g)\varphi_2, \varphi_2 \rangle = \int_{\mathrm{U}(W_n)(k')} \overline{\langle \sigma_n(h) f_n, f_n \rangle} \langle \omega_{W_n,V,\psi',\mu}(g,h)\phi_n, \phi_n \rangle \mathrm{d}h.$$

Then

$$\alpha(\varphi_1, \varphi_2, \phi) = \int_{\mathrm{U}(W_n)(k')} \langle \sigma_{n+1}(\iota(h)) f_{n+1}, f_{n+1} \rangle \langle \sigma_n(h) f_n, f_n \rangle \mathrm{d}h \geq 0$$

by [Har12], where

$$\langle \sigma_{n+1}(\iota(h)) f_{n+1}, f_{n+1} \rangle = \int_{\mathrm{U}(V)(k')} \langle \pi_1(g)\varphi_1, \varphi_2 \rangle \langle \omega_{W_{n+1},V,\psi',\mu}(g, \iota(h))\phi_n \otimes \phi, \phi_n \otimes \phi \rangle \mathrm{d}g.$$

□

D.3. **Unramified computations.**

Proof of Proposition 1.1.1 (3). Assume that we are in the unramified situation as described in Proposition 1.1.1 (3). We keep the notation from the previous subsection. The groups $\mathrm{U}(W_n)$ and $\mathrm{U}(W_{n+1})$ are unramified. We may choose the Lagrangian subspaces $\mathbf{L}_n$, $\mathbf{L}_1$ and $\mathbf{L}_{n+1}$ so that they are all defined over $\mathfrak{o}'$. The representations σ_n and σ_{n+1} are both unramified. Let $f_n \in \sigma_n$ (resp. $f_{n+1} \in \sigma_{n+1}$) be the normalized spherical vector in σ_n (resp. σ_{n+1}), i.e. f_n (resp. f_{n+1}) is $\mathrm{U}(W_n)(\mathfrak{o}')$ (resp. $\mathrm{U}(W_{n+1})(\mathfrak{o}')$) fixed and $\langle f_n, f_n \rangle = 1$ (resp. $\langle f_{n+1}, f_{n+1} \rangle = 1$). Let $\varphi_1^0 \in \pi_1$ and $\varphi_2^0 \in \pi_2$ be normalized spherical vectors.

Let $\phi_n = \mathbf{1}_{\mathbf{L}_n(\mathfrak{o}')}$ and $\phi_1 = \mathbf{1}_{\mathbf{L}_1(\mathfrak{o}')}$, then $\phi_{n+1} = \phi_n \otimes \phi_1 = \mathbf{1}_{\mathbf{L}_{n+1}(\mathfrak{o}')}$. Let $\mathrm{d}g$ (resp. $\mathrm{d}h$) be the measure on $\mathrm{U}(V)$ (resp. $\mathrm{U}(W_n)$) so that $\mathrm{vol}\,\mathrm{U}(V)(\mathfrak{o}') = 1$ (resp. $\mathrm{vol}\,\mathrm{U}(W_n)(\mathfrak{o}') = 1$). Note that $\mathrm{d}g$ (and similarly $\mathrm{d}h$) equals $L(1,\eta)\zeta_{k'}(2)\cdots L(n,\eta^n)$ times the (unnormalized) measure $|\omega|_{k'}$ that we have fixed in Section 2.

By [LR05, § 7], we have

$$\int_{\mathrm{U}(W_n)(k')} \langle \sigma_n(h) f_n, f_n \rangle \langle \omega_{W_n,V,\psi',\mu}(g,h)\phi_n, \phi_n \rangle \mathrm{d}h = \frac{L\left(\frac{1}{2}, \mathrm{BC}(\sigma_n) \times \mu^n\right)}{\prod_{j=1}^n L(j, \eta^j)} \langle \pi_2(g)\varphi_2^0, \varphi_2^0 \rangle,$$

and

$$\int_{\mathrm{U}(V)(k')} \langle \pi_1(g)\varphi_1^0, \varphi_1^0 \rangle \langle \omega_{W_{n+1},V,\psi',\mu}(g,h)\phi_{n+1}, \phi_{n+1} \rangle \mathrm{d}g = \frac{L(1, \mathrm{BC}(\pi_1) \times \mu^{n+1})}{\prod_{j=1}^n L(j+1, \eta^{j+1})} \langle \sigma_{n+1}(h) f_{n+1}, f_{n+1} \rangle.$$

Consider the absolutely convergent integral (c.f. Lemma A.3.1)

$$\text{(D.3.1)} \qquad \int_{\mathrm{U}(W_n)(k')} \int_{\mathrm{U}(V)(k')} \langle \pi_1(g)\varphi_1^0, \varphi_1^0 \rangle \langle \omega_{W_{n+1},V,\psi',\mu}(g, \iota(h))\phi_{n+1}, \phi_{n+1}\rangle \langle \sigma_n(h) f_n, f_n \rangle \mathrm{d}g\mathrm{d}h.$$

Integrating over g first and applying [Har12, Theorem 2.12], we get

$$(\text{D.3.1}) = \frac{L(1, \mathrm{BC}(\pi_1) \times \mu^{n+1})}{\prod_{j=1}^n L(j+1, \eta^{j+1})} \times \frac{L\left(\frac{1}{2}, \mathrm{BC}(\theta(\pi_1)) \times \mathrm{BC}(\pi_2)\right)}{L(1, \theta(\pi_1), \mathrm{Ad}) L(1, \theta(\pi_2), \mathrm{Ad})} \times \prod_{j=1}^{n+1} L(j, \eta^j).$$

Integrating over h first, we get

$$(\text{D.3.1}) = \frac{L\left(\frac{1}{2}, \mathrm{BC}(\sigma_n) \times \mu^n\right)}{\prod_{j=1}^n L(j, \eta^j)} \times \alpha(\varphi_1^0, \varphi_2^0, \phi_1).$$

Looking at the Satake parameters of the theta lifting of unramified representations [Liu, Appendix], one sees that $\mathrm{BC}(\sigma_{n+1}) = \mathrm{BC}(\pi_1) \otimes \mu^{-1} \boxplus \mu^n$ and $\mathrm{BC}(\sigma_n) = \mathrm{BC}(\pi_2)$. It follows from the definition of the adjoint L-function that

$$L(s, \theta(\pi_1), \mathrm{Ad}) = L(s, \eta) L(s, \pi_1, \mathrm{Ad}) L(s, \mathrm{BC}(\pi_1) \otimes \mu^{n+1}).$$

Proposition 1.1.1 (3) then follows. □

References

[Bar03] E. M. Baruch, *A proof of Kirillov's conjecture*, Ann. of Math. (2) **158** (2003), no. 1, 207–252, DOI 10.4007/annals.2003.158.207. MR1999922 (2004f:22012) ↑13

[Ber84] J. N. Bernstein, *P-invariant distributions on* GL(N) *and the classification of unitary representations of* GL(N) *(non-Archimedean case)*, Lie group representations, II (College Park, Md., 1982/1983), Lecture Notes in Math., vol. 1041, Springer, Berlin, 1984, pp. 50–102. MR748505 (86b:22028) ↑13

[BR10] J. Bernstein and A. Reznikov, *Subconvexity bounds for triple L-functions and representation theory*, Ann. of Math. (2) **172** (2010), no. 3, 1679–1718, DOI 10.4007/annals.2010.172.1679. MR2726097 (2012i:11052) ↑3

[BP] R. Beuzart-Plessis, *La conjecture locale de Gross-Prasad pour les représentations tempérées des groupes unitaires*, available at http://arxiv.org/pdf/1205.2987v2.pdf. preprint. ↑7

[DR14] H. Darmon and V. Rotger, *Diagonal cycles and Euler systems I: a p-adic Gross-Zagier formula*, Ann. Sci. Éc. Norm. Supér. (4) **47** (2014), no. 4, 779–832 (English, with English and French summaries). MR3250064 ↑3

[Fin06a] T. Finis, *Divisibility of anticyclotomic L-functions and theta functions with complex multiplication*, Ann. of Math. (2) **163** (2006), no. 3, 767–807, DOI 10.4007/annals.2006.163.767. MR2215134 (2007m:11151) ↑3

[Fin06b] ———, *The μ-invariant of anticyclotomic L-functions of imaginary quadratic fields*, J. Reine Angew. Math. **596** (2006), 131–152, DOI 10.1515/CRELLE.2006.056. MR2254809 (2007g:11139) ↑3

[Fli91] Y. Z. Flicker, *On distinguished representations*, J. Reine Angew. Math. **418** (1991), 139–172, DOI 10.1515/crll.1991.418.139. MR1111204 (92i:22019) ↑13

[GGP12] W. T. Gan, B. H. Gross, and D. Prasad, *Symplectic local root numbers, central critical L values, and restriction problems in the representation theory of classical groups*, Astérisque **346** (2012), 1–109 (English, with English and French summaries). Sur les conjectures de Gross et Prasad. I. MR3202556 ↑1, 2, 4, 7

[GI11] W. T. Gan and A. Ichino, *On endoscopy and the refined Gross-Prasad conjecture for* ($\mathrm{SO}_5, \mathrm{SO}_4$), J. Inst. Math. Jussieu **10** (2011), no. 2, 235–324, DOI 10.1017/S1474748010000198. MR2787690 ↑39, 40

[GI14] ———, *Formal degrees and local theta correspondence*, Invent. Math. **195** (2014), no. 3, 509–672, DOI 10.1007/s00222-013-0460-5. MR3166215 ↑39, 46

[GI] ———, *The Gross-Prasad conjecture and local theta correspondence*, available at http://www.math.nus.edu.sg/~matgwt/GP-theta.pdf. preprint. ↑4, 7, 9, 38

[GQT] W.-T. Gan, Y. Qiu, and S. Takeda, *The Regularized Siegel-Weil Formula (The Second Term Identity) and the Rallis Inner Product Formula*, Invent. Math., DOI 10.1007/s00222-014-0509-0, (to appear in print). ↑38, 40

[GTa] W. T. Gan and S. Takeda, *On the Howe duality conjecture in classical theta correspondence*, available at http://arxiv.org/abs/1405.2626v2. preprint. ↑38, 39

[GTb] ———, *A proof of the Howe duality conjecture*, available at http://arxiv.org/abs/1407.1995. preprint. ↑38, 39

[GJR01] S. Gelbart, H. Jacquet, and J. Rogawski, *Generic representations for the unitary group in three variables*, Israel J. Math. **126** (2001), 173–237, DOI 10.1007/BF02784154. MR1882037 (2003k:11081) ↑13

[GRS93] S. Gelbart, J. Rogawski, and D. Soudry, *On periods of cusp forms and algebraic cycles for* U(3), Israel J. Math. **83** (1993), no. 1-2, 213–252, DOI 10.1007/BF02764643. MR1239723 (95a:11047) ↑40

[Gro80] B. H. Gross, *Arithmetic on elliptic curves with complex multiplication*, Lecture Notes in Mathematics, vol. 776, Springer, Berlin, 1980. With an appendix by B. Mazur. MR563921 (81f:10041) ↑3

[Gro97] ———, *On the motive of a reductive group*, Invent. Math. **130** (1997), no. 2, 287–313, DOI 10.1007/s002220050186. MR1474159 (98m:20060) ↑3

[HKS96] M. Harris, S. S. Kudla, and W. J. Sweet, *Theta dichotomy for unitary groups*, J. Amer. Math. Soc. **9** (1996), no. 4, 941–1004, DOI 10.1090/S0894-0347-96-00198-1. MR1327161 (96m:11041) ↑9, 46

[HL04] M. Harris and J.-P. Labesse, *Conditional base change for unitary groups*, Asian J. Math. **8** (2004), no. 4, 653–683. MR2127943 (2006g:11098) ↑2

[Har12] N. Harris, *The Refined Gross–Prasad Conjecture for Unitary Groups*, Int Math Res Notices, posted on 2012, DOI 10.1093/imrn/rns219, (to appear in print), available at `http://imrn.oxfordjournals.org/content/early/2012/10/12/imrn.rns219.full.pdf+html`. ↑1, 3, 4, 7, 8, 9, 47, 48

[How89] R. Howe, *Transcending classical invariant theory*, J. Amer. Math. Soc. **2** (1989), no. 3, 535–552, DOI 10.2307/1990942. MR985172 (90k:22016) ↑38, 39

[Hsi14] M.-L. Hsieh, *Eisenstein congruence on unitary groups and Iwasawa main conjectures for CM fields*, J. Amer. Math. Soc. **27** (2014), no. 3, 753–862, DOI 10.1090/S0894-0347-2014-00786-4. MR3194494 ↑3

[Ich08] A. Ichino, *Trilinear forms and the central values of triple product L-functions*, Duke Math. J. **145** (2008), no. 2, 281–307, DOI 10.1215/00127094-2008-052. MR2449948 (2009i:11066) ↑1, 3

[II10] A. Ichino and T. Ikeda, *On the periods of automorphic forms on special orthogonal groups and the Gross-Prasad conjecture*, Geom. Funct. Anal. **19** (2010), no. 5, 1378–1425, DOI 10.1007/s00039-009-0040-4. MR2585578 (2011a:11100) ↑1, 3, 4, 9, 41

[JPSS83] H. Jacquet, I. I. Piatetskii-Shapiro, and J. A. Shalika, *Rankin-Selberg convolutions*, Amer. J. Math. **105** (1983), no. 2, 367–464, DOI 10.2307/2374264. MR701565 (85g:11044) ↑14

[JR11] H. Jacquet and S. Rallis, *On the Gross–Prasad conjecture for unitary groups*, On certain L-functions, Clay Math. Proc., vol. 13, Amer. Math. Soc., Providence, RI, 2011, pp. 205–264. MR2767518 (2012d:22026) ↑1, 5

[JS81] H. Jacquet and J. A. Shalika, *On Euler products and the classification of automorphic representations. I*, Amer. J. Math. **103** (1981), no. 3, 499–558, DOI 10.2307/2374103. MR618323 (82m:10050a) ↑14

[Kud94] S. S. Kudla, *Splitting metaplectic covers of dual reductive pairs*, Israel J. Math. **87** (1994), no. 1-3, 361–401, DOI 10.1007/BF02773003. MR1286835 (95h:22019) ↑38

[LM15] E. Lapid and Z. Mao, *A conjecture on Whittaker-Fourier coefficients of cusp forms*, J. Number Theory **146** (2015), 448–505, DOI 10.1016/j.jnt.2013.10.003. MR3267120 ↑3

[LM] E. Lapid and Z. Mao, *On Whittaker-Fourier coefficients of automorphic forms on* $\mathrm{Mp}(n)$, available at `http://www.math.huji.ac.il/~erezla/papers/master010413.pdf`. ↑4

[LR05] E. M. Lapid and S. Rallis, *On the local factors of representations of classical groups*, Automorphic representations, L-functions and applications: progress and prospects, Ohio State Univ. Math. Res. Inst. Publ., vol. 11, de Gruyter, Berlin, 2005, pp. 309–359, DOI 10.1515/9783110892703.309, (to appear in print). MR2192828 (2006j:11071) ↑40, 41, 47

[Liu14] Y. Liu, *Relative trace formulae toward Bessel and Fourier-Jacobi periods on unitary groups*, Manuscripta Math. **145** (2014), no. 1-2, 1–69, DOI 10.1007/s00229-014-0666-x. MR3244725 ↑1, 5, 6, 16, 18

[Liu] ———, *Refined global Gan-Gross-Prasad conjecture for Bessel periods*, Journal fr die reine und angewandte Mathematik (Crelles Journal), DOI 10.1515/crelle-2014-0016, (to appear in print). ↑4, 8, 48

[MY11] R. Masri and T. Yang, *Nonvanishing of Hecke L-functions for CM fields and ranks of abelian varieties*, Geom. Funct. Anal. **21** (2011), no. 3, 648–679, DOI 10.1007/s00039-011-0121-z. MR2810860 (2012f:11099) ↑3

[MV10] P. Michel and A. Venkatesh, *The subconvexity problem for* GL_2, Publ. Math. Inst. Hautes Études Sci. **111** (2010), 171–271, DOI 10.1007/s10240-010-0025-8. MR2653249 (2012c:11111) ↑3

[Mok] C. P. Mok, *Endoscopic classification of representations of quasi-split unitary groups*. To appear in Memoirs of the American Mathematical Society. ↑4

[MVW87] C. Mœglin, M.-F. Vignéras, and J.-L. Waldspurger, *Correspondances de Howe sur un corps p-adique*, Lecture Notes in Mathematics, vol. 1291, Springer-Verlag, Berlin, 1987 (French). MR1041060 (91f:11040) ↑38

[NPS14] P. D. Nelson, A. Pitale, and A. Saha, *Bounds for Rankin-Selberg integrals and quantum unique ergodicity for powerful levels*, J. Amer. Math. Soc. **27** (2014), no. 1, 147–191, DOI 10.1090/S0894-0347-2013-00779-1. MR3110797 ↑3

[Qiu] Y. Qiu, *The Bessel period functional on* $\mathrm{SO}(5)$*: the nontempered case*, available at `http://math.columbia.edu/~qiu/qiu4b_pt.pdf`. Preprint. ↑9

[RVY99] F. Rodriguez Villegas and T. Yang, *Central values of Hecke L-functions of CM number fields*, Duke Math. J. **98** (1999), no. 3, 541–564, DOI 10.1215/S0012-7094-99-09817-4. MR1695801 (2000j:11074) ↑3

[Sun12] B. Sun, *Multiplicity one theorems for Fourier-Jacobi models*, Amer. J. Math. **134** (2012), no. 6, 1655–1678, DOI 10.1353/ajm.2012.0044. MR2999291 ↑4

[SV] Y. Sakellaridis and A. Venkatesh, *Periods and harmonic analysis on spherical varieties*, available at `arXiv:1203.0039`. preprint. ↑46

[SZ12] B. Sun and C.-B. Zhu, *Multiplicity one theorems: the Archimedean case*, Ann. of Math. (2) **175** (2012), no. 1, 23–44, DOI 10.4007/annals.2012.175.1.2. MR2874638 ↑4

[Wal85] J.-L. Waldspurger, *Sur les valeurs de certaines fonctions L automorphes en leur centre de symétrie*, Compositio Math. **54** (1985), no. 2, 173–242 (French). MR783511 (87g:11061b) ↑1

[Wal90] ———, *Démonstration d'une conjecture de dualité de Howe dans le cas p-adique, $p \neq 2$*, Festschrift in honor of I. I. Piatetski-Shapiro on the occasion of his sixtieth birthday, Part I (Ramat Aviv, 1989), Israel Math. Conf. Proc., vol. 2, Weizmann, Jerusalem, 1990, pp. 267–324 (French). MR1159105 (93h:22035) ↑38, 39

[Xue14] H. Xue, *The Gan-Gross-Prasad conjecture for* $\mathrm{U}(n) \times \mathrm{U}(n)$, Adv. Math. **262** (2014), 1130–1191, DOI 10.1016/j.aim.2014.06.010. MR3228451 ↑1, 2, 5, 6, 17, 18, 20, 21, 35, 36, 37, 42, 43

[Yam14] S. Yamana, *L-functions and theta correspondence for classical groups*, Invent. Math. **196** (2014), no. 3, 651–732, DOI 10.1007/s00222-013-0476-x. MR3211043 ↑38, 39, 40

[Yan97] T. Yang, *Theta liftings and Hecke L-functions*, J. Reine Angew. Math. **485** (1997), 25–53, DOI 10.1515/crll.1997.485.25. MR1442188 (98e:11056) ↑3

[Yan99] ———, *Nonvanishing of central Hecke L-values and rank of certain elliptic curves*, Compositio Math. **117** (1999), no. 3, 337–359, DOI 10.1023/A:1000934108242. MR1702416 (2001a:11093) ↑3

[Yun11] Z. Yun, *The fundamental lemma of Jacquet and Rallis*, Duke Math. J. **156** (2011), no. 2, 167–227, DOI 10.1215/00127094-2010-210. With an appendix by Julia Gordon. MR2769216 (2012b:22009) ↑1, 5

[Zha14a] W. Zhang, *Fourier transform and the global Gan-Gross-Prasad conjecture for unitary groups*, Ann. of Math. (2) **180** (2014), no. 3, 971–1049. MR3245011 ↑1, 7, 16, 35, 36

[Zha14b] ———, *Automorphic period and the central value of Rankin-Selberg L-function*, J. Amer. Math. Soc. **27** (2014), no. 2, 541–612, DOI 10.1090/S0894-0347-2014-00784-0. MR3164988 ↑1, 5, 6, 7, 13, 21, 22, 23, 24, 26, 27, 28, 29, 30, 34

Department of Mathematics, Columbia University, New York, NY, 10027
E-mail address: xuehang@math.columbia.edu