Fourier–Jacobi periods and local spherical character identities

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1 Introduction

This is a continuation of [Xue14, Xue16], which are devoted to the study the Gan–Gross–Prasad conjecture about the Fourier–Jacobi periods on $\mathrm{U}(n) \times \mathrm{U}(n)$ and their relation with the central value of certain Rankin–Selberg L-functions. In those papers, we proved the Gan–Gross–Prasad conjecture and its refinement for $\mathrm{U}(n) \times \mathrm{U}(n)$ under certain local conditions on the representations. The main tool is the relative trace formulae developed in [Liu14, Xue14]. It is observed in [Xue16] that this relative trace formula and the Jacquet–Rallis trace formulae often reduce to local harmonic analysis problems which can be tackled by similar techniques, though this similarity does not show

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up in the global setup. In this paper, by adapting some recent developments in the Jacquet–Rallis relative trace formulae [BPc, Xue] to our current situation, we improve our previous results from [Xue14, Xue16].

1.1 The main results

We state our main results in this subsection. We also take this opportunity to fix some notation which will be used throughout this paper.

Let E/F be a quadratic extension of number fields, $\mathbb{A}_F$, $\mathbb{A}_E$ their ring of adeles. The Galois involution of an element $g \in E$ is denoted by $\overline{g}$. Let E^- be the set of purely imaginary elements, viewed as an additive group over F. We fix a nonzero element $\tau \in E^-$ throughout. Let $\mathbb{A}_{F,\mathrm{f}}$ and $\mathbb{A}_{F,\infty}$ be the ring of finite adeles and infinite adeles respectively. We fix a nontrivial additive character $\psi = \otimes\psi_v : F\backslash\mathbb{A}_F \to \mathbb{C}^\times$. We use F^n (resp. F_n) to denote the space of n dimension vector space over F consisting of column (resp. row) vectors. Similarly, we have E^n, E_n etc. Let $\psi_E = \psi \circ \frac{1}{2}\operatorname{Tr}_{E/F}$ be an additive character of $\mathbb{A}_E$. Then $\psi_E(x) = \psi(x)$ if $x \in \mathbb{A}_F$. Let $\eta = \otimes\eta_v : F^\times\backslash\mathbb{A}_F^\times \to \{\pm 1\}$ be the quadratic character attached to the extension E/F via the class field theory. We fix a character $\mu = \otimes\mu_v : E^\times\backslash\mathbb{A}_E^\times \to \mathbb{C}^\times$ so that $\mu|_{\mathbb{A}_F^\times} = \eta$.

Let W be a hermitian space over E, with skew-hermitian form $\langle -, -\rangle$. Let $W_{/F}$ be the symplectic space over F whose underlying vector space is W, viewed as a vector space over F and whose symplectic pairing is given by $[-,-] = \frac{1}{2}\operatorname{Tr}_{E/F}\langle -,-\rangle$. Let $W_{/F}^\vee$ be the dual symplectic space of $W_{/F}$ and $L, L^\vee \subset W_{/F}^\vee$ be maximal isotropic subspaces of $W_{/F}^\vee$ so that the pairing $[-,-]|_{L\times L^\vee}$ is nondegenerate. Recall that we have fixed the additive character ψ and multiplicative character μ. Thus we have a Weil representation $\omega_{\psi,\mu}$ of $\mathrm{U}(W)(\mathbb{A}_F)$, realized on $\mathcal{S}(L(\mathbb{A}_F))$, and we may form the theta function $\theta_{\psi,\mu}(g,\phi)$ on the unitary group $\mathrm{U}(W)(\mathbb{A}_F)$ where $\phi \in \mathcal{S}(L(\mathbb{A}_F))$ is a Schwartz function on $L(\mathbb{A}_F)$.

Let $H = \mathrm{U}(W)$ and $G = \mathrm{U}(W) \times \mathrm{U}(W)$ and H embeds in G diagonally. Let $\pi = \pi_1 \boxtimes \pi_2$ be an irreducible cuspidal automorphic representation of $G(\mathbb{A}_F)$ where π_1, π_2 are irreducible cuspidal automorphic representations of $H(\mathbb{A}_F)$. Let $\varphi \in \pi$ and $\phi \in \mathcal{S}(L)$. We define the Fourier–Jacobi period

$$\mathcal{FJ}_{\psi,\mu}(\varphi,\phi) = \int_{[H]} \varphi(h)\overline{\theta_{\psi,\mu}(h,\phi)}\mathrm{d}h.$$

Here for any reduction group R, we put $[R] = R(F)\backslash R(\mathbb{A}_F)$.

The first main result of this paper is the following, which improves [Xue14, Theorem 1.1.1]. It proves the Gan–Gross–Prasad conjecture for $\mathrm{U}(n) \times \mathrm{U}(n)$ under some very mild local hypothesis.

Theorem 1.1.1. *Let $\pi = \pi_1 \boxtimes \pi_2$ be irreducible cuspidal automorphic representations of $G(\mathbb{A}_F)$. Assume that there are two nonarchimedean place v_1, v_2 of F, so that $\mathrm{BC}(\pi_{v_1})$ is supercuspidal and*

π_{v_2} *is tempered. Then the following are equivalent.*

1. $L(\frac{1}{2}, \mathrm{BC}(\pi_1) \times \mathrm{BC}(\pi_2) \otimes \mu^{-1}) \neq 0$*, where* $\mathrm{BC}(\pi_i)$ *stands for the base change of* π_i $(i = 1, 2)$*.*
2. *There is an* n*-dimensional skew-hermitian space and an irreducible cuspidal automorphic representation* π' *of* $\mathrm{U}(W')(\mathbb{A}_F) \times \mathrm{U}(W')(\mathbb{A}_F)$ *that is nearly equivalent to* π*, so that* $\mathcal{FJ}_{\psi,\mu}$ *is not identically zero on* π'*.*

Here for almost all place v of F, we fix an isomorphism $W'_v \simeq W_v$ which identifies $\mathrm{U}(W')(F_v)$ and $\mathrm{U}(W)(F_v)$. By π' being nearly equivalent to π, we mean that for almost all place v of F, the local components π'_v and π_v are isomorphic under this identification.

Compared with [Xue14, Theorem 1.1.1], the most significant improvement is that the assumption that E/F splits at all archimedean places are dropped. This is crucial for almost all arithmetic applications.

To state the second result, we need more notation. We take the Tamagawa measure on $G(\mathbb{A}_F)$ and on $H(\mathbb{A}_F)$ and use them to define the Petersson inner product $\langle -, - \rangle_{\mathrm{Pet}}$ on π and the Fourier–Jacobi periods $\mathcal{FJ}_{\psi,\mu}$. We use the self-dual measure with respect to ψ (resp. ψ_E) on $\mathbb{A}_{F,n}$ (resp. $\mathbb{A}_{E,n}$). Note that this gives a measure on $\mathbb{A}^-_{E,n}$. Similarly for each place v of F, we have the self-dual measure on $F_{v,n}$ etc. We normalize the norm $|-|_{\mathbb{A}_F}$ (resp. $|-|_{\mathbb{A}_E}$) so that $\mathrm{d}(ax) = |a|_{\mathbb{A}_F}\mathrm{d}x$ (resp. $\mathrm{d}(ax) = |a|_{\mathbb{A}_E}\mathrm{d}x$) where $\mathrm{d}x$ is the self-dual measure on $\mathbb{A}_F$ (resp. $\mathbb{A}_E$). Note that if $a \in \mathbb{A}_F$, then $|a|_{\mathbb{A}_E} = |a|^2_{\mathbb{A}_F}$. Similarly for each place v of F, we have the norm $|-|_{F_v}$, etc.

We fix a decomposition $\mathrm{d}h = \prod_v \mathrm{d}h_v$ of the Tamagawa measure on $H(\mathbb{A}_F)$ where $\mathrm{d}h_v$ is a measure on $H(F_v)$. We also fix a decomposition $\langle -, - \rangle_{\mathrm{Pet}} = \prod_v \langle -, - \rangle_v$ where $\langle -, - \rangle_v$ is an inner product on π_v. Let v be a place of F. The inner product on $\mathcal{S}(L(F_v))$ is given by $\langle \phi_v, \phi_v^\vee \rangle_v = \int_{L(F_v)} \phi_v(x)\overline{\phi_v^\vee(x)}\mathrm{d}x$. We put

$$\alpha_v(\varphi_v, \varphi_v^\vee, \phi_v, \phi_v^\vee) = \int_{H(F_v)} \langle \pi_v(h)\varphi_v, \varphi_v^\vee \rangle_v \overline{\langle \omega_{\psi,\mu}(h)\phi_v, \phi_v^\vee \rangle}\mathrm{d}h, \quad \varphi_v, \varphi_v^\vee \in \pi_v, \ \phi_v, \phi_v^\vee \in \mathcal{S}(L(F_v)),$$

and $\alpha_v(\varphi_v, \phi_v) = \alpha_v(\varphi_v, \varphi_v, \phi_v, \phi_v)$. We have shown in [Xue16, Proposition 1.1.1] that the defining integral of α_v is convergent if π_v is tempered. Moreover, for almost all places v, we have

$$\alpha_v(\varphi_v, \phi_v) = L(1, \eta_v)\zeta_{F_v}(2) \cdots L(n, \eta_v^n) \frac{L(\frac{1}{2}, \mathrm{BC}(\pi_{1,v}) \times \mathrm{BC}(\pi_{2,v}) \otimes \mu_v^{-1})}{L(1, \pi_{1,v}, \mathrm{Ad})L(1, \pi_{2,v}, \mathrm{Ad})}.$$

Thus we define

$$\alpha_v^\natural(\varphi_v, \phi_v) = \left(L(1, \eta_v)\zeta_{F_v}(2) \cdots L(n, \eta_v^n) \frac{L(\frac{1}{2}, \mathrm{BC}(\pi_{1,v}) \times \mathrm{BC}(\pi_{2,v}) \otimes \mu_v^{-1})}{L(1, \pi_{1,v}, \mathrm{Ad})L(1, \pi_{2,v}, \mathrm{Ad})} \right)^{-1} \alpha_v(\varphi_v, \phi_v).$$

We now state the second main result of this paper.

Theorem 1.1.2. *Let the notation be as above. Assume that E/F is split at all archimedean places, π is tempered, and there is a place v of F so that $\mathrm{BC}(\pi_v)$ is supercuspidal. Let $\varphi = \otimes\varphi_v \in \pi$ and $\phi = \otimes\phi_v \in \mathcal{S}(L(\mathbb{A}_F))$. Then*

$$|\mathcal{FJ}_{\psi,\mu}(\varphi,\phi)|^2 = \frac{1}{4}L(1,\eta)L(2,\eta^2)\cdots L(n,\eta^n)\frac{L(\frac{1}{2},\mathrm{BC}(\pi_1)\times\mathrm{BC}(\pi_2)\otimes\mu^{-1})}{L(1,\pi_1,\mathrm{Ad})L(1,\pi_2,\mathrm{Ad})}\prod_v \alpha_v^{\natural}(\varphi_v,\phi_v). \quad (1.1)$$

The conditions in [Xue16, Theorem 1.2.3] essentially require that π_v is either supercuspidal or unramified, which is a very strong assumption. Theorem 1.1.2 is an improvement to [Xue16, Theorem 1.2.3] to a large extend. It proves [Xue16, Conjecture 1.1.2] in many cases. One unsatisfactory point in this theorem is that it requires that E/F is split at all archimedean places. The author is working on removing such a hypothesis. As observed by Michael Harris, without this hypothesis, using the technique in this paper, one can proves a weaker statement that the identity in the theorem holds up to some nonzero constant c_∞ depending only on the archimedean components of π. In some situations, e.g. W is definite at all archimedean places, it can be proved that c_∞ is an algebraic number with some additional work (of course conjecturally $c_\infty = 1$). This result then might have some arithmetic applications.

1.2 The main ingredients

The proof of Theorem 1.1.1 and 1.1.2 is an adaption of the arguments from [Xue, BPb]. There are two main ingredients: a weak form of the existence of smooth transfer, cf. Proposition 2.4.2, and a local spherical character identity, cf. Theorem 4.1.2. The smooth transfer is relatively easy to obtain. It follows directly from the main results of [Xue]. As in [Xue], Theorem 1.1.1 follows from the existence of smooth transfer under the stronger hypothesis that v_1 and v_2 are both split. However, to remove this additional hypothesis needs the second ingredient: the local spherical character identity. This is an identity which relates the spherical character attached to π_v on the unitary groups to the spherical character attached to $\mathrm{BC}(\pi_v)$ on the general linear groups. The proof of this identity is more involved and proceeds in the following three steps.

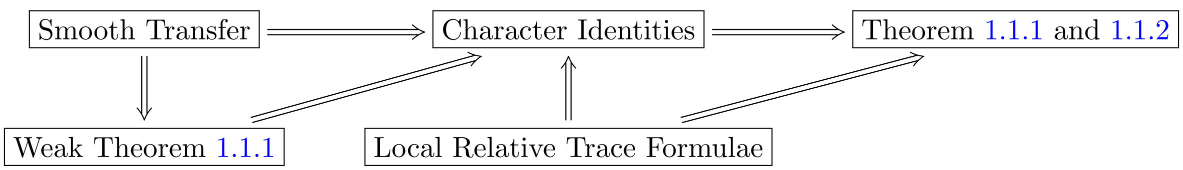
1. First we prove that if π_v is a local component of an irreducible cuspidal automorphic representations with nonzero Fourier–Jacobi periods, then the spherical character holds up to some constant depending only on π_v. Note that here we need to make use of the weak smooth transfer and weak version of Theorem 1.1.1 that we have just established.

2. The second step is to prove that the spherical character holds up to a constant for all π_v which admits a Fourier–Jacobi model. Here we use a global-to-local argument. The hard part is to prove that we can always globalize a dense part of the space of tempered representations

with nontrivial Fourier–Jacobi models to cuspidal automorphic representations with nonzero Fourier–Jacobi periods. This requires some work as we need to control the support of the Fourier transform of Schwartz functions of a certain special type.

3. The third step is to use a local relative trace formula to deduce the desired local spherical character identity.

This local spherical character identity together with the local relative trace formula will in turn provide a spectral criterion for the matching of test functions. It also yields information on the support of the local spherical character on the general linear groups. With all these ingredients, we will get Theorem 1.1.1 and 1.1.2. The above line of the argument can be summarized in the following diagram.



So even if we are only interested in Theorem 1.1.1, we still need the local spherical character identity. Even if we are only interested in Theorem 1.1.2 where the archimedean places are assumed to be split, we still need the "smooth transfer" at the nonsplit archimedean places!

1.3 Measures

We gather in this subsection our convention for the measures.

Let v be a place of F and $E_v = E \otimes_F F_v$. Let f be a Schwartz function on E_v. Let

$$\widehat{f}(y) = \int_{E_v} f(x)\psi(xy)\mathrm{d}x,$$

be its Fourier transform. Let $\mathrm{d}x$ be the self-dual measure on E_v for this Fourier transform. We define the measures on F_v in a similar way. We take the product measure $\mathrm{d}x = \prod_v \mathrm{d}x_v$ on $\mathbb{A}_E$ where $\mathrm{d}x_v$ is the self-dual measure on E_v. Similarly we define the self-dual measures on $\mathbb{A}_F$.

Let v be a place of F. Define the normalized multiplicative measure on $F_v^\times$ by

$$\mathrm{d}^\times x = \zeta_{F_v}(1)\frac{\mathrm{d}x}{|x|_{F_v}},$$

where $\mathrm{d}x$ is the self-dual measure on F_v. We also define the unnormalized one by

$$\mathrm{d}^* x = \frac{\mathrm{d}x}{|x|_{F_v}}.$$

Similarly we define the normalized and unnormalized multiplicative measures on $E_v^\times$. On $\mathrm{GL}_n(F_v)$, we take the measure

$$\mathrm{d}g = \zeta_{F_v}(1)\frac{\prod_{ij}\mathrm{d}x_{ij}}{|\det g|_{F_v}^n}, \quad g = (x_{ij}) \in \mathrm{GL}_n(F_v),$$

and similarly on $\mathrm{GL}_n(E_v)$. We take the product measure

$$\mathrm{d}g = \prod_v \mathrm{d}g_v$$

on $\mathrm{GL}_n(\mathbb{A}_F)$ where $\mathrm{d}g_v$ is the measure on $\mathrm{GL}_n(F_v)$ defined above. Similarly we take the product measure on $\mathrm{GL}_n(\mathbb{A}_E)$.

Let V be a skew-hermitian space over E. Let $\mathrm{U}(V)$ be the corresponding unitary group and $\mathfrak{u}(V)$ its Lie algebra. Let $\mathrm{d}X$ on $\mathfrak{u}(V)(\mathbb{A}_F)$ be the self-dual measure for the Fourier transform

$$\widehat{f}(Y) = \int_{\mathfrak{u}(V)(\mathbb{A}_F)} f(X)\psi(\mathrm{Tr}\, XY)\mathrm{d}X,$$

where f is a Schwartz function on $\mathfrak{u}(V)(\mathbb{A}_F)$. Similarly we can define the self-dual measure $\mathrm{d}X_v$ on $\mathfrak{u}(V)(F_v)$ for any place v of F. Define the Cayley transform $\mathfrak{c} : \mathfrak{u}(V) \to \mathrm{U}(V)$ by $X \mapsto (1+X)(1-X)^{-1}$ whenever it makes sense. Let ω be the top invariant differential form on $\mathrm{U}(V)$ so that its pullback $\mathfrak{c}^*\omega$ via the Caylay transform gives rise to the self-dual measure on $\mathfrak{u}(V)$. We choose the measure on $\mathrm{U}(V)(F_v)$ as $|\omega|_{F_v}$. It is the unique Haar measure on $\mathrm{U}(V)(F_v)$ such that the Caylay transform is measure preserving when restricted to a small neighborhood of $0 \in \mathfrak{u}(V)(F_v)$. We define the measure on $\mathrm{U}(V)(\mathbb{A}_F)$ by

$$\mathrm{d}g = \prod_v L(1,\eta_v)|\omega|_{F_v}.$$

The Tamagawa measure on $\mathrm{U}(V)(\mathbb{A}_F)$ is $L(1,\eta)^{-1}\mathrm{d}g$.

1.4 Spaces of test functions

We will be using the following spaces of test functions.

Let F be a local field of characteristic zero. Let G be a reductive group over F. Then we fix a (logarithmic) height function $\sigma : G(F) \to [1,\infty)$ on G. We it as follows. Let us fix an embedding $\iota : G \to \mathrm{GL}_N$ and put $\|g\| = \max\{1, \iota(g)_{ij}, \iota(g^{-1})_{ij}\}$. Then we define $\sigma(g) = 1 + \log\|g\|$. We let Ξ^G or Ξ when there is no confusion about the group G be the Harish-Chandra Ξ function of G.

1. Let M be an locally compact totally disconnected space if F is nonarchimedean and a Nash manifold if F is real. We denote by $\mathcal{S}(M)$ the space of Schwartz functions on M. If F is nonarchimedean, this is the space of locally constant and compactly supported functions on

M. If F is archimedean, this is the space of functions such that itself and all its derivatives are of fast decay. We refer the readers to [AG08] for the notion of Schwartz functions on Nash manifolds.

2. Let G be a reductive group. We denote by $\mathcal{C}(G(F))$ the Harish-Chandra Schwartz space. If F is nonarchimedean, then this is the space of locally constant functions f on $G(F)$ so that $|f(g)| \ll \Xi^G(g)\sigma(g)^{-d}$ for all $d > 0$. If F is archimedean, then this is the space of functions f on $G(F)$ so that $|X.f(g)| \ll \Xi^G(g)\sigma(g)^{-d}$ for all $d > 0$ and all differential operators X in the enveloping algebra of the Lie algebra of G. Note that $\mathcal{S}(G)$ is dense in $\mathcal{C}(G)$.

Let F be a number field. Let V be a vector space over F. We denote by $\mathcal{S}(V(\mathbb{A}_F))$ be the usual space of Schwartz–Bruhat functions on $V(\mathbb{A}_F)$. Let G be a reductive group over F and we fix a hyperspecial maximal compact subgroup K_v of $G(F_v)$ for almost all v. We define $\mathcal{S}(G(\mathbb{A}_F)) = \otimes' \mathcal{S}(G(F_v))$, where the restricted tensor product is taken with respect to the characteristic function of K_v for almost all v.

1.5 Additional notation and conventions

Here is some additional notation and conventions.

– Let G be a group. Let f be a function on G and $g \in G$. We put $\mathsf{L}(g)f(x) = f(g^{-1}x)$ and $\mathsf{R}(g)f(x) = f(xg)$.

– We will speak of the hermitian pairings on vectors space. By this, we always mean a pairing which is linear in the first variable and anti-linear in the second variable.

– We will write ω for $\omega_{\psi,\mu}$. The characters ψ and μ will be fixed throughout this paper. Similarly we write $\mathcal{FJ}$, θ, etc.

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2 Simple relative trace formulae

In this section, we first state and prove the simple relative trace formulae. They are slightly more general than the ones in [Xue14]. We give complete proofs for the simple relative trace formula on the general linear groups since the treatment in [Xue14] is rather *ad hoc* with the issue arising from the centers of the groups. The nice test functions that we choose here are slightly different from the ones in [Xue14]. The other proofs are only sketched, as once the statements are correctly formulated, their proofs are almost identical to the ones in [Xue14].

2.1 Group actions and norms

Let us recall some notion from group actions. Let F be a number field. Let A be a reductive group and V an affine variety over F on which A acts. We say an element $v \in V$ is regular semisimple if the following two conditions are satisfied.

– The orbit of v is closed.

– The stablizer of v is of minimal dimension.

In this case, we also say that the orbit of v is regular semisimple. If V is a vector space, then we say that $v \in V$ is nilpotent if the closure of its orbit contains zero. We denote by $V \to V//A$ or simply $V//A$ the categorical quotient (if exists). We denote by $V(F)_{\mathrm{rs}}$ the regular semisimple element in $V(F)$ and $V(F)/A(F)$ (resp. $V(F)_{\mathrm{rs}}/A(F)$) the set of (resp. regular semisimple) orbits.

For any algebraic variety X over F, we fix a (equivalence class of) norm on $X(\mathbb{A}_F)$ as in [BPc, Appendix A.1], denoted by $\|-\|_X$ or $\|\cdot\|$ when there is no confusion with the variety X. We refer the readers to [BPc, Proposition A.1.1] for various properties of the norm. We will need the following lemma, which is a slight improvement of [BPc, Proposition A.1.1 (v) (vii)], which corresponds to $s = 0$.

Lemma 2.1.1. *Suppose that G is an affine algebraic group and X is an affine algebraic varieties which carries a G-action and that the quotient map $p : X \to Y$ is a G-torsor. Let $S \subset \mathbb{C}$ be a bounded vertical strip. Assume that the natural map $G(F)\backslash X(F) \to Y(F)$ is injective. Let $\alpha : G \to F^\times$ be a character. Fix a right invariant Haar measure $\mathrm{d}g$ on $G(\mathbb{A}_F)$. Then for sufficiently large d, we have*

$$\sum_{x \in G(F)\backslash X(F)} \int_{G(\mathbb{A}_F)} \|g.x\|_X^{-d} |\alpha(g)|^s \mathrm{d}g$$

is absolutely convergent uniformly for $s \in S$.

Proof. The lemma is clear if the cover $X \to Y$ splits, i.e. $X \simeq Y \times G$. In general, let us cover Y by a finite number of open subsets $\{U_i\}$ so that for each U_i, the cover $p^{-1}(U_i) \to U_i$ splits. Moreover by [BPc, Proposition A.1.1 (iv)], if $x \in p^{-1}(U_i)(F)$, then $\|g.x\|_X \simeq \|g.x\|_{p^{-1}(U_i)}$ since $p(x) \in Y(F)$. We are then reduce to the split case. $\square$

2.2 General linear groups

Let $H' = \mathrm{Res}_{E/F}\,\mathrm{GL}_n$, $G' = H' \times H'$. Let $H'_1 = H'$ viewed as a subgroup of G' via the diagonal embedding, $H'_2 = \mathrm{GL}_n \times \mathrm{GL}_n$ (over F), viewed as a subgroup of G' as embedded componentwise. If R is a group, we denote by Z_R its center and A_R its split center. Thus R/A_R is anisotropic. Put $\widetilde{G'} = G'/A_{G'}$, $\widetilde{H'_1} = H'_1/A_{H'_1}$ and $\widetilde{H'_2} = H'_2/A_{H'_2}$. Note that $A_{G'} = A_{H'_2} = Z_{H'_2}$.

The group $H'(\mathbb{A}_E)$ acts on $\mathcal{S}(\mathbb{A}_{E,n})$ by

$$\Omega_\mu(g)\Phi(x) = \mu(\det g)|\det g|^{\frac{1}{2}}\Phi(xg). \tag{2.1}$$

Let $\Phi \in \mathcal{S}(\mathbb{A}_{E,n})$ be a Schwartz function and we define the mirabolic Eisenstein series by

$$E(g,\mu,\Phi,s) = |\det g|^{s-\frac{1}{2}} \sum_{x\in E_n} \int_{a\in A_{H_1'}(\mathbb{A}_F)} \Omega_\mu(ag)\Phi(x)|a|^{n(s-\frac{1}{2})}\mathrm{d}a.$$

This is absolutely convergent if $\Re s \gg 0$ and has a meromorphic continuation to the complex plane. It is holomorphic at $s = \frac{1}{2}$ (c.f. [JS81]).

We define a character $\eta_n : H_2' \to \{\pm 1\}$ by $\eta_n(h_1,h_2) = \eta^{n+1}(\det h_1 \det h_2)$. Let $\Pi = \Pi_1 \boxtimes \Pi_2$ be the irreducible cuspidal automorphic representations of $G'(\mathbb{A}_F)$, where Π_1, Π_2 are irreducible cuspidal automorphic representations of $H'(\mathbb{A}_F)$. We assume that the central character of Π is trivial on $Z_{H_2'}(\mathbb{A}_F)$. Let $\varphi \in \Pi$ and $\Phi \in \mathcal{S}(\mathbb{A}_{E,n})$. Define the following linear forms

$$\lambda(\varphi,\Phi,s) = \int_{[\widetilde{H_1'}]} \varphi(h)E(h,\mu^{-1},\Phi,s)\mathrm{d}h, \quad \beta(\varphi) = \int_{[\widetilde{H_2'}]} \varphi(h)\eta_n(h)\mathrm{d}h.$$

Let $S_n = \{\gamma \in \mathrm{Res}_{E/F}\mathrm{GL}_n \mid \gamma\overline{\gamma} = 1\}$ and $X_n = S_n \times F_n \times E^{-,n}$. This is an algebraic variety over F and the group GL_n acts on it via

$$h.[\gamma,x,y] = [h^{-1}\gamma h, xh, h^{-1}y].$$

The invariant of $[\gamma,x,y]$ is the following $2n$-tuple

$$\mathrm{Tr}\wedge^i\gamma,\ x\gamma^j y, \quad i = 1,\cdots,n,\ j = 0,\cdots,n-1.$$

We define a partial Fourier transform

$$\mathcal{S}(\mathbb{A}_{E,n}) \to \mathcal{S}(\mathbb{A}_{F,n} \times \mathbb{A}_E^{-,n}), \quad \Phi \mapsto \Phi^\dagger \tag{2.2}$$

where

$$\Phi^\dagger(x,y) = \int_{\mathbb{A}_{E,n}^-} \Phi(x+x^-)\psi(x^-y)\mathrm{d}x^-, \quad x \in \mathbb{A}_{F,n},\ y \in \mathbb{A}_E^{-,n}.$$

We define a map

$$\mathcal{S}(G'(\mathbb{A}_F) \times \mathbb{A}_{E,n}) \to \mathcal{S}(X_n(\mathbb{A}_F)), \quad (f',\Phi) \mapsto \Upsilon_{f',\Phi}$$

by

$$\Upsilon_{f',\Phi}(\gamma,x,y) = \begin{cases} \displaystyle\int_{\mathrm{GL}_n(\mathbb{A}_F)}\int_{\mathrm{GL}_n(\mathbb{A}_E)} f'(g^{-1},g^{-1}ah)\overline{(\Omega_\mu(g)\Phi)^\dagger(x,y)}\mathrm{d}g\mathrm{d}h, & n \text{ is odd,} \\ \displaystyle\int_{\mathrm{GL}_n(\mathbb{A}_F)}\int_{\mathrm{GL}_n(\mathbb{A}_E)} f'(g^{-1},g^{-1}ah)\mu(\det ah)\overline{(\Omega_\mu(g)\Phi)^\dagger(x,y)}\mathrm{d}g\mathrm{d}h, & n \text{ is even,} \end{cases} \tag{2.3}$$

and $\gamma = aa^{\tau,-1} \in S_n(\mathbb{A}_F)$, $a \in \mathrm{GL}_n(\mathbb{A}_E)$.

Let $[\gamma, x, y] \in X_n(F)$ be a regular semisimple element, $f' \in \mathcal{S}(G'(\mathbb{A}_F))$ and $\Phi \in \mathcal{S}(\mathbb{A}_{E,n})$. Define

$$O([\gamma,x,y], f', \Phi, s) = \int_{\mathrm{GL}_n(\mathbb{A}_F)} \Upsilon_{f',\Phi}(h.[\gamma,x,y])\eta(\det h)|\det h|^s \mathrm{d}h,$$

and put $O([\gamma,x,y], f', \Phi) = O([\gamma,x,y], f', \Phi, \frac{1}{2})$. Note that since $[\gamma,x,y]$ is regular semisimple, this integral is absolutely convergent and defines a holomorphic function in s.

Let v be a place of F. Then we have the local counterpart of the Fourier transform $\mathcal{S}(E_{v,n}) \to \mathcal{S}(F_{v,n} \times E^{-,n})$, the map $\mathcal{S}(G'(F_v) \times E_{v,n}) \to \mathcal{S}(X_n(F))$, which we denote using the same notation as in the global case. We define the local orbital integral using the same formula, integrating over $\mathrm{GL}_n(F_v)$ instead.

We say that $(f', \Phi) \in \mathcal{S}(G'(\mathbb{A}_F) \times \mathbb{A}_{E,n})$ is a nice test function, if the following conditions hold.

1. $f' = \otimes f'_v$, $\Phi = \otimes \Phi_v$ are factorizable.

2. There is a nonarchimedean place v_1 of F and a finite union of cuspidal Bernstein components Ω of $G'(F_{v_1})$ such that $f'_{v_1} \in \mathcal{S}(G'(F_v))_\Omega$ (c.f. [Ber84]). In concrete terms, this means that the integral of f'_{v_1} over $A_{G'}(\mathbb{A}_F)$ is a finite linear combination of matrix coefficients of supercuspidal representation of $\widetilde{G'}(F_{v_1})$.

3. There is a place $v_2 \neq v_1$ of F so that $\Upsilon_{f'_{v_2},\Phi_{v_2}}$ is supported in the regular semisimple locus of $X_n(F_{v_2})$.

Let $(f', \Phi) \in \mathcal{S}(G'(\mathbb{A}_F) \times \mathbb{A}_{E,n})$ be a test function. Define $\widetilde{f}' \in \mathcal{S}(\widetilde{G'}(\mathbb{A}_F))$ by

$$\widetilde{f}'(g) = \int_{Z_{H'_2}(\mathbb{A}_F)} f'(zg)\mathrm{d}z.$$

Let $K'_0 \subset G'(\mathbb{A}_{F,\mathrm{f}})$ be an open compact subgroup so that f' is bi-K'_0-invariant. We define the automorphic kernel function on $\widetilde{G'}(\mathbb{A}_F) \times \widetilde{G'}(\mathbb{A}_F)$ as usual

$$K_{\widetilde{f}'}(x,y) = \sum_{\gamma \in \widetilde{G'}(F)} \widetilde{f}'(x^{-1}\gamma y), \quad x, y \in \widetilde{G'}(\mathbb{A}_F).$$

Define

$$I(f', \Phi, s) = \int_{[\widetilde{H'_1}]} \int_{[\widetilde{H'_2}]} K_{\widetilde{f}'}(g,h) E(g, \mu^{-1}, \Phi, s)\eta_n(h)\mathrm{d}g\mathrm{d}h, \tag{2.4}$$

and we put $I(f', \Phi) = I(f', \Phi, \frac{1}{2})$.

Fix a maximal compact subgroup K'_∞ of $G'(\mathbb{A}_{F,\infty})$ with the Lie algebras $\mathfrak{k}'_\infty$ and $\mathfrak{g}'_\infty$. Let $C_{G'} \in \mathcal{U}(\mathfrak{g}'_\infty)$ and $C_{K'} \in \mathcal{U}(\mathfrak{k}'_\infty)$ be the Casimir elements of $\mathfrak{g}'_\infty$ and $\mathfrak{k}'_\infty$ respectively. Let Π be an irreducible cuspidal automorphic representation of $G'(\mathbb{A}_F)$ whose central character is trivial when restricted to $A_{H_2'}(\mathbb{A}_F)$. The Petersson inner product on Π is given by

$$\langle \varphi_1, \varphi_2 \rangle_{\mathrm{Pet}} = \int_{Z_{G'}(\mathbb{A}_F)G'(F)\backslash G'(\mathbb{A}_F)} \varphi_1(g)\overline{\varphi_2(g)}\mathrm{d}g. \tag{2.5}$$

Here and below, when we say "summing over an orthonormal basis of Π" or "something runs over an orthonormal basis of Π", we mean that we choose a sufficiently small open compact subgroup K_0' of $G'(\mathbb{A}_{F,\mathrm{f}})$ and sum over an orthonormal basis of $\Pi^{K_0'}$ consisting of $C_{G'}$ and $C_{K'}$ eigenvectors. Define

$$I_\Pi(f', \Phi) = \sum_{\varphi} \lambda(\Pi(f')\varphi, \Phi)\overline{\beta(\varphi)}, \tag{2.6}$$

where φ runs over an orthonormal basis of Π.

Proposition 2.2.1. *Suppose that (f', Φ) is a nice test function. Then*

1. *The sum*

$$\int_{[\widetilde{H_1'}]}\int_{[\widetilde{H_2'}]}\sum_{\varphi}|\mathrm{R}(\widetilde{f'})\varphi(g)||\varphi(h)||E(g,\mu^{-1},\Phi,s)|\mathrm{d}g\mathrm{d}h$$

   is absolutely convergent when $E(g,\mu^{-1},\Phi,s)$ is holomorphic, where φ runs over an orthonormal basis of Π. The convergence is locally uniformly for s. In particular, the sum (2.6) is absolutely convergent.

2. *The integral (2.4) is absolutely convergent and defines a meromorphic function in s. It is holomorphic when $E(g,\mu^{-1},\Phi,s)$ is holomorphic. Moreover*

$$I(f', \Phi) = \frac{1}{4L(1,\eta)^2}\sum_{\Pi} I_\Pi(f', \Phi).$$

3. *Let $\widetilde{X} = G' \times F_n \times E^{-,n}$ and the group $H_1' \times H_2'$ acts on it as follows. Let $[\gamma_1, \gamma_2, x, y] \in \widetilde{X}$, $g \in H_1'$ and $(h_1, h_2) \in H_2'$. Then $(g; h_1, h_2).[\gamma_1, \gamma_2, x, y] = [g^{-1}\gamma_1 h_1, g^{-1}\gamma_2 h_2, xh_1, h_1^{-1}y]$. The expression*

$$\sum_{H_1'(F)\backslash\widetilde{X}(F)_{\mathrm{rs}}/H_2'(F)} \int_{\mathrm{GL}_n(\mathbb{A}_F)}\int_{\mathrm{GL}_n(\mathbb{A}_F)}\int_{\mathrm{GL}_n(\mathbb{A}_E)} f'(g^{-1}\gamma_1 h_1, g^{-1}\gamma_2 h_2)$$
$$\mu(\det h_1)^{-1}(\Omega_{\mu^{-1}}(h_1^{-1}\gamma_1^{-1}g)\Phi)^\dagger(xh_1, h_1^{-1}y)|\det g|^{s-\frac{1}{2}}\mathrm{d}g\mathrm{d}h_1\mathrm{d}h_2,$$

   is absolutely convergent for all s, uniformly for s in any bounded vertical strip. In particular, it defines a holomorphic function in s.

4. *We have*

$$\sum_{[\gamma,x,y]\in X_n(F)_{\mathrm{rs}}/\operatorname{GL}_n(F)} O([\gamma,x,y],f',\Phi) = \frac{1}{4L(1,\eta)^2}\sum_{\Pi} I_\Pi(f',\Phi), \tag{2.7}$$

where the sum on the right hand side ranges over the set of irreducible cuspidal automorphic representations of $G'(\mathbb{A}_F)$. Both sides are absolutely convergent.

Proof. 1. By [BPc, Appendix (7)], we have

$$\sum_{\varphi\in\mathcal{B}} |\mathrm{R}(\widetilde{f'})\varphi(g_1)||\varphi(g_2)| \ll \|g_1\|_{G'}^{-d}\|g_2\|_{G'}^{-d}$$

for all $d>0$. It is well-known (c.f. [JS81, Lemma 4.2]) that $E(g,\mu^{-1},\Phi,s)$ is of moderate growth, i.e. there is a $d>0$ so that

$$|E(g,\mu^{-1},\Phi,s)| \ll \|g\|_{H'}^{d},$$

and this estimate is locally uniform in s. Now the first assertion follows from [BPc, Proposition A.1.1(ix)]. Note that this in particular implies that the sum (2.6) and the sum in the right hand side of (2.7) are absolutely convergent.

2. We now prove the second assertion. Put

$$K^{\mathrm{cusp}}_{\widetilde{f'},\Phi}(x,y) = \sum_{\Pi}\sum_{\varphi\in\Pi} \Pi(\widetilde{f'})\varphi(x)\overline{\varphi(y)}, \quad x,y\in G'(\mathbb{A}_F),$$

where the outer sum runs over all irreducible cuspidal automorphic representation of $\widetilde{G'}(\mathbb{A}_F)$, and the inner sum runs over an orthonormal basis of Π. Note that here the inner product on Π is given by the L^2-norm on $[\widetilde{G'}]$. Since f' is nice, $\widetilde{f'}_{v_1}$ is a sum of matrix coefficients of supercuspidal representations. Thus $K_{\widetilde{f'},\Phi} = K^{\mathrm{cusp}}_{\widetilde{f'},\Phi}$ and

$$I(f',\Phi,s) = \int_{[\widetilde{H'_1}]}\int_{[\widetilde{H'_2}]}\sum_{\Pi}\sum_{\varphi\in\Pi}\Pi(\widetilde{f'})\varphi(h_1)\overline{\varphi(h_2)}E(h_1,\mu^{-1},\Phi,s)\mathrm{d}h_1\mathrm{d}h_2.$$

By the first assertion, the right hand side is absolutely convergent locally uniformly in s when $E(g,\mu^{-1},\Phi,s)$ is holomorphic. Therefore $I(f',\Phi,s)$ is holomorphic in s when $E(g,\mu^{-1},\Phi,s)$ is holomorphic. Moreover we can switch the order of integration and summation and conclude that

$$I(f',\Phi) = \sum_{\Pi}\sum_{\varphi\in\Pi}\int_{[\widetilde{H'_1}]}\int_{[\widetilde{H'_2}]}\Pi(\widetilde{f'})\varphi(h_1)\overline{\varphi(h_2)}E(h_1,\mu^{-1},\Phi,\frac{1}{2})\mathrm{d}h_1\mathrm{d}h_2.$$

We denote the term on the right hand side indexed by Π by $I'_\Pi(f',\Phi)$. Note that the Petersson inner product on Π when we define I_Π is given by the integral over $Z_{G'}(\mathbb{A}_F)G'(F)\backslash G'(\mathbb{A}_F)$ while here the inner product is given by the integral over $Z_{H'_2}(\mathbb{A}_F)G'(F)\backslash G'(\mathbb{A}_F)$. Therefore we conclude that

$$I_\Pi(f',\Phi) = 4L(1,\eta)^2 I'_\Pi(f',\Phi).$$

Therefore

$$I(f', \Phi) = \frac{1}{4L(1,\eta)^2} \sum_{\Pi} I_\Pi(f', \Phi).$$

3. For the third assertion, observe that the function

$$(g_1, g_2, x, y) \mapsto f'(g_1, g_2)(\Omega_{\mu^{-1}}(g_1))^\dagger(x, y)$$

is a Schwartz function on $\widetilde{X}(\mathbb{A}_F)$. It follows that

$$|f'(g^{-1}\gamma_1 h_1, g^{-1}\gamma_2 h_2)(\Omega_{\mu^{-1}}(h_1^{-1}\gamma_1^{-1}g)\Phi)^\dagger(xh_1, h_1^{-1}y)| \ll \|(g; h_1, h_2).[\gamma_1, \gamma_2, x, y]\|_{\widetilde{X}}^{-d}$$

for all $d > 0$. We thus only need to prove that for s being in a bounded vertial strip, we may choose sufficiently large d so that

$$\sum_{H_1'(F)\backslash \widetilde{X}(F)_{\rm rs}/H_2'(F)} \int_{\mathrm{GL}_n(\mathbb{A}_F)} \int_{\mathrm{GL}_n(\mathbb{A}_F)} \int_{\mathrm{GL}_n(\mathbb{A}_E)} \|(g; h_1, h_2).[\gamma_1, \gamma_2, x, y]\|_{\widetilde{X}}^{-d} |\det g|^s \mathrm{d}g \mathrm{d}h_1 \mathrm{d}h_2 < \infty. \tag{2.8}$$

Let $Q_{\rm rs} = H_1' \backslash \widetilde{X}_{\rm rs}/H_2'$ be the geometric quotient. This is again an affine variety over F and the morphism $\widetilde{X}_{\rm rs} \to Q_{\rm rs}$ is an $H_1' \times H_2'$-torsor. By [BPc, Proposition A.1.1(iv)], we have

$$\|(g; h_1, h_2).[\gamma_1, \gamma_2, x, y]\|_{\widetilde{X}} \sim \|(g; h_1, h_2).[\gamma_1, \gamma_2, x, y]\|_{\widetilde{X}_{\rm rs}}.$$

Then the convergence of (2.8) follows from Lemma 2.1.1.

4. We now prove the fourth assertion. By definition

$$E(g, \mu^{-1}, \Phi, s) = |\det g|^{s-\frac{1}{2}} \sum_{x \in E_n/F^\times} \int_{a \in \mathbb{A}_F^\times} \Omega_{\mu^{-1}}(ag)\Phi(x)|a|^{n(s-\frac{1}{2})} \mathrm{d}a,$$

if $\Re s \gg 0$. Here the action of $F^\times$ on E_n is componentwise multiplication. In this case, we may switch the order of integration and summation and conclude that

$$E(g, \mu^{-1}, \Phi, s) = |\det g|^{s-\frac{1}{2}} \int_{F^\times \backslash \mathbb{A}_F^\times} \sum_{x \in E_n} \Omega_{\mu^{-1}}(ag)\Phi(x)|a|^{n(s-\frac{1}{2})} \mathrm{d}a.$$

Applying the Poisson summations formula, for any $h_1 \in \mathrm{GL}_n(\mathbb{A}_F)$ and $\gamma_1 \in \mathrm{GL}_n(F)$, we have

$$\sum_{v \in E_n} \Omega_{\mu^{-1}}(ag)\Phi(v) = \sum_{(x,y) \in F_n \times E^{-,n}} \mu(\det h_1)^{-1}(\Omega_{\mu^{-1}}(h_1^{-1}\gamma_1^{-1}g)\Phi)^\dagger(xh_1 a, a^{-1}h_1^{-1}y).$$

By definition, if $\Re s \gg 0$, we have

$$\begin{aligned} I(f', \Phi, s) = \iint_{[\widetilde{H_2'}]} \int_{[\widetilde{H_1'}]} \sum_{(\gamma_1,\gamma_2) \in \mathrm{GL}_n(E) \times \mathrm{GL}_n(E)} \int_{[A_{H_1'}]} \sum_{(x,y) \in F_n \times E^{-,n}} \widetilde{f}'(g^{-1}\gamma_1 h_1, g^{-1}\gamma_2 h_2) \\ \mu(\det h_1)^{-1}(\Omega_{\mu^{-1}}(h_1^{-1}\gamma_1^{-1}g)\Phi)^\dagger(xh_1 a, a^{-1}h_1^{-1}y)|\det ag|^{s-\frac{1}{2}} \mathrm{d}a \mathrm{d}g \mathrm{d}h_1 \mathrm{d}h_2. \end{aligned} \tag{2.9}$$

If we switch the order of integration and summation, we have

$$\begin{aligned} I(f', \Phi, s) = & \sum_{[\gamma_1, \gamma_2, x, y] \in H_1'(F) \backslash \widetilde{X}(F)_{\mathrm{rs}} / H_2'(F)} \int_{\mathrm{GL}_n(\mathbb{A}_F)} \int_{\mathrm{GL}_n(\mathbb{A}_F)} \int_{\mathrm{GL}_n(\mathbb{A}_E)} f'(g^{-1}\gamma_1 h_1, g^{-1}\gamma_2 h_2) \\ & \mu(\det h_1)^{-1} (\Omega_{\mu^{-1}}(h_1^{-1}\gamma_1^{-1} g)\Phi)^{\dagger}(x h_1, h_1^{-1} y) |\det g|^{s-\frac{1}{2}} \mathrm{d}g \mathrm{d}h_1 \mathrm{d}h_2. \end{aligned} \tag{2.10}$$

Note that what we sum over is noting but $X(F)_{\mathrm{rs}}/\mathrm{GL}_n(F)$ and the summand is the orbital integral $O([\gamma, x, y], f', \Phi, s)$. By the third assertion, the right hand side of (2.10) is absolutely convergent and defines a holomorphic function in s. This also shows that the change of order of integration and summation in (2.10) is legitimate. Since both sides of (2.10) are meromorphic functions in s and are holomorphic at $s = \frac{1}{2}$, the fourth assertion is then proved by evaluating (2.10) at $s = \frac{1}{2}$. □

2.3 Unitary groups

The simple relative trace formulae on the unitary groups can be derived in the same fashion as the ones on the general linear groups. We only state the results and leave the proof to the interested reader.

Let $f \in \mathcal{S}(G(\mathbb{A}_F))$ and $\phi_1, \phi_2 \in \mathcal{S}(L(\mathbb{A}_F))$. Let $Y = \mathrm{U}(W) \times W^{\vee}$ and $\mathrm{U}(W)$ acts on Y via

$$h.[\delta, w^{\vee}] = [h^{-1}\delta h, w^{\vee} h].$$

The invariant of $[\delta, w^{\vee}]$ is defined to be the $2n$-tuple

$$\mathrm{Tr} \wedge^i \delta, \ [w^{\vee}\delta^j, w^{\vee}], \quad i = 1, \cdots, n, \ j = 0 \cdots, n-1.$$

We define a Fourier transform

$$(\phi_1 \otimes \phi_2)^{\ddagger}(w^{\vee}) = \int_{L(\mathbb{A}_F)} \phi_1(x+z)\phi_2(x-z)\psi([z, y])\mathrm{d}z,$$

where $w^{\vee} = x + y$, $x \in L$, $y \in L^{\vee}$. We define a map

$$\mathcal{S}(G(\mathbb{A}_F) \times L(\mathbb{A}_F) \times L(\mathbb{A}_F)) \to \mathcal{S}(Y(\mathbb{A}_F)), \quad (f, \phi_1, \phi_2) \mapsto \Psi_{f, \phi_1, \phi_2},$$

where

$$\Psi_{f, \phi_1, \phi_2}([\delta, w^{\vee}]) = \int_{\mathrm{U}(W)(\mathbb{A}_F)} f(g^{-1}, g^{-1}\delta)(\overline{\omega(g)\phi_1} \otimes \phi_2)^{\ddagger}(w^{\vee})\mathrm{d}g. \tag{2.11}$$

Let $[\delta, w^{\vee}] \in Y(F)$, $f \in \mathcal{S}(G(\mathbb{A}_F))$ and $\phi_1, \phi_2 \in \mathcal{S}(L(\mathbb{A}_F))$. We define the orbital integral

$$O([\delta, w^{\vee}], f, \phi_1, \phi_2) = \int_{\mathrm{U}(W)(\mathbb{A}_F)} \Psi_{f, \phi_1, \phi_2}(h.[\delta, w^{\vee}])\mathrm{d}h.$$

It is absolutely convergent if $[\delta, w^{\vee}]$ is regular semisimple. For any place v of F, we also have the local counterpart of the Fourier transform and the map which we denote using the same notation as in the global case. We also define the local orbital integrals using the same formula, integrating over $\mathrm{U}(W)(F_v)$ instead.

We say that $(f, \phi_1, \phi_2) \in \mathcal{S}(G(\mathbb{A}_F) \times L(\mathbb{A}_F) \times L(\mathbb{A}_F))$ is a nice test function, if the following conditions hold.

1. $f = \otimes f_v$, $\phi_1 = \otimes \phi_{1,v}, \phi_2 = \otimes \phi_{2,v}$ are factorizable.

2. There is a nonarchimedean place v_1 and a finite union of Bernstein components Ω of $G(F_{v_1})$ such $f_{v_1} \in \mathcal{S}(G(F_v))_\Omega$.

3. There is a place $v_2 \neq v_1$ of F so that $\Psi_{f_{v_2}, \phi_{1,v_2}, \phi_{2,v_2}}$ is supported in the regular semisimple locus of $Y(F_{v_2})$.

Let π be an irreducible cuspidal automorphic representation of $G(\mathbb{A}_F)$. The Petersson inner product on π is given by

$$\langle \varphi_1, \varphi_2 \rangle_{\mathrm{Pet}} = \int_{G(F)\backslash G(\mathbb{A}_F)} \varphi_1(g)\overline{\varphi_2(g)}\mathrm{d}g. \tag{2.12}$$

Define

$$J_\pi(f, \phi_1, \phi_2) = \sum_{\varphi} \mathcal{FJ}(\pi(f)\varphi, \phi_1)\overline{\mathcal{FJ}(\varphi, \phi_2)}, \tag{2.13}$$

where φ runs over an orthonormal basis of π (which is interpreted in an analogous way as in the previous subsection).

Proposition 2.3.1. *We have*

$$\sum_{[\delta, w^\vee] \in H(F)\backslash Y_{\mathrm{rs}}(F)/H(F)} O([\delta, w^\vee], f, \phi_1, \phi_2) = \sum_{\pi} J_\pi(f, \phi_1, \phi_2).$$

Both sides are absolutely convergent.

This proposition can be proved in the same way as Proposition 2.2.1.

2.4 Comparison

In this subsection, we are going to vary the skew-hermitian space W. So we denote the objects from the previous subsection by $H(W)$, $G(W)$, $Y(W)$, f^W, ϕ_1^W, ϕ_2^W etc. to stress the dependance on W.

For any field F' containing F, there is a one-to-one correspondence between the regular semisimple orbits

$$H_1'(F')\backslash X_n(F')_{\mathrm{rs}}/H_2'(F') \simeq \coprod_W H(W)(F')\backslash Y(W)(F')_{\mathrm{rs}}/H(W)(F'),$$

where on the right hand side, the space W ranges over all isomorphism classes of skew-hermitian spaces of dimension n. Two elements correspond or match if their invariants are the same.

Let v be a place of F. We define a transfer factor $\mathbf{t}$ as follows. Let $[\gamma, x, y] \in X_n(F_v)$. Define

$$\mathbf{T}_{[\gamma,x,y]} = \det \begin{pmatrix} x \\ x\gamma \\ \vdots \\ x\gamma^{n-1} \end{pmatrix},$$

and define the transfer factor $\mathbf{t}([\gamma, x, y]) = \mu\left(\mathbf{T}_{[\gamma,x,y]}\right)$.

Let $(f', \Phi) \in \mathcal{S}(X_n(F_v))$ and $(f^W, \phi_1^W, \phi_2^W) \in \mathcal{S}(Y(W)(F_v))$ for each skew-hermitian space W. We say that (f', Φ) and the collection $\{(f^W, \phi_1^W, \phi_2^W)\}$ match, or they are smooth transfer of each other, if for all matching $[\gamma, x, y] \in X_n(F_v)$ and $[\delta, w^\vee] \in Y(W)(F_v)$, we have

$$\mathbf{t}([\gamma, x, y])O([\gamma, x, y], f', \Phi) = O([\delta, w^\vee], f^W, \phi_1^W, \phi_2^W).$$

We say that the test functions (f', Φ) and $(f^W, \phi_1^W, \phi_2^W)$ match, if for each $W' \not\simeq W$, we can find $(f^{W'}, \phi_1^{W'}, \phi_2^{W'})$ such that (f', Φ) and the completed collection $\{(f^W, \phi_1^W, \phi_2^W); (f^{W'}, \phi_1^{W'}, \phi_2^{W'})\}$ match. We say that (f', Φ) is transferable if its smooth transfer exists. We say that $(f^W, \phi_1^W, \phi_2^W)$ is transferable if the smooth transfer of the collection $\{(f^W, \phi_1^W, \phi_2^W); 0\}$ exits. Here 0 means that the test function is zero if $W' \neq W$.

We also need the orbital integrals on the Lie algebras. Let $\mathfrak{s}_n = \{\gamma \in \mathfrak{gl}_n(E) \mid \gamma + \overline{\gamma} = 0\}$ and $\mathfrak{x}_n = \mathfrak{s}_n \times F_n \times E^{-,n}$. This is the "Lie algebra" of X_n. The group $\mathrm{GL}_{n,F}$ acts on $\mathfrak{x}_n$ via $h.[\gamma, x, y] = [h^{-1}\gamma h, xh, h^{-1}y]$. We also put $\mathfrak{y}(W) = \mathfrak{u}(W) \times W^\vee$. The group $\mathrm{U}(W)$ acts on $\mathfrak{y}(W)$ by $h.[\delta, w^\vee] = [h^{-1}\delta h, w^\vee h]$. We need the orbital integral on the Lie algebras only in the local situation. So we fix a place v of F. For $\varphi' \in \mathcal{S}(\mathfrak{x}_n(F_v))$ and $[\gamma, x, y] \in \mathfrak{x}_n(F_v)$, we define

$$O([\gamma, x, y], \varphi') = \int_{\mathrm{GL}_n(F_v)} \varphi'(h.[\gamma, x, y])\eta(\det h)\mathrm{d}h.$$

Similarly for $\varphi^W \in \mathcal{S}(\mathfrak{y}(W)(F_v))$ and $[\delta, w^\vee] \in \mathfrak{y}(W)(F_v)$, we define

$$O([\delta, w^\vee], \varphi^W) = \int_{\mathrm{U}(W)(F_v)} \varphi^W(h.[\delta, w^\vee])\mathrm{d}h.$$

We define the transfer factor on the level of Lie algebras as follows. Let $[\gamma, x, y] \in \mathfrak{x}_n(F_v)$. Let

$$\mathbf{T}_{[\gamma,x,y]} = \det \begin{pmatrix} x \\ x\gamma \\ \vdots \\ x\gamma^{n-1} \end{pmatrix}.$$

If $[\gamma, x, y] \in \mathfrak{x}_n(F_v)$ and $\mathbf{T}_{[\gamma,x,y]} \neq 0$, we define the transfer factor $\mathbf{t}([\gamma, x, y]) = \mu(\mathbf{T}_{[\gamma,x,y]})$. We say that $\varphi' \in \mathcal{S}(\mathfrak{x}_n(k'))$ and a collection of test functions $\{\varphi^W \in \mathcal{S}(\mathfrak{y}(W)(k'))\}$ match, or they are smooth matching of each other if for all matching regular semisimple orbits $[\gamma, x, y] \in \mathfrak{x}_n(F)$ and $[\delta, w^\vee] \in \mathfrak{y}(W)(F)$, we have

$$\mathfrak{t}([\gamma, x, y])O([\gamma, x, y], \varphi') = O([\delta, w^\vee], \varphi^W).$$

We need to the understand the relation between the smooth transfer on the level of groups and on the level of Lie algebras. Define the Cayley transform

$$\mathfrak{c} : \mathfrak{a} \to A, \quad X \mapsto (1 + X)(1 - X)^{-1},$$

whenever it makes sense, where $A = S_n$ or $\mathrm{U}(W)$ and $\mathfrak{a}$ is its Lie algebra. It induces a map $\mathfrak{x}_n \to X_n$ on the general linear group side or $\mathfrak{y}(W) \to Y(W)$ on the unitary group side. We also denote it by $\mathfrak{c}$. Let $\mathbf{n} \subset \mathfrak{s}_n(F)$ (resp. $\mathbf{n}^W \subset \mathfrak{u}(W)(F)$) be an $\mathrm{GL}_n(F)$ (resp. $\mathrm{U}(W)(F)$)-invariant open neighbourhood of 0 such that it is relatively compact modulo conjugation and that $\mathfrak{c}$ restricts to an analytic measure preserving map from $\mathbf{n}$ (resp. $\mathbf{n}^W$) to its image. Suppose that $\varphi' \in \mathcal{S}(X_n(F_v))$. Then we put

$$\varphi'_\natural(X) = \begin{cases} \varphi'(\mathfrak{c}(X)), & X \in \mathbf{n} \times F_n \times E^{-,n} \\ 0, & X \notin \mathbf{n} \times F_n \times E^{-,n}. \end{cases}$$

Similarly for $\varphi^W \in \mathcal{S}(Y(W)(F_v))$, we define $\varphi^W_\natural \in \mathcal{S}(\mathfrak{y}(W)(F_v))$. The following is [Xue16, Lemma 6.3.1].

Lemma 2.4.1. *Suppose that $\varphi' \in \mathcal{S}(X_n)(F_v)$ (resp. $\{\varphi^W \in \mathcal{S}(Y(W)(F_v))\}$) is a test function (resp. a collection of test functions) such that it is supported in $\mathbf{n}$ (resp. $\mathbf{n}^W$). If φ' and $\{\varphi^W\}$ match, then $\eta(2)^{\frac{n(n-1)}{2}}\eta(-1)^{[\frac{n}{2}]}\varphi'_\natural$ and $\{\varphi^W_\natural\}$ match.*

Proposition 2.4.2. *We have the following assertions.*

1. *Suppose that v is a finite place of F. Then any $(f', \Phi) \in \mathcal{S}(X_n(F_v))$ is transferable. The same holds for the test functions on the unitary groups.*

2. *Suppose that v is an archimedean place of F. Then the space of transferable test functions in $\mathcal{S}(X_n(F_v))$ is dense. The same holds for the test functions on the unitary groups.*

Proof. The first statement is proved in [Xue14, Proposition 5.2.1]. The second statement can be proved in the same way as [Xue, Theorem 2.7]. Namely, by using the Cayley transform, the statement is reduced to an analogous statement on the level of Lie algebras. This analogous statement on the Lie algebra is precisely [Xue, Theorem 3.3]. □

We have the following result whose proof is identical to [Xue14, Section 6].

Proposition 2.4.3. *Suppose that (f', Φ) and the collection $\{(f^W, \phi_1^W, \phi_2^W)\}$ are matching test functions. Assume that they are all nice test functions. Let W_0 be a skew-hermitian space of dimension n. Let π_0 be an irreducible cuspidal automorphic representation of $\mathrm{U}(W_0)(\mathbb{A}_F)$ and Assume that $\mathrm{BC}(\pi_0)i$ is cuspidal. Then*

$$I_{\mathrm{BC}(\pi_0)}(f', \Phi) = 4L(1, \eta)^2 \sum_W \sum_\pi J_\pi(f^W, \phi_1^W, \phi_2^W), \tag{2.14}$$

where the out sum ranges over all skew-hermitian spaces of dimension n and the inner sum runs over all irreducible automorphic cuspidal automorphic representations π of $\mathrm{U}(W)(\mathbb{A}_F)$ which is nearly equivalent to π_0.

The following proposition improves the main result of [Xue14] by dropping the condition at the archimedean places. The proof is identical to that in [Xue14, Section 6]. The only difference is that we use Proposition 2.4.2 to pick up the correct test functions at the archimedean places.

Proposition 2.4.4. *Let $\pi = \pi_1 \boxtimes \pi_2$ be irreducible cuspidal automorphic representations of $G(\mathbb{A}_F)$. Assume that there are two finite nonarchimedean place v_1, v_2 of F so that π_{v_1} and π_{v_2} are supercuspidal. Then the following are equivalent.*

1. *$L(\frac{1}{2}, \mathrm{BC}(\pi_1) \times \mathrm{BC}(\pi_2) \otimes \mu^{-1}) \neq 0$, where $\mathrm{BC}(\pi_i)$ stands for the base change of π_i $(i = 1, 2)$.*
2. *There is an n-dimensional skew-hermitian space and an irreducible cuspidal automorphic representation π' of $\mathrm{U}(W')(\mathbb{A}_F) \times \mathrm{U}(W')(\mathbb{A}_F)$ that is nearly equivalent to π, so that $\mathcal{FJ}$ is not identically zero on π'.*

3 Spherical characters on the general linear groups

3.1 Decomposition

Let $\Pi = \Pi_1 \boxtimes \Pi_2$ be an irreducible cuspidal automorphic representation of $G'(\mathbb{A}_F)$ where Π_1, Π_2 are irreducible cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$. Then $\Pi_i = \otimes \Pi_{i,v}$, $i = 1, 2$ where $\Pi_{i,v}$ is an irreducible admissible representation of $\mathrm{GL}_n(E_v)$. We assume in this subsection that Π is tempered. We have the Whittaker model $\mathcal{W}(\Pi_1, \psi)$ (resp. $\mathcal{W}(\Pi_2, \overline{\psi})$) of Π_1 (resp. Π_2). Let v be a place of F. Then we have the local Whittaker model $\mathcal{W}(\Pi_{1,v}, \psi_v)$ (resp. $\mathcal{W}(\Pi_{2,v}, \overline{\psi_v})$) of $\Pi_{1,v}$ (resp. $\Pi_{2,v}$). Then Let $\varphi_1 \in \Pi_1$. Let N_n be the subgroup of GL_n consisting of upper triangular unipotent matrices. We define

$$W_{\varphi_1}(g) = \int_{N_n(E)\backslash N_n(\mathbb{A}_E)} \varphi_1(ng)\overline{\psi(n)}\mathrm{d}n \in \mathcal{W}(\Pi_1, \psi).$$

Let $\varphi_2 \in \Pi_2$ and we define $W_{\varphi_2} \in \mathcal{W}(\Pi_2, \overline{\psi})$ in an analogues way. We fix a decomposition $W_{\varphi_1} = \otimes W_{1,v}$ (resp. $W_{\varphi_2} = \otimes W_{2,v}$) where $W_{1,v} \in \mathcal{W}(\Pi_{1,v}, \psi_v)$ (resp. $W_{2,v} \in \mathcal{W}(\Pi_{2,v}, \overline{\psi_v})$).

Let v be a place of F. We define the linear form $\beta_n : \mathcal{W}(\Pi_{1,v}, \psi_v) \to \mathbb{C}$ and the inner product θ_n on $\mathcal{W}(\Pi_{1,v})$ by

$$\beta_n(W_{1,v}) = \int_{N_{n-1}(F_v)\backslash \mathrm{GL}_{n-1}(F_v)} W_{1,v}(\epsilon_n(\tau)h)\eta(deth)^{n+1}\mathrm{d}h$$
$$\theta_n(W_{1,v}, W'_{1,v}) = \int_{N_{n-1}(F_v)\backslash \mathrm{GL}_{n-1}(F_v)} W_{1,v}(h)\overline{W'_{1,v}(h)}\mathrm{d}h,$$

where GL_{n-1} is viewed as a subgroup of GL_n via $h \mapsto \mathrm{diag}[h, 1]$ and $\epsilon_n(\tau) = \mathrm{diag}[\tau^{n-1}, \cdots, \tau] \in \mathrm{GL}_{n-1}(E_v)$. We define the liner form β_n and the inner product θ_n on $\mathcal{W}(\Pi_{2,v}\overline{\psi_v})$ by the same formulae. Put $\mathcal{W}_v = \mathcal{W}(\Pi_{1,v}, \psi_v) \otimes \mathcal{W}(\Pi_{2,v}, \overline{\psi_v})$, $\beta = \beta_n \otimes \beta_n$, $\theta = \theta_n \otimes \theta_n$, which is a linear form on $\mathcal{W}_v$ and an inner product on $\mathcal{W}_v$ respectively. We moreover define the linear form $\lambda_v : \mathcal{W}_v \times \mathcal{S}(E_{v,n}) \to \mathbb{C}$ by

$$\lambda_v(W_v, \Phi_v) = \int_{N_n(E_v)\backslash H'_1(E_v)} W_v(h)\Phi_v(e_n h)\mu^{-1}(\det h)|\det h|_{E_v}^{\frac{1}{2}}\mathrm{d}h.$$

This integral is absolutely convergent since Π_v is tempered.

We define the local linear form on $\mathcal{S}(G'(E_v)) \otimes \mathcal{S}(E_{v,n})$ as

$$I_{\Pi_v}(f'_v, \Phi_v) = \sum_{W_v} \lambda_v(\Pi_v(f')W_v, \Phi_v)\overline{\beta(W_v)}.$$

where the sum is over an orthonormal basis of Π_v, as explained before Proposition 2.2.1. In both cases, the sum is absolutely convergent and is independent of the choices that we made.

We have proved the following proposition in [Xue16, Proposition 3.2.1].

Proposition 3.1.1. *Let the notation be as above. Suppose that* $f' = \otimes f'_v \in \mathcal{S}(G'(\mathbb{A}_E))$ *and* $\Phi = \otimes \Phi_v \in \mathcal{S}(\mathbb{A}_E)$. *Then*

$$I_\Pi(f', \Phi) = L(1, \eta)^2 \frac{L(\frac{1}{2}, \Pi_1 \times \Pi_2 \otimes \mu^{-1})}{L(1, \Pi_1, \mathrm{As}^{(-1)^n})L(1, \Pi_2, \mathrm{As}^{(-1)^n})} \prod_v I^\natural_{\Pi_v}(f'_v, \Phi_v).$$

3.2 The partial germ expansion

In this subsection, we fix a nonsplit nonarchimedean place v of F and suppress it from all the notation. Thus F is a nonarchimedean local field of characteristic zero and E is a quadratic field extension of F.

Let $\widehat{\mathfrak{x}_n}(F) = \mathfrak{s}_n \times F^n \times E_n^-$. We define a Fourier transform of $\varphi' \in \mathcal{S}(\mathfrak{x}_n(F))$ as a Schwartz function on $\widehat{\mathfrak{x}_n}(F)$ given by

$$\widehat{\varphi'}(\gamma', x', y') = \int_{[\gamma,x,y]\in\mathfrak{x}_n(F)} \varphi'([\gamma, x, y])\psi(\mathrm{Tr}\,\gamma\gamma' + xx' + y'y)\mathrm{d}\gamma\mathrm{d}x\mathrm{d}y.$$

The group GL_n acts on $\widehat{\mathfrak{x}_n}$ in an analogues way as on $\mathfrak{x}_n$. For any function $\varphi' \in \mathcal{S}(\widehat{\mathfrak{x}_n}(F))$, we define its regular semisimple orbital integral in the same way as functions on $\mathfrak{x}_n(F)$. If $[\gamma, x, y] \in \widehat{\mathfrak{x}_n}(F)$, then we put

$$\widehat{\mathbf{T}}([\gamma, x, y]) = \det \begin{pmatrix} y \\ \vdots \\ y\gamma^{n-1} \end{pmatrix},$$

and $\widehat{\mathbf{t}}([\gamma, x, y]) = \mu(\widehat{\mathbf{T}}([\gamma, x, y]))$.

Let $Q = \widehat{\mathfrak{x}_n} /\!/ \mathrm{GL}_n$ be the categorical quotient. Then Q is isomorphic to the affine space of dimension $2n$. Let

$$\xi_- = \left[\tau \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}, {}^t(0, \cdots, 0), (0, \cdots, 0, \tau) \right] \in \widehat{\mathfrak{x}_n}(F).$$

It is explained in [Zha14b, Lemma 6.9] and [Xue16, Lemma 5.5.3] that for any $\varphi' \in \mathcal{S}(\widehat{\mathfrak{x}_n}(F))$, the following integral, as a function of s

$$\int_{\mathrm{GL}_n(F)} \varphi'(h.\xi_-)\eta(\det h)|\det h|_F^s \mathrm{d}h,$$

is holomorphic when $\Re s$ is large and has a meromorphic continuation to the whole complex plane and it holomorphic at $s = 0$. We define $O(\xi_-, \varphi')$ to be its value at $s = 0$.

Let $r > m_2 > m_1 > m > 0$ be integers. In [Xue16, Section 5], we have defined a space of test functions (f', Φ) which we call "(m, m_1, m_2, r)-admissible test functions". These admissible test functions span a finite dimensional subspace of $\mathcal{S}(G'(F)) \otimes \mathcal{S}(E_n)$. The precise definition of admissibility, which we will not recall here, is not essentially to us. The important fact is the following proposition, which is proved in [Xue16, Lemma 5.6.3, Proposition 5.6.1]. We say that (f', Φ) is sufficiently admissible, if it is (m, m_1, m_2, r)-admissible for $r \gg m_2 \gg m_1 \gg m \gg 0$. The following proposition is [Xue16, Proposition 5.7.1], which is the main technical result of that paper.

Proposition 3.2.1. *Let $\mathcal{Y}$ be an open compact neighbourhood of $0 \in Q(F)$. There is a sufficiently admissible test function (f', Φ), such that the following two statements hold.*

1. *The function*

$$X \mapsto \widehat{\mathbf{t}}(X) O(X, \widehat{\Upsilon_{f',\Phi,\natural}})$$

is a nonzero constant when restricted to the inverse image of $\mathcal{Y}$ in $\widehat{\mathfrak{x}_n}(F)_{\mathrm{rs}}$. This constant equals $\widehat{\mathbf{t}}(\xi_-) O(\xi_-, \widehat{\Upsilon_{f',\Phi,\natural}})$.

2. *We have*

$$I_\Pi(f', \Phi) = |\tau|_E^{d_n} \chi_\Pi(\tau) \mu(\tau)^{-\frac{n(n+1)}{2}} O(\xi_-, \widehat{\Upsilon_{f',\Phi,\natural}}),$$

where $d_n = \binom{n}{3}$, χ_Π is the central character of Π.

We derive the following consequence of this proposition.

Corollary 3.2.2. *Let $C \subset \mathrm{Temp}(G'(F))$ be a compact set. Let U and $\mathcal{Y}$ be open compact neighbourhoods of $[0, 0, {}^t(0, \cdots, 0, \tau)] \in X_n(F)$ and $0 \in Q(F)$ respectively. There is a sufficiently admissible test function (f', Φ), such that the following two statements hold.*

1. *The support of $\Upsilon_{f',\Phi,\natural}$ is contained in U.*

2. *The function*
$$X \mapsto \widehat{\mathbf{t}}(X) O(X, \widehat{\Upsilon_{f',\Phi,\natural}})$$
is a nonzero constant when restricted to the inverse image of $\mathcal{Y}$ in $\mathfrak{x}_n$. This constant equals $\widehat{\mathbf{t}}(\xi_-) O(\xi_-, \widehat{\Upsilon_{f',\Phi,\natural}})$.

3. *For all $\Pi \in C$, we have*
$$I_\Pi(f', \Phi) = |\tau|_E^{d_n} \chi_\Pi(\tau) \mu(\tau)^{-\frac{n(n+1)}{2}} O(\xi_-, \widehat{\Upsilon_{f',\Phi,\natural}}),$$
where $d_n = \binom{n}{3}$, χ_Π is the central character of Π.

Proof. The proof is identical to [BPc, Corollary 4.1.4] by making use of Proposition 3.2.1 and the Baire category theorem. □

4 Spherical characters on the unitary groups

4.1 Decomposition

Recall that W is a skew-hermitian space of dimension n and $H = \mathrm{U}(W)$ and $G = H \times H$. The group H is viewed as a subgroup of G via the diagonal embedding.

We first define the local spherical character on the unitary groups. Let v be a place of F. Let $\pi_v = \pi_{1,v} \boxtimes \pi_{2,v}$ be an irreducible admissible tempered representation of $G(F_v)$ where $\pi_{1,v}$ and $\pi_{2,v}$ are irreducible admissible tempered representations of $H(F_v)$. Let $\Pi_{i,v}$ be the base change of $\pi_{i,v}$ to $\mathrm{GL}_n(E_v)$. Let $f_v \in \mathcal{S}(G(F_v))$ and $\phi_{1,v}, \phi_{2,v} \in \mathcal{S}(L(F_v))$ be Schwartz functions. Define
$$J_{\pi_v}(f_v, \phi_{1,v}, \phi_{2,v}) = \int_{H(F_v)} \mathrm{Tr}(\pi_v(h)\pi_v(f_v)) \overline{\langle \omega_v(h)\phi_{1,v}, \phi_{2,v} \rangle} \mathrm{d}h.$$

We have shown in [Xue16, Proposition 1.1.1] the following facts.

1. The defining integral of J_{π_v} is absolutely convergent for any $f_v \in \mathcal{S}(G(F_v))$ and $\phi_{1,v}, \phi_{2,v} \in \mathcal{S}(L(F_v))$.

2. The space $\mathrm{Hom}_{H(F_v)}(\pi_v \otimes \overline{\omega_v}, \mathbb{C}) \neq 0$ if and only if J_{π_v} is not identically zero.

Define a normalized local spherical character (c.f. [Xue16, Proposition 1.1.1 and Definition 1.3.2] for the explanation)

$$J^{\natural}_{\pi_v}(f_v,\phi_{1,v},\phi_{2,v}) = \left(\prod_{i=1}^{n} L(i,\eta_v^i)\frac{L(\frac{1}{2},\Pi_{1,v}\times\Pi_{2,v}\otimes\mu_v^{-1})}{L(1,\pi_{1,v},\mathrm{Ad})L(1,\pi_{2,v},\mathrm{Ad})}\right)^{-1} J_{\pi_v}(f_v,\phi_{1,v},\phi_{2,v}).$$

We now switch to the global situation. Let π be an irreducible cuspidal tempered automorphic representation of $G(\mathbb{A}_F)$. Recall that we have defined the global spherical character J_π in Subsection 2.3. By [Sun12, SZ12], we know that $\dim_{H(\mathbb{A}_F)}(\pi\otimes\overline{\omega},\mathbb{C})\leq 1$. It follows that there is a constant $C(\pi)$, depending on π only, such that

$$J_\pi(f,\phi_1,\phi_2) = C(\pi)\prod_v J^{\natural}_{\pi_v}(f_v,\phi_{1,v},\phi_{2,v}). \tag{4.1}$$

As in [Xue16, Lemma 1.3.5], Theorem 1.1.2 follows directly from the following theorem.

Theorem 4.1.1. *Let the notation be as above. Assume the following conditions.*

1. *E/F is split at all archimedean places;*
2. *There is a finite place v such that $\mathrm{BC}(\pi_v)$ is supercuspidal.*

Then

$$C(\pi) = \frac{1}{4}\prod_{i=1}^{n} L(i,\eta^i)\frac{L(\frac{1}{2},\Pi_1\times\Pi_2\otimes\mu^{-1})}{L(1,\pi_1,\mathrm{Ad})L(1,\pi_2,\mathrm{Ad})}.$$

We will deduce this theorem from the following local result.

Theorem 4.1.2. *Let v be a place of F. Put*

$$\kappa_v = |\tau|_{E_v}^{d_n}\epsilon\left(\frac{1}{2},\eta_v,\psi_v\right)^{\frac{n(n+1)}{2}}\chi_\Pi(\tau)\mu_v(\mathrm{disc}\,W)\eta_v(2)^{\frac{n(n-1)}{2}},$$

where $d_n=\binom{n}{3}$ and $\mathrm{disc}\,W_v\in E_v^-$ is the discriminant of W_v. Suppose that π_v is an irreducible tempered representation of $G(F_v)$ such that $J_{\pi_v}\neq 0$. Then for all matching test functions (f'_v,Φ_v) and $(f_v,\phi_{1,v},\phi_{2,v})$, we have

$$I_{\Pi_v}(f'_v,\Phi_v) = L(1,\eta_v)\kappa_v J_{\pi_v}(f_v,\phi_{1,v},\phi_{2,v}). \tag{4.2}$$

The proof of these two theorems will be given in Section 6.

4.2 Elementary properties

From now on till the end of this section, we fix a nonsplit place v of F (could be infinite) and suppress it from all the notation. If v is nonarchimedean, then we work in the category of smooth representations of finite length. If v is archimedean, we work in the category of Frechet representations of finite length and of moderate growth, i.e. the Casselmann–Wallach representations. To simplify notation, we also write G for $G(F)$, H for $H(F)$ etc.

Let us introduce a more general version of J_π. We denote by $\mathrm{Temp}(G)$ the set of isomorphism classes of irreducible tempered representations of $G(F)$. We denote by $\mathcal{X}_{\mathrm{temp}}(G)$ the isomorphism classes of representations of $G(F)$ of the form $i_P^G \sigma$, where $P = MN$ is a parabolic subgroup with the Levi subgroup M, σ is a square-integrable representation of $M(F)$ and i_P^G is the normalized induction. Note that the isomorphism class of $i_P^G\sigma$ depends only on M and σ, but not on P, so we sometimes write $i_M^G\sigma$ instead. The set $\mathcal{X}_{\mathrm{temp}}(G)$ has a natural structure of a smooth manifold (with infinitely many connected components), whose components are indexed by the pairs (M, σ) (up to conjugation) where M is a Levi subgroup of G and σ is a discrete series representation of $M(F)$. If $\lambda \in i\mathfrak{a}_{M,\mathbb{R}}^*$, then we put $\sigma_\lambda = \sigma \otimes \lambda$ and $\pi_\lambda = i_P^G \sigma_\lambda$.

Let $\pi \in \mathcal{X}_{\mathrm{temp}}(G)$. We denote by $\pi^{-\infty}$ its algebraic dual, i.e. the space of linear forms on π. Moreover, we denote by $\mathrm{End}(\pi)$ the space of endomorphisms of π. The group $G(F) \times G(F)$ acts on $\mathrm{End}(\pi)$. We put $\mathrm{End}(\pi)^\infty$ the subspace of smooth endomorphisms and $\mathrm{End}(\pi)^{-\infty}$ be the space of linear forms on $\mathrm{End}(\pi)^\infty$. Then the elements of the form $e \otimes e'$ $(e, e' \in \pi)$ span a dense subspace of $\mathrm{End}(\pi)^\infty$, where $e \otimes e'(e'') = \langle e'', e' \rangle e$. If $f \in \mathcal{C}(G(F))$, then $\pi(f) \in \mathrm{End}(\pi)^\infty$.

Let $T \in \mathrm{End}(\pi)^\infty$ and $\phi_1, \phi_2 \in \mathcal{S}(L)$. We define

$$\mathcal{L}_\pi^{\phi_1,\phi_2}(T) = \int_H \mathrm{Tr}(\pi(h)T)\overline{\langle \omega(h)\phi_1, \phi_2 \rangle} \mathrm{d}h.$$

One can prove, as in [Xue16, Proposition 1.1.1], without much difficulty that the defining integral of $\mathcal{L}_\pi^{\phi_1,\phi_2}$ is absolutely convergent.

The linear form $\mathcal{L}_\pi^{\phi_1,\phi_2}$ on $\mathrm{End}(\pi)^\infty$, defines a map

$$L_\pi^{\phi_1,\phi_2} : \pi \mapsto \overline{\pi^{-\infty}}, \quad e \mapsto \left(e' \mapsto \mathcal{L}_\pi^{\phi_1,\phi_2}(e \otimes e')\right).$$

If $T \in \mathrm{End}(\pi)^\infty$, then T extends continuously to a map $T : \overline{\pi^{-\infty}} \to \pi$. Thus we have

$$T L_\pi^{\phi_1,\phi_2} : \pi \to \pi, \quad L_\pi^{\phi_1,\phi_2} : \overline{\pi^{-\infty}} \to \overline{\pi^{-\infty}}.$$

It follows from the definition that

$$\mathrm{Tr}\, T L_\pi^{\phi_1,\phi_2} = \mathrm{Tr}\, L_\pi^{\phi_1,\phi_2} T = \mathcal{L}_\pi^{\phi_1,\phi_2}(T).$$

Lemma 4.2.1. *Let $f_1, f_2 \in \mathcal{C}(G)$ and $\phi_1, \phi_2, \phi_3, \phi_4 \in \mathcal{S}(L)$.*

1. *The maps*

$$\pi \in \mathcal{X}_{\mathrm{temp}}(G) \mapsto \mathcal{L}_\pi^{\phi_1,\phi_2} \in \mathrm{End}(\pi)^{-\infty}, \quad \pi \in \mathcal{X}_{\mathrm{temp}}(G) \mapsto L_\pi^{\phi_1,\phi_2} \in \mathrm{Hom}(\pi, \overline{\pi^{-\infty}})$$

   are smooth.

2. *Suppose* $\pi \in \mathrm{Temp}(G)$ *or* $\mathcal{X}_{\mathrm{temp}}(G)$*. If* $S, T \in \mathrm{End}(\pi)^\infty$*, then* $SL_\pi^{\phi_1,\phi_2}T \in \mathrm{End}(\pi)^\infty$*. Moreover,*

$$\mathcal{L}_\pi^{\phi_1,\phi_2}(SL_\pi^{\phi_3,\phi_4}T) = \mathcal{L}_\pi^{\phi_1,\phi_4}(S)\mathcal{L}_\pi^{\phi_3,\phi_2}(T).$$

3. *Suppose* $f_1, f_2 \in \mathcal{C}(G(F))$*. There is a function* $f_0 \in \mathcal{C}(G(F))$ *(depending on* ϕ_1, ϕ_2*) so that*

$$\pi(f_1)L_\pi^{\phi_1,\phi_2}\pi(f_2) = \pi(f_0).$$

Proof. Most part of the proof of this lemma is identical to the proof of [BPc, Lemma 8.2.1]. Only the second assertion when $\pi \in \mathrm{Temp}(G)$ needs a little bit more care. We only prove this part and leave the rest to the interested reader.

Assume that $\pi \in \mathrm{Temp}(G)$ and we now prove the equality

$$\mathcal{L}_\pi^{\phi_1,\phi_2}(SL_\pi^{\phi_3,\phi_4}T) = \mathcal{L}_\pi^{\phi_1,\phi_4}(S)\mathcal{L}_\pi^{\phi_3,\phi_2}(T).$$

Suppose that $\pi \in \mathrm{Temp}(G)$. Let $e_1, e_1^\vee, e_2, e_2^\vee \in \pi$. By continuity, we only need to prove this equality for $S = e_1 \otimes e_1^\vee$ and $T = e_2 \otimes e_2^\vee$. Computing both sides directly, we have

$$\mathcal{L}_\pi^{\phi_1,\phi_2}(SL_\pi^{\phi_3,\phi_4}T) = \alpha(e_2, e_1^\vee, \phi_3, \phi_4)\alpha(e_1, e_2^\vee, \phi_1, \phi_2),$$
$$\mathcal{L}_\pi^{\phi_1,\phi_4}(S)\mathcal{L}_\pi^{\phi_3,\phi_2}(T) = \alpha(e_1, e_1^\vee, \phi_1, \phi_4)\alpha(e_2, e_2^\vee, \phi_3, \phi_2).$$

Note that the maps

$$(e, \phi) \mapsto \alpha(e, e_1^\vee, \phi, \phi_4), \quad (e, \phi) \mapsto \alpha(e, e_2^\vee, \phi, \phi_2)$$

belong to the one dimensional space $\mathrm{Hom}_{H(F)}(\pi \otimes \overline{\omega}, \mathbb{C})$, and thus are proportional. The desired identity then follows. □

Lemma 4.2.2. *For any* $\pi \in \mathcal{X}_{\mathrm{temp}}(G)$*, there is at most one subrepresentation* π_1 *so that* $J_{\pi_1} \neq 0$*.*

Proof. Let us assume that there are two distinct subrepresentations π_1, π_2 such that $J_{\pi_i} \neq 0$, $i = 1, 2$. Then by [Xue16, Proposition 1.2.1], we may find $T_1 \in \mathrm{End}(\pi_1)^\infty \subset \mathrm{End}(\pi)^\infty$, $T_2 \in \mathrm{End}(\pi_2)^\infty \subset \mathrm{End}(\pi)^\infty$ and $\phi_1, \phi_2, \phi_3, \phi_4 \in \mathcal{S}(L)$, such that

$$\mathcal{L}_{\pi_1}^{\phi_1,\phi_4}(T_1) \neq 0, \quad \mathcal{L}_{\pi_2}^{\phi_3,\phi_2}(T_2) \neq 0.$$

Recall that π is a unitary representation and π_1 and π_2 are orthogonal to each other. Now by Lemma 4.2.1 (2), we have

$$0 \neq \mathcal{L}_{\pi_1}^{\phi_1,\phi_4}(T_1)\mathcal{L}_{\pi_2}^{\phi_3,\phi_3}(T_2) = \mathcal{L}_{\pi}^{\phi_1,\phi_4}(T_1)\mathcal{L}_{\pi}^{\phi_3,\phi_3}(T_2) = \mathcal{L}_{\pi}^{\phi_1,\phi_2}(T_1 L_{\pi}^{\phi_3,\phi_4} T_2).$$

But it is clear that $T_1 L_\pi^{\phi_3,\phi_4} T_2 = 0$ as $\pi_1 \not\simeq \pi_2$. This is a contradiction. □

Let $\pi \in \mathrm{Temp}_H(G)$. There is a unique (up to conjugation) pair (M, σ) where M is a Levi subgroup of G and σ is a discrete series representation of $M(F)$ so that π is a subrepresentation of $\pi' = i_M^G \sigma$. For $f \in \mathcal{C}(G(F))$, $\phi_1, \phi_2 \in \mathcal{S}(L)$, by the lemma above, we have

$$J_\pi(f, \phi_1, \phi_2) = \mathcal{L}_{\pi'}^{\phi_1,\phi_2}(\pi'(f)).$$

For fixed f, ϕ_1, ϕ_2, we may regard $J_\pi(f, \phi_1, \phi_2)$ as a function on $\mathcal{X}_{\mathrm{temp}}(G)$. Then the map $\pi \mapsto J_\pi(f, \phi_1, \phi_2)$ is smooth by Lemma 4.2.1.

4.3 Induced representations and Fourier–Jacobi models

Let $P = MN$ be a parabolic subgroup of G with the Levi component M. Let σ be a discrete series representation of M and $\lambda \in i\mathfrak{a}_{M,\mathbb{R}}^*$. Then $i_P^G \sigma_\lambda \in \mathcal{X}_{\mathrm{temp}}(G)$. The goal of this subsection is to prove the following lemma.

Lemma 4.3.1. *Suppose that σ is a discrete series representation of $M(F)$. If*

$$\dim_H \mathrm{Hom}(i_P^G \sigma_\lambda \otimes \overline{\omega}, \mathbb{C}) = 1$$

for some $\lambda = \lambda_0 \in i\mathfrak{a}_{M,\mathbb{R}}^*$, then the same holds for all $\lambda \in i\mathfrak{a}_{M,\mathbb{R}}^*$.

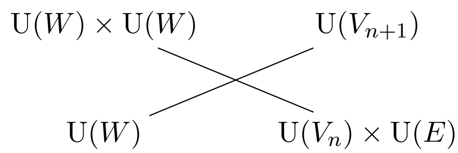
One could mimic the argument in [BPb, Section 8] to prove this lemma. We take a different approach here which reduces the lemma to what has been proved in [BPb, Section 8]. The main technique is the local theta correspondence for unitary groups. Let us recall some basic setup of it before we plunge into the proof of Lemma 4.3.1. For a detailed discussion, the readers may refer to [GI, GI14].

Let us temporarily denote by W (resp. V) a skew-hermitian (resp. hermitian) space. We have a Weil representation ω_{WV} of $\mathrm{U}(W) \times \mathrm{U}(V)$. This depends on an additive character of F and a pair of characters (χ_W, χ_V) of $E^\times$ such that $\chi_W|_{F^\times} = \eta^{\dim V}$ and $\chi_V|_{F^\times} = \eta^{\dim W}$. We always choose the additive character ψ and the pair of characters $(\mu^{\dim V}, \mu^{\dim W})$ in this subsection. For any irreducible admissible representation π of $\mathrm{U}(W)$, let $\pi \boxtimes \Theta_{WV}(\pi)$ be the maximal π-isotypic component of ω_{WV} and $\theta_{WV}(\pi)$ the maximal semisimple quotient of $\Theta_{WV}(\pi)$. It is known that $\theta(\pi)$ is irreducible by the work of Gan–Takeda and Waldspurger. By [GI14, Theorem 4.1 and 4.4], if π is tempered and $\dim V - \dim W = 0$ or 1, then $\Theta_{WV}(\pi) = \theta_{WV}(\pi)$ if they are not zero, hence $\Theta_{WV}(\pi)$ is irreducible. We drop the subscripts when the space involved is clear.

We now let W (resp. V_n) be a skew-hermitian (resp. hermitian) space of dimension n (resp. n). Put $V_{n+1} = V_n \oplus E$ where E is the one dimensional hermitian space over E with the hermitian form given by $(x, y) \mapsto x\overline{y}$. We have the decomposition

$$\omega_{W,V_{n+1}}|_{\mathrm{U}(W)\times\mathrm{U}(V_n)} \simeq \omega_{W,V_n} \otimes \omega_{W,E}.$$

There is a seesaw diagram



and the associated seesaw identity reads

$$\mathrm{Hom}_{\mathrm{U}(W)}(\sigma \otimes \overline{\Theta_{V_n,W}(\pi)} \otimes \overline{\omega}, \mathbb{C}) = \mathrm{Hom}_{\mathrm{U}(V_n)}(\Theta_{W,V_{n+1}}(\sigma) \otimes \pi, \mathbb{C}). \tag{4.3}$$

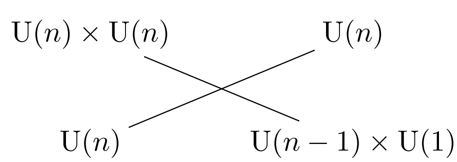
Proof of Lemma 4.3.1. The group $P = MN$ is of the form $P_1 \times P_2$ where $P_1 = M_1N_1$ (resp. $P_2 = M_2N_2$) is a parabolic subgroup of H. The representation $\sigma = \sigma_1 \boxtimes \sigma_2$ where σ_1, σ_2 are irreducible discrete series representations of M_1 and M_2 respectively. The groups M_1 and M_2 are of the form

$$M_1 \simeq \mathrm{GL}_{r_1,E} \times \cdots \times \mathrm{GL}_{r_a,E} \times \mathrm{U}(W_1), \quad M_2 \simeq \mathrm{GL}_{s_1,E} \times \cdots \times \mathrm{GL}_{s_b,E} \times \mathrm{U}(W_2)$$

where W_1 (resp. W_2) is a skew-hermitian space of dimension $n - 2(r_1 + \cdots r_a)$ (resp. $n - 2(s_1 + \cdots + s_b)$). Thus we may further write

$$\sigma_1 \simeq \xi_1 \boxtimes \sigma_1^0, \quad \sigma_2 \simeq \xi_2 \boxtimes \sigma_2^0,$$

where ξ_1 is an irreducible discrete series representation of $\mathrm{GL}_{r_1}(E) \times \cdots \times \mathrm{GL}_{r_a}(E)$, ξ_2 is an irreducible discrete series representation of $\mathrm{GL}_{s_1}(E) \times \cdots \times \mathrm{GL}_{s_b}(E)$, σ_1^0 is an irreducible discrete series representation of $\mathrm{U}(W_1)$ and σ_2^0 is an irreducible discrete series representation of $\mathrm{U}(W_2)$. We assume that σ_2^0 is not a theta lift from a unitary group attached to a hermitian space of dimension $\dim W_2 - 1$. Otherwise, we may argue in the similar way, except that we make use of the seesaw diagram



We write $\lambda = (\lambda_1, \lambda_2)$, where $\lambda_1 \in i\mathfrak{a}_{M_1,\mathbb{R}}^*$ and $\lambda_2 \in i\mathfrak{a}_{M_2,\mathbb{R}}^*$.

Without loss of generality, we may assume that $\lambda_0 = 0$. Suppose that ρ_1 and ρ_2 are subrepresentations of $i^H_{P_1}\xi_1 \boxtimes \sigma^0_1$ and $i^H_{P_2}\xi_2 \boxtimes \sigma^0_2$ respectively so that

$$\dim \mathrm{Hom}_H(\rho_1 \otimes \rho_2 \otimes \overline{\omega_{\psi,\mu}}, \mathbb{C}) = 1.$$

By the theta dichotomy [GI14, Appendix C], we may find a unique hermitian space V_n and an irreducible tempered representation π_n of $\mathrm{U}(V_n)$ so that $\rho_1 = \overline{\theta_{V_n,W}(\pi_n)}$. Put $V_{n+1} = V_n \oplus E$ and $\pi_{n+1} = \theta_{W,V_{n+1}}(\rho_2)$. We now make use of the seesaw identity (4.3) and conclude that

$$\dim \mathrm{Hom}_{\mathrm{U}(V_n)}(\pi_{n+1} \otimes \pi_n, \mathbb{C}) = 1.$$

By the description of the theta lift of tempered representations [GI, GI14], there is a hermitian subspace $V'_{n+1} \subset V_{n+1}$ with $\dim V'_{n+1} = \dim W_2 + 1$ so that π_{n+1} is a subrepresentation of

$$i^{\mathrm{U}(V_{n+1})}_{P_{n+1}}\xi_2 \boxtimes \pi^0_{n+1},$$

where $P_{n+1} = L_{n+1}U_{n+1}$ is a parabolic subgroup of $\mathrm{U}(V_{n+1})$ whose Levi component L_{n+1} is isomorphic to

$$\mathrm{GL}_{s_1,E} \times \cdots \times \mathrm{GL}_{s_b,E} \times \mathrm{U}(W_2),$$

and $\pi^0_{n+1} = \theta(\sigma^0_1)$. Put $\tau_{n+1} = \xi_2 \boxtimes \pi^0_{n+1}$, which is an irreducible discrete representation of $L_{n+1}(F)$ since by assumption, σ_2 is not a theta lift from a unitary groups attached to a hermitian space of dimension $\dim W_2 - 1$. Similarly there is a hermitian subspace $V'_n \subset V_n$ with $\dim V'_n = \dim W_1$ and an irreducible discrete series representation π^0_n of $\mathrm{U}(V'_n)$ such that $\sigma^0_1 = \theta(\pi^0_n)$ and π_n is a subrepresentation of

$$i^{\mathrm{U}(V_n)}_{P_n}\xi_1 \boxtimes \pi^0_n,$$

where $P_n = L_nU_n$ is a parabolic subgroup of $\mathrm{U}(V_n)$ whose Levi component L_n is isomorphic to

$$\mathrm{GL}_{r_1,E} \times \cdots \times \mathrm{GL}_{r_a,E} \times \mathrm{U}(V'_n).$$

Put $\tau_n = \xi_1 \boxtimes \pi^0_n$, which is an irreducible discrete series representation of L_n.

We now apply [BPb, Corollary 8.6.1] to conclude that for any $\lambda_1 \in \mathfrak{a}^*_{L_n,\mathbb{R}}$, $\lambda_2 \in i\mathfrak{a}^*_{L_{n+1},\mathbb{R}}$, we have

$$\dim \mathrm{Hom}_{\mathrm{U}(V_n)}(i^{\mathrm{U}(V_{n+1})}_{Pn+1}\tau_{n+1,\lambda_2} \otimes i^{\mathrm{U}(V_n)}_{P_n}\tau_{n,\lambda_1}, \mathbb{C}) = 1.$$

Let $\pi_{n+1,\lambda_2} \subset i^{\mathrm{U}(V_{n+1})}_{Pn+1}\tau_{n+1,\lambda_2}$ and $\pi_{n,\lambda_1} \subset i^{\mathrm{U}(V_n)}_{P_n}\tau_{n,\lambda_1}$ be the (unique) subrepresentations such that

$$\dim \mathrm{Hom}_{\mathrm{U}(V_n)}(\pi_{n+1,\lambda_2} \otimes \pi_{n,\lambda_1}, \mathbb{C}) = 1.$$

By the description of theta lift of tempered representations, we $\rho_{1,\lambda_1} = \theta(\pi_{n,\lambda_1})$ is a nonzero irreducible representation of $\mathrm{U}(W)$ and there is an irreducible tempered representation ρ_{2,λ_2} of $\mathrm{U}(W)$ such that $\theta(\rho_2) = \pi_{n+1,\lambda_2}$. By the seesaw identity (4.3) again, we have

$$\dim \mathrm{Hom}_H(\rho_{1,\lambda_1} \otimes \rho_{2,\lambda_2} \otimes \overline{\omega}, \mathbb{C}) = 1.$$

We note that $i\mathfrak{a}^*_{M_1,\mathbb{R}}$ and $i\mathfrak{a}^*_{L_n,\mathbb{R}}$ are canonically identified, so are $i\mathfrak{a}^*_{M_2,\mathbb{R}}$ and $i\mathfrak{a}^*_{L_{n+1},\mathbb{R}}$. We also note that ρ_{1,λ_1} is a subrepresentation of $i^H_{P_1}\sigma_{1,\lambda_1}$ and ρ_{2,λ_2} is a subrepresentation of $i^H_{P_2}\sigma_{2,\lambda_2}$. Thus

$$\dim \mathrm{Hom}_H(i^H_{P_1}\sigma_{1,\lambda_1} \otimes i^H_{P_2}\sigma_{2,\lambda_2} \otimes \overline{\omega}, \mathbb{C}) = 1,$$

for all $\lambda_1 \in i\mathfrak{a}^*_{M_1,\mathbb{R}}$ and $\lambda_2 \in i\mathfrak{a}^*_{M_2,\mathbb{R}}$. This is precisely what we are after. □

4.4 A local relative trace formula

Let us denote by $\mathrm{d}\pi$ the Plancherel measure on $\mathcal{X}_{\mathrm{temp}}(G)$. Let $\mathrm{Temp}_H(G)$ be the subset of $\mathrm{Temp}(G)$ consisting of irreducible tempered representations of G such that $\mathrm{Hom}_H(\pi \otimes \overline{\omega}, \mathbb{C}) \neq 0$. By Lemma 4.2.2, the map $\mathrm{Temp}_H(G) \to \mathcal{X}_{\mathrm{temp}}(G)$ which sends π to $i^G_M\sigma$ such that π is a subrepresentation of $i^G_M\sigma$ is injective. By Lemma 4.3.1, the image of $\mathrm{Temp}_H(G)$ is a union of connected components in $\mathcal{X}_{\mathrm{temp}}(G)$. Let us denote also by $\mathrm{d}\pi$ the restriction of the Plancherel measure of $\mathcal{X}_{\mathrm{temp}}(G)$ to $\mathrm{Temp}_H(G)$ via this map.

Lemma 4.4.1. *Let $(f, \phi_1, \phi_2) \in \mathcal{S}(G \times L \times L)$. Suppose that the map*

$$\pi \mapsto \pi(f) \in \mathrm{End}(\pi)^\infty$$

is compactly supported. Then

$$\Psi_{f,\phi_1,\phi_2}(1,0) = \int_H f(h^{-1})\overline{\langle \omega(h)\phi_1, \phi_2\rangle}\mathrm{d}h = \int_{\mathrm{Temp}_H(G)} J_\pi(f, \phi_1, \phi_2)\mathrm{d}\pi.$$

Both integrals are absolutely convergent.

Proof. By the assumption on f, the (vector-valued) function

$$\pi \in \mathrm{Temp}(G) \mapsto (g \mapsto \mathrm{Tr}(\pi(g^{-1})\pi(f))))$$

is continuous and compactly supported. It follows that this function is absolutely integrable (c.f. [BPb, Appendix A.2]). By the Plancherel formulae, as explained in [BPb, Theorem 2.6.1], we have

$$f^\vee = \int_{\mathcal{X}_{\mathrm{temp}}(G)} \mathrm{Tr}(\pi(\cdot)\pi(f))\mathrm{d}\pi, \quad f^\vee(g) = f(g^{-1})$$

as elements in $\mathcal{S}(G)$ ($\subset \mathcal{C}(G)$). Indeed, it is true if we evaluate both side at any $g \in G$. Thus for any continuous linear form ℓ on $\mathcal{C}(G)$, we have

$$\ell(f^\vee) = \int_{\mathcal{X}_{\mathrm{temp}}(G)} \ell(\mathrm{Tr}(\pi(\cdot)\pi(f)))\mathrm{d}\pi.$$

Since for $\phi_1, \phi_2 \in \mathcal{S}(L)$, the map

$$f \mapsto \int_H f(h)\overline{\langle \omega(h)\phi_1, \phi_2\rangle}\mathrm{d}h$$

is a continuous linear form, we conclude that

$$\int_H f(h^{-1})\overline{\langle \omega(h)\phi_1, \phi_2\rangle}\mathrm{d}h = \int_{\mathcal{X}_{\mathrm{temp}}(G)} J_\pi(f, \phi_1, \phi_2)\mathrm{d}\pi = \int_{\mathrm{Temp}_H(G)} J_\pi(f, \phi_1, \phi_2)\mathrm{d}\pi.$$

This proves the lemma. $\square$

Let $f_1, f_2 \in \mathcal{S}(G)$ and $\phi_1, \phi_2, \phi_3, \phi_4 \in \mathcal{S}(L)$. We consider the integral

$$\begin{aligned} T(f_1, f_2, \phi_1, \phi_2, \phi_3, \phi_4) = \int_H \int_H \int_G \int_{W^\vee} & f_1(h_1 g h_2) f_2(g)(\omega(g_1^{-1})\overline{\phi_3} \otimes \phi_4)^\ddagger(w^\vee) \\ & \overline{(\omega(h_2^{-1}g_1^{-1}h_1^{-1})\phi_1 \otimes \phi_2)}^\ddagger(w^\vee h_2)\mathrm{d}g\mathrm{d}w^\vee\mathrm{d}h_1\mathrm{d}h_2, \end{aligned} \tag{4.4}$$

where we write $g = (g_1, g_2) \in G$, $g_1, g_2 \in H$.

Lemma 4.4.2. *This integral is absolutely convergent.*

Proof. We note first that the function

$$(g, w^\vee) \mapsto f_2(g)(\omega(g_1^{-1})\overline{\phi_3} \otimes \phi_4)^\ddagger(w^\vee)$$

is a Schwartz function on $G(F) \times V^\vee$. Since $\mathcal{S}(G) \subset \mathcal{C}(G)$, for all $d > 0$, there is a nonnegative Schwartz function $\Phi_2 \in \mathcal{S}(V^\vee)$ so that

$$|f_2(g)(\omega(g_1^{-1})\overline{\phi_3} \otimes \phi_4)^\ddagger(w^\vee)| \ll \Xi^G(g)\sigma(g)^{-d}\Phi_2(w^\vee).$$

Similarly for all $d > 0$, there is a nonnegative Schwartz function $\Phi_1 \in \mathcal{S}(V^\vee)$ so that

$$|f_1(g)\overline{(\omega(g_1^{-1})\phi_1} \otimes \phi_2)^\ddagger(w^\vee)| \ll \Xi^G(g)\sigma(g)^{-d}\Phi_1(w^\vee).$$

Therefore we are reduced to prove that there is a $d > 0$, so that for all $\Phi_1, \Phi_2 \in \mathcal{S}(V^\vee)$, the integral

$$\int_{H(F)} \int_{H(F)} \int_{G(F)} \int_{W^\vee} \Xi^G(h_1 g h_2)\Xi^G(g)\sigma(g)^{-d}\sigma(h_1 g h_2)^{-d}\Phi_1(w^\vee h_2)\Phi_2(w^\vee)\mathrm{d}w^\vee\mathrm{d}g\mathrm{d}h_1\mathrm{d}h_2$$

is convergent.

We make a change of variables $h_1 \mapsto h_1 h_2^{-1} g^{-1}$. Then the integrand of the above integral becomes

$$\Xi^H(h_1)\Xi^H(h_1 h_2^{-1} g_1^{-1} g_2 h_2)\Xi^H(g_1)\Xi^H(g_2)\sigma(g_1)^{-d}\sigma(g_2)^{-d}\sigma(h_1)^{-d}\Phi_1(w^\vee h_2)\Phi_2(w^\vee).$$

Then we make use of the doubling principle for Ξ^H and conclude that the integral equals

$$\int_{H(F)^2} \Xi^H(g_1)^2\Xi^H(g_2)^2\sigma(g_1)^{-d}\sigma(g_2)^{-d}\mathrm{d}g_1\mathrm{d}g_2 \times \int_{H(F)} \Xi^H(h_1)^2\sigma(h_1)^{-d}\mathrm{d}h_1$$
$$\times \int_{H(F)}\int_{W^\vee} \Xi^H(h_2)^2\Phi_1(w^\vee h_2)\Phi_2(w^\vee)\mathrm{d}w^\vee\mathrm{d}h_2.$$

The first two integrals are convergent for sufficiently large d. The last integral is convergent for any Φ_1 and Φ_2 by [Xue16, Proposition 1.1.1]. □

Let $\varphi \in \mathcal{S}(\mathfrak{y}(F)) = \mathcal{S}(\mathfrak{y}(W)(F))$. We define its Fourier transform by

$$\widehat{\varphi}([\delta, w^\vee]) = \int_{[\delta', w'^\vee]\in\mathfrak{y}(W)(F)} \varphi([\delta', w'^\vee])\psi(\mathrm{Tr}\,\delta\delta' + [w^\vee, w'^\vee])\mathrm{d}\delta'\mathrm{d}w'^\vee.$$

We use the notation $\check{\varphi}$ to denote the inverse Fourier transform. The local relative trace formula alluded in the title of this subsection is the following proposition.

Proposition 4.4.3. *Let $f_1, f_2 \in \mathcal{S}(G(F))$ and $\phi_1, \phi_2, \phi_3, \phi_4 \in \mathcal{S}(L)$. Assume that the support of Ψ_{f_2,ϕ_3,ϕ_4} is contained in $\mathfrak{c}(\mathbf{n}) \times W^\vee$ (recall that $\mathfrak{c}$ is the Cayley transform and $\mathbf{n}$ is a small neighbourhood of $0 \in \mathfrak{u}(W)(F)$ we have fixed so that the Cayley transform is analytic and measure preserving). Assume that the map $\pi \mapsto \pi(f_2)$ is compactly support in* Temp(G). *Then*

$$\int_{\mathcal{X}_{\mathrm{temp}}(G)} J_\pi(f_1,\phi_1,\phi_2)J_{\pi^\vee}(f_2,\phi_3,\phi_4)\mathrm{d}\pi = \int_{\mathfrak{y}(F)} \widecheck{\Psi_{f_2,\phi_3,\phi_4,\natural}}(y)O(y, \widehat{\Psi_{f_1,\phi_1,\phi_2,\natural}})\mathrm{d}y,$$

where $\pi^\vee$ stands for the contragredient of π and the Weil representation in the definition of $J_{\pi^\vee}$ and Ψ_{f_2,ϕ_3,ϕ_4} is $\overline{\omega}$.

Proof. We compute the expression $T(f_1, f_2, \phi_1, \phi_2, \phi_3, \phi_4)$ in two ways.

We first compute it geometrically. Thanks to Lemma 4.4.2, we may change the order of integration in (4.4) and make change of variables. We make the change of variables in the following order:

$$g_1 \mapsto h_1^{-1}g_1, \quad g_2 \mapsto h_1^{-1}g_2, \quad h_1 \mapsto g_1h_1, \quad g_2 \mapsto g_2g_1, \quad w^\vee \mapsto w^\vee g_1, \quad h_2 \mapsto g_1^{-1}h_2.$$

We then end up with

$$T(f_1, f_2, \phi_1, \phi_2, \phi_3, \phi_4) = \int_{Y(W)(F)}\int_{H(F)} \Psi_{f_1,\phi_1,\phi_2}(g_1.y)\Psi_{f_2,\phi_3,\phi_4}(y)\mathrm{d}g_1\mathrm{d}y. \tag{4.5}$$

By the support condition of Ψ_{f_2,ϕ_3,ϕ_4}, it equals

$$\int_{\mathfrak{y}(W)(F)}\int_{H(F)}\Psi_{f_1,\phi_1,\phi_2,\natural}(g_1.y)\Psi_{f_2,\phi_3,\phi_4,\natural}(y)\mathrm{d}g_1\mathrm{d}y.$$

By [Zha14a, Corollary 4.5] and [Xue, Lemma 5.8], this integral is absolutely convergent and equals

$$\int_{\mathfrak{y}(W)(F)}\widetilde{\Psi_{f_1,\phi_1,\phi_2}}(y)O(y,\widehat{\Psi_{f_2,\phi_3,\phi_4}})\mathrm{d}y.$$

In conclusion, we have proved

$$T(f_1,f_2,\phi_1,\phi_2,\phi_3,\phi_4)=\int_{\mathfrak{y}(W)(F)}\widetilde{\Psi_{f_1,\phi_1,\phi_2}}(y)O(y,\widehat{\Psi_{f_2,\phi_3,\phi_4}})\mathrm{d}y. \tag{4.6}$$

We now compute $T(f_1,f_2,\phi_1,\phi_2,\phi_3,\phi_4)$ spectrally. First of all, since Fourier transform preserves the L^2-norm of Schwartz functions, we have

$$\begin{aligned}
&T(f_1,f_2,\phi_1,\phi_2,\phi_3,\phi_4)\\
=&\int_{H(F)}\int_{H(F)}\int_{G(F)}f_1(h_1gh_2)f_2(g)\overline{\langle\omega(h_1^{-1})\phi_1,\phi_3\rangle}\langle\omega(h_2)\phi_2,\overline{\phi_4}\rangle\mathrm{d}g\mathrm{d}h_2\mathrm{d}h_1\\
=&\int_{H(F)}\int_{H(F)}(f_2^\vee*\mathsf{L}(h_1^{-1})f_1)(h_2)\overline{\langle\omega(h_1^{-1})\phi_1,\phi_3\rangle}\langle\omega(h_2)\phi_2,\overline{\phi_4}\rangle\mathrm{d}h_2\mathrm{d}h_1,
\end{aligned}$$

where $*$ stands for the convolutions of functions on $G(F)$. Applying Lemma 4.4.1 to the inner integral, this equals

$$\begin{aligned}
&\int_{H(F)}\int_{\mathcal{X}_{\mathrm{temp}}(G)}\mathcal{L}_\pi^{\overline{\phi_4},\phi_2}(\pi(f_2^\vee*\mathsf{L}(h_1^{-1})f_1))\overline{\langle\omega(h_1^{-1})\phi_1,\phi_3\rangle}\mathrm{d}\pi\mathrm{d}h_1\\
=&\int_{H(F)}\int_{\mathcal{X}_{\mathrm{temp}}(G)}\mathrm{Tr}(\pi(h_1^{-1})\pi(f_1)L_\pi^{\overline{\phi_4},\phi_2}\pi(f_2^\vee))\overline{\langle\omega(h_1^{-1})\phi_1,\phi_3\rangle}\mathrm{d}\pi\mathrm{d}h_1.
\end{aligned}$$

By Lemma 4.2.1(1), there is a function $f\in\mathcal{C}(G(F))$ so that $\pi(f_1)L_\pi^{\overline{\phi_4},\phi_2}\pi(f_2^\vee)=\pi(f_0)$. By the Paley–Wiener theorem, the inner integral equals $f_0(h_1)$. Apply Lemma 4.4.1 again to f_0, the outer integral then equals

$$\int_{\mathcal{X}_{\mathrm{temp}}(G)}\mathcal{L}_\pi^{\phi_1,\overline{\phi_3}}(\pi(f_1)L_\pi^{\overline{\phi_4},\phi_2}\pi(f_2^\vee))\mathrm{d}\pi.$$

Then by Lemma 4.2.1(2) the integrand equals $\mathcal{L}_\pi^{\phi_1,\phi_2}(\pi(f_1))\mathcal{L}_\pi^{\overline{\phi_4},\overline{\phi_3}}(\pi_2(f_2^\vee))$. Thus we conclude

$$T(f_1,f_2,\phi_1,\phi_2,\phi_3,\phi_4)=\int_{\mathcal{X}_{\mathrm{temp}}(G)}J_\pi(f_1,\phi_1,\phi_2)J_{\pi^\vee}(f_2,\phi_3,\phi_4)\mathrm{d}\pi. \tag{4.7}$$

Combining (4.7) and (4.6), we get Proposition 4.4.3. □

5 A globalization result

5.1 Statement of the results

We come back to the global situation in this section. Thus E/F is quadratic extension of number fields. Let W be a skew-hermitian space over E of dimension n and $H = \mathrm{U}(W)$ and $G = H \times H$ as before. Throughout this section, we make the following assumption:

$$W \text{ is definite at all archimedean places of } E.$$

We let S be a finite set of nonarchimedean split places of F and we put $F_S = \prod_{v \in S} F_v$. For simplicity, for any algebraic group A over F, we write A_S for $\prod_{v \in S} A(F_v)$. We fix a supercuspidal representation σ of G_S. Let v_0 be a nonarchimedean nonsplit place of F. Let $\mathrm{Irr}_{v_0,\sigma,H}(G)$ be the subset of $\mathrm{Irr}_{\mathrm{unit}}(G_{v_0})$ consisting of representations π_{v_0} such that there is an irreducible cuspidal automorphic representation π of $G(\mathbb{A}_F)$ so that

- the local component of π at v_0 is isomorphic to π_{v_0};
- the local component of π at S is isomorphic to an unramified twist of σ, i.e. $\sigma \otimes \chi$ where $\chi \in i\mathfrak{a}^*_{G_S,\mathbb{R}}$.
- π admits nontrivial Fourier–Jacobi periods, i.e. $J_\pi \neq 0$.

Recall that $\mathrm{Temp}_{H_{v_0}}(G_{v_0})$ is the space of irreducible tempered representations of G_{v_0} which admit Fourier–Jacobi models.

The space $\mathrm{Irr}_{\mathrm{unit}}(G_{v_0})$ is equipped with the Fell topology and its subspaces are equipped with the subspace topology. Recall that this means the following. For any representations (not necessarily irreducible) π and σ, we say that π is weakly contained in σ if any diagonal matrix coefficient of π can be approximated by sums of diagonal matrix coefficients of σ uniformly on compact subsets of G, i.e. for any compact subset K of G_{v_0}, any $v \in \pi$ and any $\epsilon > 0$, we can find $\xi_1, \cdots, \xi_r$ in σ, such that

$$\left| \langle \pi(g)v, v \rangle - \sum_{i=1}^{r} \langle \sigma(g)\xi_i, \xi_i \rangle \right| < \epsilon,$$

for all $g \in K$. The Fell topology is the unique topology on $\mathrm{Irr}_{\mathrm{unit}}(G_{v_0})$ such that the closure of any subset T is the set of irreducible unitary representation of G_{v_0} that is weakly contained in $\oplus_{\sigma \in T} \sigma$.

The main result of this section is the following proposition whose proof will be given in subsequent subsections.

Proposition 5.1.1. *The set* $\mathrm{Irr}_{v_0,\sigma,H}(G) \cap \mathrm{Temp}(G_{v_0})$ *is dense in* $\mathrm{Temp}_{H_{v_0}}(G_{v_0})$.

5.2 Schwartz functions of positive type

In this subsection, we prove some technical lemmas on the Fourier transform of Schwartz functions.

Let us fix a nonarchimedean place v of F and suppress it from all notation. Recall that we have a skew-hermitian space W and a maximal isotropic subspace L of $W_{/F}$. We say that a Schwartz function $\Psi \in \mathcal{S}(L^2)$ is of positive type if $\Psi(x,y) = \overline{\Psi(y,x)}$ for all $x, y \in L$ and for any $x_1, \cdots, x_n \in L$, the matrix

$$\{\Psi(x_i, x_j)\}_{1 \le i,j \le n}$$

is semipositive-definite.

Lemma 5.2.1. *The function $\Psi(x,y)$ is semipositive definite if and only if it can be written in the form*

$$\sum_i \phi_i(x)\overline{\phi_i(y)}, \quad \phi_i \in \mathcal{S}(L).$$

Proof. The "if" part is clear. Now assume that Ψ is of positive type. Then we may find $\phi_1, \cdots, \phi_n$ so that

$$\Psi(x,y) = \sum_{i,j=1}^{n} \lambda_{ij}\phi_i(x)\overline{\phi_j(y)}.$$

We may further assume that $\phi_1, \cdots, \phi_n$ are linearly independent. The condition that $\Psi(x,y) = \Psi(y,x)$ implies that $\lambda_{ij} = \overline{\lambda_{ji}}$. We choose $s_1, \cdots, s_n$ so that the matrix

$$S = \{\phi_i(s_j)\}_{1 \le i,j \le n}$$

is invertible. Let $\underline{\Psi}$ be the matrix $\{\Psi(x_i, x_j)\}_{1 \le i,j \le n}$ and $\Lambda = \{\lambda_{ij}\}_{1 \le i,j \le n}$. Then

$$\underline{\Psi} = S\Lambda\, {}^t\overline{S}.$$

The semipositivity of $\underline{\Psi}$ implies that Λ is semipositive definite. Therefore we may find a matrix P so that $\Lambda = P\, {}^t\overline{P}$. Put $(\phi'_1, \cdots, \phi'_n) = (\phi_1, \cdots, \phi_n)P$. We conclude that

$$\Psi = \sum_i \phi'_i \otimes \overline{\phi'_i}.$$

This proves the "only if" part. □

Lemma 5.2.2. *Let $N \subset L$ be an open compact neighbourhood of $0 \in L$ such that $y \in N$ if and only if $-y \in N$. Choose a large integer m so that if $x, y \in \varpi^m N$, then $x \pm y \in \varpi^m N$. Then the form*

$$\Psi(x,y) = \mathbf{1}_{\varpi^m N}(x-y)\, \mathbf{1}_N(x+y)$$

is of positive type.

Proof. Let $s_1, \cdots, s_n \in L$. We claim that we can re-label them so that the matrix

$$\{\mathbf{1}_{\varpi^m N}(s_i - s_j)\}_{1 \le i,j \le n}$$

is of the form

$$\begin{pmatrix} E_{r_1} & & \\ & \ddots & \\ & & E_{r_k} \end{pmatrix},$$

where E_i stands for the $i \times i$ matrix whose entries are all 1. Indeed, if $s_n - s_i \in \varpi^m N$ for all i, then we are done. Otherwise, we let $S \subset \{s_1, \cdots, s_n\}$ be the subset of all s_i's with $s_n - s_i \in \varpi^m N$. Let T be the complement of S. Then for any $s \in S$ and $t \in T$, we have $s - t \notin \varpi^m N$. Otherwise $s_n - t \in \varpi^m N$ since both $s - s_n, s - t \in \varpi^m N$, which contradicts the choice of S. Then we repeat this argument for T. This proves the claim.

Let us observe that

$$\mathbf{1}_{\varpi^m N}(x - y)\, \mathbf{1}_N(x + y) = \mathbf{1}_{\varpi^m N}(x - y)\, \mathbf{1}_N(x)\, \mathbf{1}_N(y).$$

Indeed, if $x - y \in \varpi^m N$, then $x + y \in N$ if and only if both $x, y \in \varpi^m N$. It follows that the matrix

$$\{\mathbf{1}_{\varpi^m N}(s_i - s_j)\, \mathbf{1}_N(s_i + s_j)\}_{1 \le i,j \le n}$$

is a blocked diagonal matrix, with the diagonal blocks all of the form ${}^t(y_1, \cdots, y_r)(y_1, \cdots, y_r)$, which is semipositive definite. This proves the lemma. □

Lemma 5.2.3. *Let N be an open compact neighbourhood of $0 \in W$. Then there exist a Schwartz function $\Psi \in \mathcal{S}(L^2)$ of positive type so that* $\operatorname{supp} \Psi^\ddagger \subset N$.

Proof. Let us fix an open compact neighbourhood N' of 0 in L and a large integer m so that $\varpi^m N' \subset N$ and the Fourier transform of the characteristic function of $\mathbf{1}_{N'}$ is the characteristic function of an open compact neighbourhood of 0 which is contained in N. Then

$$\Psi(x, y) = \mathbf{1}_{\varpi^m N'}(x + y)\, \mathbf{1}_{N'}(x - y)$$

is the desired function. □

5.3 Approximating matrix coefficients

The goal of this subsection is to prove the following proposition. This is the key to the proof of Proposition 5.1.1.

Proposition 5.3.1. *Let S be any finite set of finite places of F. Let $f_S \in \mathcal{S}(G_S)$ and $\phi_S \in \mathcal{S}(L_S)$ so that*

$$\int_{H_S}\int_{G_S} f(h^{-1}g)\overline{f(g)}\langle \omega(h)\phi_S, \phi_S\rangle \mathrm{d}g\mathrm{d}h \neq 0.$$

Then for any $\epsilon > 0$ and any compact subset K_S of G_S, there is a finite set of automorphic representations π_i of $G(\mathbb{A}_F)$ with $J_{\pi_i} \neq 0$ and $\varphi_i \in \pi_i$, so that

$$\left| \int_{H_S}\int_{G_S} f(h^{-1}gy)\overline{f(g)}\langle \omega(h)\phi_S, \phi_S\rangle \mathrm{d}g\mathrm{d}h - \sum_i \langle \pi_{i,S}(y)\varphi_i, \varphi_i\rangle \right| \leq \epsilon, \quad y \in K_S.$$

Proof. For any $f \in \mathcal{S}(G(\mathbb{A}_F))$ and $\phi \in \mathcal{S}(L(\mathbb{A}_F)$, define

$$K_{f,\phi}(g) = \int_{[H]} \left(\sum_{\gamma \in G(F)} f(h^{-1}\gamma g) \right) \overline{\theta(h,\phi)} \mathrm{d}h.$$

Since $[G]$ is compact, we have

$$\sum_{\gamma \in G(F)} f(h^{-1}\gamma g) = \sum_{\pi}\sum_{\varphi \in \pi} \pi(f)\varphi(h)\overline{\varphi(g)}.$$

where π runs over irreducible cuspidal automorphic representations of $G(\mathbb{A}_F)$ and φ runs overs an orthonormal basis of π. Therefore

$$\langle R(y)K_{f,\phi}, K_{f,\phi}\rangle = \sum_{\pi}\sum_{\varphi,\varphi' \in \pi} \mathcal{FJ}(\pi(f)\varphi, \phi)\overline{\mathcal{FJ}(\pi(f)\varphi', \phi)\langle \pi(y)\varphi, \varphi'\rangle}, \tag{5.1}$$

where φ and φ' independently run through an orthonormal basis of π. Note that for a fixed f, there are only finitely many π's appearing in the sum.

We may also compute $\langle R(y)K_{f,\phi}, K_{f,\phi}\rangle$ geometrically. Indeed, usual unfolding procedure yields

$$\begin{aligned}\langle R(y)K_{f,\phi}, K_{f,\phi}\rangle = & \sum_{[\gamma,w] \in ((H(F)\backslash G(F)) \times W^\vee(F))//H(F)} \mathrm{vol}[\mathrm{Stab}_{[\gamma,w]}] \int_{\mathrm{Stab}_{[\gamma,w]}(\mathbb{A}_F)\backslash H(\mathbb{A}_F)} \int_{G(\mathbb{A}_F)} \int_{H(\mathbb{A}_F)} \\ & f(h_1^{-1}h_2^{-1}\gamma h_2 gy)\overline{f(g)}(\overline{\omega(h_1)\phi} \otimes \phi)^\dagger(wh_2)\mathrm{d}h_1\mathrm{d}g\mathrm{d}h_2.\end{aligned} \tag{5.2}$$

Here and below we denote by $\mathrm{Stab}_?$ the stabilizer of ?. The term corresponding to $[\gamma, w] = [1, 0]$ equals

$$\mathrm{vol}[H] \int_{H(\mathbb{A}_F)}\int_{G(\mathbb{A}_F)} f(h^{-1}gy)\overline{f(g)}\langle \overline{\omega(h_1)\phi}, \phi\rangle \mathrm{d}g\mathrm{d}h_1.$$

We fix a nonarchimedean place v_1 not in S. We are going to choose the test function $f = \otimes f_v$ and $\phi^{(i)} = \otimes \phi_v^{(i)}$, $i = 1, \cdots, r$. At the places $v \neq v_1$, we do the following.

- If v is archimedean, then put $f_v = (\mathrm{vol}\, G_v)^{-\frac{1}{2}} \mathbf{1}_{G_v}$ and choose ϕ_v to be G_v-invariant and $\langle \phi_v, \phi_v\rangle = 1$. This is possible since G_v is compact.

– If $v \in S$, then put $f_v = f_{S,v}$ and $\phi_v = \phi_{S,v}$.

– If $v \notin S \cup \{v_1\} \cup \{\infty\}$, then we choose any f_v and ϕ_v so that

$$\int_{G_v} \int_{H_v} f_v(h_1^{-1}g)\overline{f_v(g)}\langle\overline{\omega(h_1)\phi_v}, \phi_v\rangle \mathrm{d}h_1 = 1.$$

<u>*Claim*</u>: for any $\epsilon > 0$, for fixed f_v and ϕ_v, $v \neq v_1$, we can choose $\phi_{v_1}^{(i)}$ and f_{v_1} so that the following conditions hold.

1. For $\gamma \in G(F)$, there exist $y \in K_S$, $g \in \operatorname{supp} f$ and $h_1, h_2 \in H(\mathbb{A}_F)$ so that

$$f(h_1^{-1}h_2^{-1}\gamma h_2 gy) \neq 0$$

   only if $\gamma = 1$.

2. We have

$$\sum_{\substack{w \in W^\vee(F)/H(F) \\ w \neq 0}} \mathrm{vol}[\mathrm{Stab}_w] \int_{\mathrm{Stab}_w(\mathbb{A}_F)\backslash H(\mathbb{A}_F)} \int_{G(\mathbb{A}_F)} \int_{H(\mathbb{A}_F)} \left| f(h_1^{-1}gy)\overline{f(g)} \sum_{i=1}^{r} (\overline{\omega(h_1)\phi^{(i)}} \otimes \phi^{(i)})^\ddagger(wh_2) \right| \mathrm{d}h_1 \mathrm{d}g \mathrm{d}h_2 \leq \epsilon$$

3. We have $\omega(h_1)\phi_{v_1}^{(i)} = \phi_{v_1}^{(i)}$ if there exists $g \in \operatorname{supp} f_{v_1}$ so that $h_1^{-1}g \in \operatorname{supp} f_{v_1}$.

4. We have

$$\int_{H(F_v)} \int_{G(F_v)} f_{v_1}(h^{-1}g)\overline{f_{v_1}(g)} \mathrm{d}g\mathrm{d}h = 1, \quad \sum_{i=1}^{r} (\overline{\phi_{v_1}^{(i)}} \otimes \phi_{v_1}^{(i)})^\ddagger(0) = 1.$$

We identity $H\backslash G$ with H by sending (γ_1, γ_2), $\gamma_1, \gamma_2 \in H$, to $\gamma_1^{-1}\gamma_2 \in H$. The categorical quotient $H//H$ is identified with the n-dimensional affine space and the natural map $H \to H//H$ is given by sending h to the coefficients of the characteristic polynomial of $h - 1_n$ (note that this is slightly different from the usual choice, which amounts to making some change of coordinates in the affine space). It sends $1_n \in H$ to $0 \in H//H$. Given the choices f_v for all $v \neq v_1$, we have for each $v \neq v_1$ an open compact subset Ω'_v of $H(F)\backslash G(F)$ so that if $g \in \operatorname{supp} f_v$, $h_1, h_2 \in H_v$, $\gamma \in H(F)\backslash G(F)$, $y \in K_v$ and $h_1^{-1}h_2^{-1}\gamma h_2 gy \in \operatorname{supp} f_v$, then $h_2^{-1}\gamma h_2 \in \Omega'_v$. Let Ω_v be the image of Ω'_v in $(H//H)(F_v)$ under the natural map as described above. We conclude that the image of γ in $H//H(F_v)$ lies in Ω_v. At the place v_1, let us choose a sufficiently small open compact neighbourhood Ω_{v_1} of $0 \in H//H(F_{v_1})$ so that

$$\prod_v \Omega_v \cap (H//H)(F) = \{0\}.$$

This is possible. Indeed if $\gamma \in \prod_v \Omega_v \cap (H//H)(F)$, we write $\gamma = (x_1, \cdots, x_n)$, $x_i \in F$. Suppose that some $x_i \neq 0$. Then $\prod_v |x_i|_{F_v} = 1$. But Ω_v is compact, thus $\prod_{v \neq v_1} |x_i|_{F_v}$ is bounded from above and this bound depend on Ω_v's only, which means that there is a positive real number δ such that $|x_i|_{F_{v_1}} \geq \delta$. We just need to choose Ω_{v_1} so that there is no such a gamma. Let Ω'_{v_1} be an open compact subset of $G(F_{v_1})$ such that if $h_1, h_2 \in H(F_{v_1})$, $g \in \Omega'_{v_1}$ and $h_1^{-1} h_2^{-1} \gamma h_2 g \in \Omega'_{v_1}$ then the image of γ in $H//H(F_{v_1})$ lies in Ω_{v_1}. Then for any f_{v_1} supported in Ω'_{v_1}, condition (1) holds. One should note that since G_∞ is compact, every rational orbit is semisimple.

We observe that the categorical quotient $W^\vee//H$ is isomorphic to the affine line and the morphism $W^\vee \to W^\vee//H$ is given by $w \mapsto [w, w]$. To simplify notation, for any place $v \neq v_1$ and any $w \in W_v^\vee$, let us temporarily put

$$O(w, f_v, \phi_v) = \int_{\mathrm{Stab}_w(F_v) \backslash H(F_v)} \int_{G(F_v)} \int_{H(F_v)} |f_v(h_1^{-1} g y_v) \overline{f_v(g_v)} (\overline{\omega_v(h)\phi_v} \otimes \phi_v)^\ddagger (w h_2)| \mathrm{d}h_1 \mathrm{d}g \mathrm{d}h_2.$$

Note that $w \mapsto O(w, f_v, \phi_v)$, as a function on $W^\vee//H(F_v)$ is compactly supported (resp. of rapid decay at infinity) if v is nonarchimedean (resp. archimedean). It might have a singularity at $w = 0$.

For any $\epsilon > 0$, we can find a enough small neighbourhood N_{v_1} of $W_{v_1}^\vee$ so that

$$\sum_{\substack{w \in W^\vee(F)/H(F) \\ w \neq 0}} \mathrm{vol}[\mathrm{Stab}_w]\, \mathbf{1}_{N_{v_1}}(w) \prod_{v \neq v_1} O(w, f_v, \phi_v) \leq \epsilon.$$

In fact, fix any open compact neighbourhood N'_{v_1}, we can find a large positive number B so that

$$\sum_{\substack{w \in W^\vee(F)/H(F) \\ |[w,w]|_{F_\infty} > B}} \mathrm{vol}[\mathrm{Stab}_w]\, \mathbf{1}_{N'_{v_1}}(w) \prod_{v \neq v_1} O(w, f_v, \phi_v) \leq \epsilon.$$

For the fixed choices of f_v and ϕ_v for nonarchimedean $v \neq v_1$, if $w \in W^\vee(F)$ and $O(w, f_v, \phi_v) \neq 0$, then $|[w, w]|_{F_v} \leq A_v^{-1}$ for some fixed positive number A_v, almost all being 1. As y varies in a compact subset, we can take A_v's to be independent of y. We may choose $N_{v_1} \subset N'_{v_1}$ to be small enough so that if $w \in W^\vee(F) \cap N_{v_1}$, then $\prod_{v \nmid \infty} |[w, w]|_{F_v} < B^{-1}$. It follows from the product formula that, with this choice, $\mathbf{1}_{N_{v_1}}(w) \prod_{v \neq v_1} O(w, f_v, \phi_v) \neq 0$ automatically implies that $|[w, w]|_{F_\infty} > B$. This N_{v_1} is the desired neighbourhood.

By Lemma 5.2.3, we can choose $\phi_{v_1}^{(i)}$'s so that $\sum_{i=1}^r (\overline{\phi_{v_1}^{(i)}} \otimes \phi_{v_1}^{(i)})^\ddagger(0) = 1$ and $\sum_{i=1}^r (\overline{\phi_{v_1}^{(i)}} \otimes \phi_{v_1}^{(i)})^\ddagger$ is bounded some multiple of the characteristic function of N_{v_1} we have just chosen. Let us fix an f_{v_1} supported in Ω'_{v_1}, such that if $h \in H_{v_1}$, $g, h^{-1} g \in \mathrm{supp}\, f_{v_1}$, then $\omega(h) \phi_{v_1}^{(i)} = \phi_{v_1}^{(i)}$ for all i and

$$\int_{H(F_{v_1})} \int_{G(F_{v_1})} f_{v_1}(h^{-1} g) \overline{f_{v_1}(g)} \mathrm{d}g \mathrm{d}h = 1.$$

So condition (3) and (4) are satisfied.

Finally, with these choices, we have that

$$\sum_{\substack{w\in W^\vee(F)/H(F)\\ w\neq 0}} \mathrm{vol}[\mathrm{Stab}_w] \int_{\mathrm{Stab}_w(\mathbb{A}_F)\backslash H(\mathbb{A}_F)} \int_{G(\mathbb{A}_F)} \int_{H(\mathbb{A}_F)} \left| f(h_1^{-1}h_2^{-1}\gamma h_2 y)\overline{f(g)} \sum_{i=1}^{r} (\overline{\omega(h_1)\phi^{(i)}} \otimes \phi^{(i)})^{\ddagger}(wh_2) \right| \mathrm{d}h_1 \mathrm{d}g \mathrm{d}h_2$$

equals

$$\sum_{\substack{w\in W^\vee(F)/H(F)\\ w\neq 0}} \mathrm{vol}[\mathrm{Stab}_w]\, \mathbf{1}_{N_{v_1}}(w) \prod_{v\neq v_1} O(w, f_v, \phi_v),$$

which is bounded by ϵ. Thus condition (2) is satisfied. This proves the claim.

With this choice of the test function, we conclude from (5.1) and (5.2) that the difference between

$$\sum_{\pi} \sum_{\varphi,\varphi'\in\pi} \sum_{i=1}^{r} \mathcal{FJ}(\pi(f)\varphi, \phi^{(i)}) \overline{\mathcal{FJ}(\pi(f)\varphi', \phi^{(i)})}\, \overline{\langle \pi(y)\varphi, \varphi'\rangle}$$

and

$$\mathrm{vol}[H] \sum_{i=1}^{r} \int_{H(\mathbb{A}_F)} \int_{G(\mathbb{A}_F)} f(h^{-1}gy)\overline{f(g)} \langle \omega(h_1)\phi^{(i)}, \phi^{(i)}\rangle \mathrm{d}g\mathrm{d}h_1$$

is bounded by ϵ. Moreover by our choice

$$\sum_{i=1}^{r} \int_{H(\mathbb{A}_F)} \int_{G(\mathbb{A}_F)} f(h^{-1}gy)\overline{f(g)} \langle \omega(h_1)\phi^{(i)}, \phi^{(i)}\rangle \mathrm{d}g\mathrm{d}h_1 = \int_{H_S} \int_{G_S} f_S(h^{-1}gy)\overline{f_S(g)} \langle \omega(h)\phi_1, \phi_1\rangle \mathrm{d}g\mathrm{d}h.$$

This proves the proposition. □

Let G_S act on $L^2(G_S)$ by right translation and on $L^2(L_S)$ by $\omega(h_1)$ where $(h_1, h_2) \in G_S$, $h_1, h_2 \in H_S$. Let $\mathscr{L}_S \subset L^2(G_S) \otimes L^2(L_S)$ be the subrepresentation spanned by functions $f \otimes \phi$ such that there exists $\phi'_S \in L^2(L_S)$ with

$$\int_{H(F_S)} f(h) \langle \omega_S(h)\phi_S, \phi'_S\rangle \mathrm{d}h \neq 0.$$

Corollary 5.3.2. *The representation $\mathscr{L}_S$ is weakly contained in $\oplus_{\pi\in T}\pi_S$ where T is the set of irreducible automorphic representation π of $G(\mathbb{A}_F)$ such that $J_\pi \neq 0$.*

This is just a reformulation of the proposition in terms of the notion of weak containment.

5.4 Globalization

The goal of this subsection is to prove Proposition 5.1.1. We begin with two lemmas.

Let π be a tempered representation of H_{v_0}. Then π is a subrepresentation of $i_P^G\sigma$ where $P = MN$ is a parabolic subgroup of G and σ is a discrete series representation of M. We say that π is regular if for any $w \in W(G, M)$ (the Weyl group), we have $w\sigma \not\simeq \sigma$. Let $\mathrm{Temp}_{\mathrm{reg}}(H_{v_0})$ be the isomrphism classes of regular tempered representations.

Let π be an irreducible representation of H_{v_0}. Recall that by the Langlands classification, there exits following data.

- A parabolic subgroup $P = MN$ of H_{v_0}, where

$$M \simeq \mathrm{Res}_{E_{v_0}/F_{v_0}} \mathrm{GL}_{n_1} \times \cdots \times \mathrm{Res}_{E_{v_0}/F_{v_0}} \mathrm{GL}_{n_r} \times \mathrm{U}(W_0)$$

  where W_0 is a skew-hermitian space of dimension $n - 2(n_1 + \cdots + n_r)$.

- Tempered representations $\tau_1, \cdots, \tau_r$ of $\mathrm{GL}_{n_1}(E_{v_0})$, $\cdots$, $\mathrm{GL}_{n_r}(E_{v_0})$ respectively and a tempered representation σ of $\mathrm{U}(W_0)(F_{v_0})$.

- A set of positive real numbers $\lambda_1 > \lambda_2 > \cdots > \lambda_r > 0$.

They satisfy the property that π is the unique irreducible quotient of the induced representation

$$i_P^H \tau_1 |\det|^{\lambda_1} \boxtimes \cdots \boxtimes \tau_r |\det|^{\lambda_r} \boxtimes \sigma.$$

Put $\mathfrak{c}(\pi) = \lambda_1$ if $r \geq 1$ and $\mathfrak{c}(\pi) = 0$ if $r = 0$, namely, π is tempered. Let $c_0 \geq 0$ be a real number. We denote by $\mathrm{Irr}_{\mathrm{unit}, \leq c_0}(H_{v_0})$ be the set of unitary representations π of H_{v_0} with the property that $\mathfrak{c}(\pi) \leq c_0$.

The following two lemmas are [BPc, Lemma 3.6.1 and Lemma 3.6.2] respectively.

Lemma 5.4.1. *Suppose that* $0 \leq c_0 < \frac{1}{2}$. *Then* $\mathrm{Temp}_{\mathrm{reg}}(H_{v_0})$ *is open in* $\mathrm{Irr}_{\mathrm{unit}, \leq c_0}(H_{v_0})$.

Lemma 5.4.2. *Suppose that π is a local component of an irreducible cuspidal automorphic representation of $H(\mathbb{A}_F)$ whose base change is also cuspidal. Then* $\pi \in \mathrm{Irr}_{\mathrm{unit}, \leq \frac{1}{2} - \frac{1}{n^2+1}}(H_{v_0})$.

Proof of Proposition 5.1.1. It follows from Lemma 4.3.1 that the closure of set $\mathrm{Temp}_{H_{v_0}}(G_{v_0})$ is a disjoint union of connected components of $\mathrm{Temp}(G_{v_0})$. Therefore the set $\mathrm{Temp}_{H_{v_0}, \mathrm{reg}}(G_{v_0}) = \mathrm{Temp}_{H_{v_0}}(G_{v_0}) \cap \mathrm{Temp}_{\mathrm{reg}}(G_{v_0})$ is dense in $\mathrm{Temp}_{H_{v_0}}(G_{v_0})$. So we only need to prove that any $\pi_{v_0} \in \mathrm{Temp}_{H_{v_0}, \mathrm{reg}}(G_{v_0})$ can be approximated by elements in $\mathrm{Irr}_{v_0, \sigma, H}(G) \cap \mathrm{Temp}(G_{v_0})$.

Put $S' = S \cup \{v_0\}$. Let $\mathscr{L}_{S'}$ be the representation of $G_{S'}$ defined before Corollary 5.3.2. Lemma 4.4.1 shows that $\pi_{v_0} \otimes \sigma$ is weakly contained in $\mathscr{L}_{S'}$. Corollary 5.3.2 implies that $\mathscr{L}_{S'}$

is weakly contained in $\oplus_{\pi\in T}\pi_{S'}$ where T is the set of irreducible automorphic representations of $G(\mathbb{A}_F)$ so that $J_\pi \neq 0$. Therefore we can find a sequence of automorphic representations π_j such that $J_{\pi_j} \neq 0$ and $\pi_{j,S'}$ is convergent to $\pi_{v_0} \otimes \sigma$ in the Fell topology. The set $\{\sigma \otimes \chi \mid \chi \in i\mathfrak{a}^*_{G_S}\}$ is a open in $\mathrm{Irr}_{\mathrm{unit}}(G_S)$, therefore for n sufficiently large we know that $\pi_{j,S}$ is an unramified twist of σ. Thus $\mathrm{BC}(\pi_j)$ is a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ and hence $\pi_{j,v_0} \in \mathrm{Irr}_{\mathrm{unit},\leq \frac{1}{2}-\frac{1}{n^2+1}}(G_{v_0})$ by Lemma 5.4.2. As π_{v_0} is regular and $\mathrm{Temp}_{H_{v_0},\mathrm{reg}}(G_{v_0})$ is open in $\mathrm{Irr}_{\mathrm{unit},\leq \frac{1}{2}-\frac{1}{n^2+1}}(G_{v_0})$, we conclude that $\pi_{j,v_0} \in \mathrm{Temp}_{H_{v_0},\mathrm{reg}}(G_{v_0})$ if j is sufficiently large. This proves Proposition 5.1.1. □

6 The main theorems

6.1 Weak comparison of local spherical characters

The goal of this subsection is to prove Theorem 4.1.2 up to some constant.

Lemma 6.1.1. *Let the notation be as in Theorem 4.1.2. Assume in addition that π_v is a local component of an irreducible cuspidal automorphic representation π of v so that*

- *$J_\pi \neq 0$;*
- *there are two split places $v_1, v_2 \neq v$ such that π_{v_1}, π_{v_2} are both supercuspidal,*
- *π_∞ is tempered.*

Then there is a nonzero constant c_{π_v}, depending only on π_v, such that for all matching test functions (f'_v, Φ_v) and $(f_v, \phi_{1,v}, \phi_{2,v})$, we have

$$c_{\pi_v} I_{\Pi_v}(f'_v, \Phi_v) = J_{\pi_v}(f_v, \phi_{1,v}, \phi_{2,v}).$$

Proof. In the present situation, the local Gan–Gross–Prasad conjecture is known by the work of Gan–Ichino [GI] and Beuzart-Plessis [BPb]. Note that we only need the multiplicity one part of the local Gan–Gross–Prasad conjecture and the method of Gan–Ichino applies also to the archimedean places. Indeed, this case is significantly easier since the hardest part of the proof, i.e. the conjecture of Prasad, which predicts the behaviour of the Langlands parameters under theta correspondences, is known. There is only one term on the right hand side of (2.14) being nonzero, namely J_π by assumption. Note that we did not assume that π is tempered. So the decomposition (4.1) is not available. But we have the following alternative. We fix an element $\ell_v \in \mathrm{Hom}_{H(F_v)}(\pi_v \otimes \overline{\omega_v}, \mathbb{C})$ for all places v. We define for all place v'

$$J'_{\pi_{v'}}(f_{v'}, \phi_{1,v'}, \phi_{2,v'}) = \sum_{\varphi_{v'} \in \pi_{v'}} \ell_{v'}(\pi_{v'}(f_{v'})\varphi_{v'}, \phi_{1,v'}) \overline{\ell_{v'}(\varphi_{v'}, \phi_{2,v'})}.$$

At the place v, we may choose ℓ_v so that $J_{\pi_v} = J_{\pi_{v'}}$. For almost all place v' we fix $\ell_{v'}$ so that $J_{\pi_v}(f_{v'}, \phi_{1,v'}, \phi_{2,v'}) = 1$. Then using the multiplicity one theorem, we again have a decomposition

$$J_\pi(f, \phi_1, \phi_2) = C(\pi)' \prod_v J'_{\pi_v}(f_v, \phi_{1,v}, \phi_{2,v}),$$

where $C(\pi)'$ is some constant (depending on the choice of ℓ_v's).

For a matching pair of test functions (f'_v, Φ_v) and $(f_v, \phi_{1,v}, \phi_{2,v})$, we claim that

$$J'_{\pi_v}(f_v, \phi_{1,v}, \phi_{2,v}) = 0 \text{ implies that } I_{\Pi_v}(f'_v, \Phi_v) = 0.$$

If $I_{\Pi_v}(f'_v, \Phi_v) \neq 0$, then for any place $w \neq v$, choose matching test functions (f'_w, Φ_w) and $(f_w, \phi_{1,w}, \phi_{2,w})$ such that $I_{\Pi_w}(f'_w, \Phi_w) \neq 0$. Since $J_\pi \neq 0$, we have $L(\frac{1}{2}, \Pi_1 \times \Pi_2) \neq 0$ by Theorem 1.1.1. It follows from Proposition 3.1.1 that $I(f', \Phi) \neq 0$ and hence $J_\pi(f, \phi_1, \phi_2) \neq 0$. This is a contradiction.

Let (f'_v, Φ_v) and $(f_v, \phi_{1,v}, \phi_{2,v})$ be a matching pair of test functions such that $J_{\pi_v}(f_v, \phi_{1,v}, \phi_{2,v}) \neq 0$. We define

$$c'_{\pi_v} = I_{\Pi_v}(f'_v, \Phi_v) J_{\pi_v}(f_v, \phi_{1,v}, \phi_{2,v})^{-1}.$$

This is independent of the choice of (f'_v, Φ_v) and $(f_v, \phi_{1,v}, \phi_{2,v})$. To see this, again for any place $w \neq v$, choose matching test functions (f'_w, Φ_w) and $(f_w, \phi_{1,w}, \phi_{2,w})$ such that $I_{\Pi_w}(f'_w, \Phi_w) \neq 0$. It follows that $J_{\pi_w}(f_w, \phi_{1,w}, \phi_{2,w}) \neq 0$. Then by the decompositions of I_{Π_v} and J_π, we conclude that

$$c'_{\pi_v} = C(\pi)' \left(\prod_{w \neq v} I_{\Pi_w}(f'_w, \Phi_w) J_{\pi_v}(f_w, \phi_{1,w}, \phi_{2,w})^{-1} \cdot \frac{L(\frac{1}{2}, \Pi_1 \times \Pi_2)}{L(1, \pi_1, \mathrm{Ad}) L(1, \pi_2, \mathrm{Ad})} \right)^{-1}.$$

This does not depend on the choice of the test functions at v.

Therefore for any matching test functions (f'_v, Φ_v) and $(f_v, \phi_{1,v}, \phi_{2,v})$, we have

$$I_{\Pi_v}(f'_v, \Phi_v) = c'_{\pi_v} J_{\pi_v}(f_v, \phi_{1,v}, \phi_{2,v}).$$

It is clear that $c'_{\pi_v} \neq 0$. Otherwise $I_{\Pi_v} = 0$ for all test functions. This is not possible. We put $c_{\pi_v} = c'^{-1}_{\pi_v}$. Thus we have proved the lemma. ☐

Lemma 6.1.2. *For all $\pi_v \in \mathrm{Temp}_{H_v}(G_v)$, there exists a nonzero constant c_{π_v}, such that for any matching test functions (f'_v, Φ_v) and $(f_v, \phi_{1,v}, \phi_{2,v})$, we have*

$$c_{\pi_v} I_{\Pi_v}(f'_v, \Phi_v) = J_{\pi_v}(f_v, \phi_{1,v}, \phi_{2,v}).$$

Proof. We use a global-to-local argument to prove this lemma. We may find the following data.

– A totally real number field $\mathbb{F}$ and a CM extension $\mathbb{E}$ of $\mathbb{F}$ and a place v_0 of $\mathbb{F}$ so that $\mathbb{F}_{v_0} = F_v$.

– A skew-hermitian space $\mathbb{W}$ over $\mathbb{E}$ of dimension n such that $\mathbb{W}_{v_0} = W_v$. Put $\mathbb{H} = \mathrm{U}(\mathbb{W})$ and $\mathbb{G} = \mathbb{H} \times \mathbb{H}$.

– A Weil representation $\underline{\underline{\omega}}$ of $\mathbb{H}(\mathbb{A}_{\mathbb{F}})$ such that $\underline{\underline{\omega}}_{v_0} \simeq \omega_v$.

– Two nonarchimedean split place v_1 and v_2 of $\mathbb{F}$, and irreducible supercuspidal representations σ_{v_1}, σ_{v_2} of $\mathbb{H}_{v_1}$ and $\mathbb{H}_{v_2}$ respectively.

With these choice of data, by Proposition 5.1.1, we can find a sequence of irreducible cuspidal automorphic representations π_j of $\mathbb{G}(\mathbb{A}_{\mathbb{F}})$ so that π_{j,v_0} is convergent to π_v in the Fell topology and $\pi_{j,v_0} \in \mathrm{Irr}_{v_0,\sigma_{v_1}\otimes\sigma_{v_2},\mathbb{H}_{v_0}}(\mathbb{G}_{v_0}) \cap \mathrm{Temp}_{\mathbb{H}_{v_0}}(\mathbb{G}_{v_0})$. Note that any of these π_j's satisfy the properties that

– $J_{\pi_j} \neq 0$;

– π_{j,v_1} (resp. π_{j,v_2}) are unramified twists of σ_{v_1} and σ_{v_2} respectively, and hence are supercuspidal;

– $\pi_{j,\infty}$ is tempered since $\mathbb{G}_\infty$ is compact.

The constant $c_{\sigma_{v_0}}$ is defined for any element $\sigma_{v_0} \in \mathrm{Irr}_{v_0,\sigma_{v_1}\otimes\sigma_{v_2},\mathbb{H}_{v_0}}(\mathbb{G}_{v_0}) \cap \mathrm{Temp}_{\mathbb{H}_{v_0}}(\mathbb{G}_{v_0})$. We observe that the map $\sigma_{v_0} \to c_{\sigma v_0}$ is continuous on $\mathrm{Irr}_{v_0,\sigma_{v_1}\otimes\sigma_{v_2},\mathbb{H}_{v_0}}(\mathbb{G}_{v_0}) \cap \mathrm{Temp}_{\mathbb{H}_{v_0}}(\mathbb{G}_{v_0})$ since both $I_{\mathrm{BC}(\sigma_{v_0})}$ and $J_{\sigma_{v_0}}$ are continuous and we can always choose test functions so that both $I_{\mathrm{BC}(\sigma_{v_0})}$ and $J_{\sigma_{v_0}}$ do not vanish. Therefore we have

$$c_{\pi_v} I_{\Pi_v}(f'_v, \Phi_v) = J_{\pi_v}(f_v, \phi_{1,v}, \phi_{2,v}), \quad c_{\pi_v} = \lim_{j\to\infty} c_{\pi_{j,v_0}}.$$

This proves the lemma. □

6.2 Local spherical character identities

The goal of this subsection is to prove Proposition 4.1.2. Therefore we work in the local situation. Thus F stands for a nonarchimedean local field. We will assume that E is a quadratic field extension of F as the case $E = F \times F$ has been proved in [Xue16]. We keep the notation from Proposition 4.1.2.

We change all the measures to the unnormalized ones, i.e. the ones without the normalization factor $\zeta_F(1)$, $L(1,\eta)$, etc.. Then as observed in [Xue16, Lemma 4.5.1], we have the following lemma.

Lemma 6.2.1. *With this new choice of the measures, the identity* (4.2) *becomes*

$$I_\Pi(f', \Phi) = \kappa J_\pi(f, \phi_1, \phi_2) \tag{6.1}$$

We make an identification $\mathfrak{x}_n \simeq \widehat{\mathfrak{x}_n}$ by $x = [\gamma, x, y] \mapsto {}^t x = [\gamma, {}^t x, {}^t y]$. A test function f' on $\widehat{\mathfrak{x}_n}(F)$ is thus identified with a function on $\mathfrak{x}_n(F)$. We say that two regular semisimple orbits $x \in \widehat{\mathfrak{x}_n}(F)$ and $y \in \mathfrak{y}(F)$ match if ${}^t x \in \mathfrak{x}_n(F)$ and $y \in \mathfrak{y}(F)$ match. We say that the test functions φ' on $\widehat{\mathfrak{x}_n}(F)$ and φ on $\mathfrak{y}(F)$ match if they match when φ' is viewed as a function on $\mathfrak{x}_n(F)$ via the above identification.

Lemma 6.2.2 ([Zha14a, Theorem 4.17]). *Suppose that $\varphi' \in \mathcal{S}(\mathfrak{x}_n(F))$ and $\varphi \in \mathcal{S}(\mathfrak{y}(F))$ match. Then so do $\widehat{\varphi'}$ and $\epsilon(\frac{1}{2}, \eta, \psi)^{\frac{n(n+1)}{2}} \widehat{\varphi}$.*

Proof of Proposition 4.1.2. We denote by Q the categorical quotient of X_n by GL_n, or the categorical quotient of $Y(W)$ by $\mathrm{U}(W)$ for any W. They are all isomorphic and we fix an isomorphism. Let $U \subset X_n(F)$ be the inverse image of the image of $\mathfrak{c}(\mathbf{n}^W \times W^\vee)$ in $Q(F)$. We may further assume that $U \subset \mathfrak{c}(\mathbf{n} \times F_n \times E^{-,n})$. Let $C \subset \mathrm{Temp}(G)$ be a compact subset which is contained in the support of the map $\pi \mapsto \pi(f_1)$. Let $C^\vee = \{\pi^\vee \mid \pi \in C\}$ and $\mathrm{BC}(C^\vee) = \{\mathrm{BC}(\pi) \mid \pi \in C^\vee\} \subset \mathrm{Temp}(G')$. They are both compact.

Let $(f_1, \phi_1, \phi_2) \in \mathcal{S}(G(F) \times L \times L)$ be a test function and let $\mathcal{Y}$ be the image of the support of $\widetilde{\Psi_{f_1,\phi_1,\phi_2,\natural}}$ in $(\mathfrak{y}(W)//\mathrm{U}(W))(F) = (\mathfrak{x}_n//\mathrm{GL}_n)(F)$.

Let us choose a test function $(f', \Phi) \in \mathcal{S}(G'(F) \times E_n)$ as in Corollary 3.2.2, namely,

1. The support of $\Upsilon_{f',\Phi}$ is contained in U.
2. The function
$$X \mapsto \widehat{\mathbf{t}}(X) O(X, \widehat{\Upsilon_{f',\Phi,\natural}})$$
is a nonzero constant when restricted to the inverse image of $\mathcal{Y}$ in $\mathfrak{x}_n(F)_{\mathrm{rs}}$. This constant equals $\widehat{\mathbf{t}}(\xi_-) O(\xi_-, \widehat{\Upsilon_{f',\Phi,\natural}})$.
3. For all $\Pi \in \mathrm{BC}(C^\vee)$, we have
$$I_\Pi(f', \Phi) = |\tau|_E^{d_n} \chi_\Pi(\tau) \mu(\tau)^{-\frac{n(n+1)}{2}} O(\xi_-, \widehat{\Upsilon_{f',\Phi,\natural}}),$$
where $d_n = \binom{n}{3}$, χ_Π is the central character of Π.

Let $(f_2, \phi_3, \phi_4) \in \mathcal{S}(G(F) \times L \times L)$ be a test function which matches (f', Φ). By condition 1 above, we may assume that Ψ_{f_2,ϕ_3,ϕ_4} is supported in $\mathfrak{c}(\mathbf{n}^W \times W^\vee)$. As in [Xue16, p. 612, Proof of Theorem 4.4.1(3)], we have for any matching $x \in \widehat{\mathfrak{x}_n}(F)$ and $y \in \mathfrak{y}(W)(F)$,

$$O(y, \widehat{\Psi_{f_2,\phi_3,\phi_4,\natural}}) = \eta(2)^{\frac{n(n-1)}{2}} \eta(-1)^{[\frac{n}{2}]} \epsilon(\frac{1}{2}, \eta, \psi)^{-\frac{n(n+1)}{2}} \mu(\operatorname{disc} W)^{-1} \mu(\tau)^{-n(n+1)} \widehat{\mathbf{t}}(x) O(x, \widehat{\Upsilon_{f',\Phi,\natural}}).$$

Proposition 4.4.3 gives

$$\int_{\mathrm{Temp}_H(G)} J_\pi(f_1, \phi_1, \phi_2) J_{\pi^\vee}(f_2, \phi_3, \phi_4) \mathrm{d}\pi = \int_{\mathfrak{y}(W)(F)} \widetilde{\Psi_{f_1,\phi_1,\phi_2,\natural}}(y) O(y, \widehat{\Psi_{f_2,\phi_3,\phi_4,\natural}}) \mathrm{d}y.$$

By our choice of (f_2,ϕ_3,ϕ_4), if $y \in \operatorname{supp} \widecheck{\Psi_{f_1,\phi_1,\phi_2,\natural}}$, then $O(y, \widehat{\Psi_{f_2,\phi_3,\phi_4,\natural}})$ is a constant and equals

$$\eta(2)^{\frac{n(n-1)}{2}}\eta(-1)^{[\frac{n}{2}]}\epsilon(\frac{1}{2},\eta,\psi)^{-\frac{n(n+1)}{2}}\mu(\operatorname{disc} W)^{-1}\mu(\tau)^{-n(n+1)}\widehat{\mathfrak{t}}(\xi_-)O(\xi_-,\widehat{\Upsilon_{f',\Phi,\natural}}).$$

It follows that

$$\begin{aligned}&\int_{\mathrm{Temp}_H(G)} J_\pi(f_1,\phi_1,\phi_2)J_{\pi^\vee}(f_2,\phi_3,\phi_4)\mathrm{d}\pi\\ =&\eta(2)^{\frac{n(n-1)}{2}}\eta(-1)^{[\frac{n}{2}]}\epsilon(\frac{1}{2},\eta,\psi)^{-\frac{n(n+1)}{2}}\mu(\operatorname{disc} W)^{-1}\mu(\tau)^{-n(n+1)}\widehat{\mathfrak{t}}(\xi_-)O(\xi_-,\widehat{\Upsilon_{f',\Phi,\natural}})\Psi_{f_1,\phi_1,\phi_2,\natural}(0)\\ =&\eta(2)^{\frac{n(n-1)}{2}}\epsilon(\frac{1}{2},\eta,\psi)^{-\frac{n(n+1)}{2}}\mu(\operatorname{disc} W)^{-1}\chi_\Pi(\tau)^{-1}|\tau|_E^{-d_n}I_\Pi(f',\Phi)\Psi_{f_1,\phi_1,\phi_2,\natural}(0)\\ =&\kappa^{-1}I_\Pi(f',\Phi)\Psi_{f_1,\phi_1,\phi_2,\natural}(0).\end{aligned} \tag{6.2}$$

for any $\Pi \in \mathrm{BC}(C^\vee)$, where the last equality follows from Condition 3.

We apply Lemma 4.4.1 to conclude that

$$\Psi_{f_1,\phi_1,\phi_2,\natural}(0) = \Psi_{f_1,\phi_1,\phi_2}(1,0) = \int_{\mathrm{Temp}_H(G)} J_\pi(f_1,\phi_1,\phi_2)\mathrm{d}\pi. \tag{6.3}$$

By Lemma 6.1.2, we have

$$c_{\pi^\vee}J_{\pi^\vee}(f_2,\phi_3,\phi_4) = I_\Pi(f',\Phi), \tag{6.4}$$

where $c_{\pi^\vee}$ is a constant depending only on $\pi^\vee$. Combining (6.2), (6.3), (6.4) and the fact that $I_\Pi(f',\Phi) \neq 0$, we finally arrive at

$$\int_{\mathrm{Temp}_H(G)} J_\pi(f_1,\phi_1,\phi_2)(\kappa^{-1}c_{\pi^\vee}-1)\mathrm{d}\pi = 0 \tag{6.5}$$

for all $(f_1,\phi_1,\phi_2) \in \mathcal{S}(G(F)\times L\times L)$. Since $\mathcal{S}(G(F))$ is dense in $\mathcal{C}(G(F))$, we conclude that (6.5) holds for all $(f_1,\phi_1,\phi_2) \in \mathcal{C}(G(F))\otimes\mathcal{S}(L\times L)$.

Let $\mathcal{Z}_{\mathrm{temp}}$ be the center of the category of tempered representations of $G(F)$ [SZ07]. An element $z \in \mathcal{Z}_{\mathrm{temp}}$ can be viewed as a function on $\mathrm{Temp}(G)$ whose restriction to $\mathrm{Temp}_H(G)$ is separating and satisfies the condition that $z(\pi) = z(\pi^\vee)$. Moreover $\mathcal{Z}_{\mathrm{temp}}$ acts on $\mathcal{C}(G(F))$, the action being denoted by $*$. It satisfies $\pi(z*f) = z(\pi)\pi(f)$. Thus for all $z \in \mathcal{Z}_{\mathrm{temp}}$, we have

$$J_\pi(z*f_1,\phi_1,\phi_2) = z(\pi)J_\pi(f_1,\phi_1,\phi_2).$$

It follows that for all $(f_1,\phi_1,\phi_2) \in \mathcal{C}(G(F))\otimes\mathcal{S}(L\times L)$,

$$\int_{\mathrm{Temp}_H G} J_\pi(f_1,\phi_1,\phi_2)(\kappa^{-1}c_{\pi^\vee}-1)z(\pi)\mathrm{d}\pi = 0.$$

As $z(\pi)$ is separating, we have $J_\pi(f_1,\phi_1,\phi_2)(\kappa^{-1}c_{\pi^\vee}-1) = 0$ for all π. For any $\pi \in \mathrm{Temp}_H(G)$, we may choose (f_1,ϕ_1,ϕ_2) so that $J_\pi(f_1,\phi_1,\phi_2) \neq 0$. We finally conclude that $c_{\pi^\vee} = \kappa$. Finally note $c_{\pi^\vee} = c_\pi$. This proves Proposition 4.1.2. □

Corollary 6.2.3. *Suppose that* (f', Φ) *and* (f, ϕ_1, ϕ_2) *are test functions. Suppose that*

$$I_\Pi(f', \Phi) = \kappa J_\pi(f, \phi_1, \phi_2)$$

for all $\pi \in \mathrm{Temp}_H(G)$*. Then* (f', Φ) *and* (f, ϕ_1, ϕ_2) *match.*

Proof. Let $(\widetilde{f}, \widetilde{\phi_1}, \widetilde{\phi_2})$ be the smooth transfer of (f', Φ). Then it follows from Theorem 4.1.2 that

$$J_\pi(f, \phi_1, \phi_2) = J_\pi(\widetilde{f}, \widetilde{\phi_1}, \widetilde{\phi_2}).$$

Therefore for any test function (f_2, ϕ_3, ϕ_4), we have

$$\int_{\mathrm{Temp}_H(G)} J_\pi(f, \phi_1, \phi_2) J_{\pi^\vee}(f_2, \phi_3, \phi_4) \mathrm{d}\pi = \int_{\mathrm{Temp}_H(G)} J_\pi(\widetilde{f}, \widetilde{\phi_1}, \widetilde{\phi_2}) J_{\pi^\vee}(f_2, \phi_3, \phi_4) \mathrm{d}\pi.$$

As in the proof of Proposition 4.4.3, we conclude

$$\int_{Y(W)(F)} O(y, \Psi_{f,\phi_1,\phi_2}) \Psi_{f_2,\phi_3,\phi_4}(y) \mathrm{d}y = \int_{Y(W)(F)} O(y, \Psi_{\widetilde{f},\widetilde{\phi_1},\widetilde{\phi_2}}) \Psi_{f_2,\phi_3,\phi_4}(y) \mathrm{d}g_1 \mathrm{d}y.$$

Let $y_0 \in Y(W)(F)$ be a regular semisimple point. We may choose (f_2, ϕ_3, ϕ_4) so that Ψ_{f_2,ϕ_3,ϕ_4} is the characteristic function of a sufficiently small neighbourhood of y_0 so that for any y in this neighbourhood the functions

$$y \mapsto O(y, \Psi_{f,\phi_1,\phi_2}), \quad y \mapsto O(y, \Psi_{\widetilde{f},\widetilde{\phi_1},\widetilde{\phi_2}})$$

are both constant, namely $O(y_0, \Psi_{f,\phi_1,\phi_2})$ and $O(y_0, \Psi_{\widetilde{f},\widetilde{\phi_1},\widetilde{\phi_2}})$. It follows that

$$O(y, \Psi_{f,\phi_1,\phi_2}) = O(y, \Psi_{\widetilde{f},\widetilde{\phi_1},\widetilde{\phi_2}}).$$

This proves what we want. □

Corollary 6.2.4. *Suppose that* π *is tempered. Then we can find a test functions* (f', Φ) *so that* $I_{\mathrm{BC}(\pi)}(f', \Phi) \neq 0$ *and* $\Upsilon_{f',\Phi}$ *is supported in the regular semisimple locus.*

Proof. By [Xue16, Appendix C], we can find a test functions (f, ϕ_1, ϕ_2) so that $J_\pi(f, \phi_1, \phi_2) \neq 0$ and Ψ_{f,ϕ_1,ϕ_2} is supported in the regular semisimple locus. Let (f', Φ) be a test functions that matches (f, ϕ_1, ϕ_2). By the support condition of (f, ϕ_1, ϕ_2), we can assume that $\Upsilon_{f',\Phi}$ is supported in the regular seimsimple locus. By Theorem 4.1.2, we have $I_\Pi(f', \Phi) \neq 0$. This proves what we want. □

6.3 The Gan–Gross–Prasad conjectures

We prove our main theorems, i.e. Theorem 1.1.1, 1.1.2 and 4.1.1 in this subsection.

Proof of Theorem 1.1.1. By assumption, there are finite places v_1 and v_2 so that $\mathrm{BC}(\pi_{v_1})$ is supercuspidal and π_{v_2} is tempered.
Proof of (1) implies (2). Assume that $L(\frac{1}{2}, \Pi_1 \times \Pi_2 \otimes \mu^{-1}) \neq 0$. Then the Rankin–Selberg period λ is identically zero. We have shown in [Xue] that under the assumption of the theorem, the Flicker–Rallis period β is not identically zero (cf. also [Zha14a, Theorem 1.4] the remark after it). Then we may choose some test function (f', Φ) so that $I_\Pi(f', \Phi) \neq 0$. By making use of Proposition 2.4.2 and Corollary 6.2.4, we may assume that (f', Φ) is a nice and transferable test function that is transferable, such that $I_\Pi(f', \Phi) \neq 0$.

Let $\{(f^W, \phi_1^W, \phi_2^W)\}$ be a collection of test functions which is the transfer of (f', Φ). However, there is no a priori reason that this collection of test functions should be nice. The only problem is at the place v_1. By assumption, there is a finite number of cuspidal Bernstein components Ω' of $G'(F_{v_1})$ so that $f' \in \mathcal{S}(G(F_{v_1}))_{\Omega'}$. Let Ω^W be the finite collection of cuspidal Bernstein components of $G^W(F_{v_1})$ consists of supercuspidal representation of whose base change lies in Ω'. Let e_{Ω^W} be the idempotent corresponding to Ω^W in the tempered center of $G^W(F_{v_1})$. Let $\{(\widetilde{f^W}, \phi_1^W, \phi_2^W)\}$ be the test function obtained by modifying f^W by replacing $f_{v_1}^W$ by $e_{\Omega^W} * f_{v_1}^W$. Then $\{(\widetilde{f^W}, \phi_1^W, \phi_2^W)\}$ and (f', Φ) still match by Corollary 6.2.3. Moreover $\{(\widetilde{f^W}, \phi_1^W, \phi_2^W)\}$ is a good test function. The second statement of Theorem 1.1.1 then follows from the first one and Proposition 2.4.3.
Proof of (2) implies (1). Now assume that we may find test functions $(f^W, \phi_1^W, \phi_2^W)$ so that $J_{\pi_W}(f^W, \phi_1^W, \phi_2^W) \neq 0$ for some W. We may assume that this test function is of positive type, i.e. $f = f_0 * f_0^*$ where $f_0^*(g) = \overline{f_0(g^{-1})}$ and $\phi_1^W = \phi_2^W = \phi^W$. It follows that if π_1 is an irreducible cuspidal automorphic representation of $G^W(\mathbb{A}_F)$ and is nearly equivalent to π, then $J_{\pi_1}(f^W, \phi^W, \phi^W) \geq 0$. It follows that

$$\sum_{\pi_1} J_{\pi_1}(f^W, \phi^W, \phi^W) > 0,$$

where π_1 runs over all cuspidal automorphic representation of $G^W(\mathbb{A}_F)$ which is nearly equivalent to π. We may modify the test function at v_1, v_2 and infinity so that $(f^W, \phi_1^W, \phi_2^W)$ is a good test function and is transferable. Let (f', Φ) be its smooth transfer. Even though (f', Φ) might not be nice (the problem at v_1), we may modify it at the place v_1 as in the proof (1) $\Rightarrow$ (2) to get a nice test function which still matches $\{f^W, \phi_1^W, \phi_2^W\}$. We then conclude again from Proposition 2.4.3 that $I_\Pi(f', \Phi) \neq 0$. This proves (1). □

Proof of Theorem 1.1.2 and Theorem 4.1.1. As we now assume that π is tempered, on the right hand side of the identity (2.14), all representations are in the same Vogan packet. By assumption,

π is tempered and E/F splits at all archimedean places. In this case, the local Gan–Gross–Prasad conjecture is known by the work of Gan–Ichino [GI] and Beuzart-Plessis [BPb]. Therefore there is at most one π' appearing on the right hand side of (2.14) such that $\mathrm{Hom}_{H(\mathbb{A}_F)}(\pi' \otimes \overline{\omega}, \mathbb{C}) \neq 0$. Suppose that $\pi \neq \pi'$. Then we have $\mathrm{Hom}_{H(\mathbb{A}_F)}(\pi \otimes \overline{\omega}, \mathbb{C}) = 0$ and thus both sides of (4.1) are zero. Suppose that $\pi = \pi'$ and π does not have a nonzero Fourier–Jacobi period. Then we conclude from (2.14) that I_Π is identically zero, which implies that $L(\frac{1}{2}, \Pi_1 \times \Pi_2 \otimes \mu^{-1}) = 0$. Thus again both sides of (4.1) are zero. So we may assume that π has a nonzero Fourier–Jacobi period.

Under this assumption, as argued in the proof of Theorem 1.1.1, we may find nice test functions (f', Φ) and (f, ϕ_1, ϕ_2) so that

$$I_\Pi(f', \Phi) = 4L(1, \eta)^2 J_\pi(f, \phi_1, \phi_2) \neq 0.$$

Then Theorem 4.1.1 follows immediately from Proposition 3.1.1, the decomposition (4.1) and Theorem 4.1.2. As we have already observed in [Xue16, Lemma 1.3.5], this implies Theorem 1.1.2. □

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