# DIAGONAL CYCLES AND THE NONTEMPERED GAN–GROSS–PRASAD CONJECTURES

HANG XUE

Contents



## 1. Introduction

The goal of this paper is to initiate a study of the connection between the nontempered Gan–Gross–Prasad (GGP) conjecture and the Tate conjecture for some product of Shimura varieties. It is largely inspired by a question raised in [RSZ20, Remark 6.16] which asks for the nonvanishing properties of the cohomology class of the diagonal cycles of product of Shimura varieties. The main input is a recent result of Boisseau, Lu and the author [BLX], which confirms the Fourier–Jacobi case of (tempered) GGP conjecture for unitary group. We do not pursue the maximal generality, but only put ourselves in the context where ideas are the most transparent and are not buried in the messy calculations. Extensions to more general situations will be considered in subsequence work.

Generally speaking the philosophy behind this work can be summarized as "*the functoriality produces Tate cycles, and period integrals can be used to detect whether these Tate cycles come from some obvious algebraic cycles*". The pioneering work of Harder, Langlands and Rapoport [HLR86] is the first in this direction. Other more recent works in this direction include the work of Ichino–Prasanna [IP23], Lemma [Lem20], Sweeting [Swe] and others. These works deal with groups of small ranks. We attempt to work with unitary groups of arbitrary size in this paper.

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1.1. **Statement of the main theorem.** Let $F$ be a totally real number field of degree $d$ (assume $d>1$ for simplicity) and $E/F$ a CM extension. Fix an embedding $\iota: E \to \mathbb{C}$. We Let $\mathbb{A}=\mathbb{A}_F$ and $\mathbb{A}_E$ be the ring of adeles of $F$ and $E$ respectively, $\mathbb{A}_{\mathrm{f}}, \mathbb{A}_{E,\mathrm{f}}$ the rings of finite adeles respectively, and $F_\infty = \otimes_{v|\infty} F_v$, $E_\infty = \otimes_{v|\infty} E_v$. Let $\eta = \eta_{E/F}$ the quadratic character of $E^\times \backslash \mathbb{A}_E^\times$ associated to the extension $E/F$. Fix an additive character $\psi = \otimes \psi_v$ of $F\backslash\mathbb{A}$ such that $\psi_v(x) = e^{-2\pi\sqrt{-1}x}$ if $v \mid \infty$.

Let $n=2r$ be an even integer. Write $L$ for the one-dimensional hermitian space $E$ with the hermitian form given by the norm. Let $W$ be a hermitian space over $E$ of dimension $n+1$, and assume that the signatures of $W$ at the archimedean places are

$$(n,1)\times(n+1,0)^{d-1}.$$

The archimedean place where the $W$ is isotropic is $\iota$. Let $V = W + L$. Then the signature of $V$ is

$$(n+1,1)\times(n+2,0)^{d-1}.$$

Put $G = \mathrm{U}(V)$ and $H = \mathrm{U}(W)$, which are algebraic groups over $F$.

Let

$$D_V = \{\text{negative lines in } V\}, \quad D_W = \{\text{negative lines in } W\}$$

be hermitian symmetric domains. Fix open compact subgroups $K_{G,\mathrm{f}}$ and $K_{H,\mathrm{f}}$ of $G(\mathbb{A}_{\mathrm{f}})$ and $H(\mathbb{A}_{\mathrm{f}})$. Then we have the Shimura varieties

$$X = G(F)\backslash(D_V \times G(\mathbb{A}_{\mathrm{f}})/K_{G,\mathrm{f}}), \quad Y = H(F)\backslash(D_W \times H(\mathbb{A}_{\mathrm{f}})/K_{H,\mathrm{f}}).$$

They are projective varieties over $E$ (considered as subfields of $\mathbb{C}$ via $\iota$) of dimension $n$ and $n+1$ respectively. We always assume that $K_{H,\mathrm{f}}$ and $K_{G,\mathrm{f}}$ are sufficiently small, and the image of former in $G(\mathbb{A}_{\mathrm{f}})$ is contained in the later, so we obtain a morphism $Y \to X$, and hence a morphism $Y \to X \times Y$. We are mainly interested in the cycle class of $Y$ in the cohomology of $X \times Y$.

Let $W'$ be the split skew-hermitian space over $E$ of dimension $2r$, and $H' = \mathrm{U}(W')$ the corresponding unitary grouop. Denote by $\mathbf{1}$ the trivial character of $\mathbb{A}_E^\times/E^\times$. Fix a character $\mu = \otimes \mu_v$ of $\mathbb{A}_E^\times/E^\times$ such that $\mu|_{\mathbb{A}_F^\times} = \eta$ and that $\mu_v(z) = z/\sqrt{z\overline{z}}$ if $v \mid \infty$. We will consider theta lifts from $H'$ to $G$ and from $H$ to $H'$. For $H' \times G$, we use the trivial character to split the metaplectic cover. For $H' \times H$ we use the character $\mu$ to split the metaplectic cover. We refer the readers to Section 2 for more details.

The following is the main theorem of this paper, and will be proved in Subsection 5.3.

**Theorem 1.1.** *Let $\pi', \sigma'$ be irreducible cuspidal tempered automorphic representations of $H'(\mathbb{A})$ such that $\pi'_\infty$ and $\sigma'_\infty$ are discrete series representations. Assume that $\pi$ is the theta lift of $\pi'$ and $\sigma^\vee$ is the theta lift of $\sigma'^\vee$ (the notation $-^\vee$ stands for the contragradient), and assume that $\pi$ and $\sigma$ are cohomological (with respect to the trivial representation). Then the cycle class of $Y$ in*

$$\mathrm{H}^{n+1,n+1}(X\times Y)[(\pi_{\mathrm{f}}^\vee)^{K_{G,\mathrm{f}}} \otimes (\sigma_{\mathrm{f}}^\vee)^{K_{H,\mathrm{f}}}]$$

*is not trivial if and only if* $\mathrm{Hom}_{H(\mathbb{A}_\mathrm{f})}(\pi_\mathrm{f}\otimes\sigma_\mathrm{f},\mathbb{C})\neq 0$ *and*

$$\frac{L(s,\pi\otimes\sigma)}{L(s+\frac{1}{2},\pi,\mathrm{Ad})L(s+\frac{1}{2},\sigma,\mathrm{Ad})}\Big|_{s=\frac{1}{2}}\neq 0.$$

*Remark* 1.2. We will show in Lemma 4.3 that

$$\mathrm{Hom}_{H_\infty}(\pi_\infty\,\widehat{\otimes}\,\sigma_\infty,\mathbb{C})\neq 0.$$

*Remark* 1.3. By [GGP20, Theorem 9.7], the $L$-function

$$\frac{L(s,\pi\otimes\sigma)}{L(s+\frac{1}{2},\pi,\mathrm{Ad})L(s+\frac{1}{2},\sigma,\mathrm{Ad})}$$

is holomorphic at $s=\frac{1}{2}$, and its value at $s=\frac{1}{2}$ differs from $L(\frac{1}{2},\pi'\otimes\sigma'\otimes\mu^{-1})$ by a nonzero factor. So we may replace this condition on the $L$-functions by $L(\frac{1}{2},\pi'\otimes\sigma'\otimes\mu^{-1})\neq 0$.

1.2. **What do Arthur, Kottwitz and Tate say?** We now examine the theorem in view of Kottwitz conjecture and Tate conjecture.

We will also make use of Arthur's conjecture on endoscopic classifications of automorphic forms on unitary groups, and in particular Arthur's multiplicity formulae. They are established for unitary groups that are either quasi-split or of (rational) rank at most one, cf. [CZ, Mok15]. All groups we consider in this paper satisfy these conditions.

We follow the convention that a (global) A-parameter (for unitary groups) is a formal sum

$$\bigoplus_d \Pi_d\otimes\mathrm{Sym}^d\,\mathbb{C}^2,$$

where $\Pi_d$ is an irreducible cuspidal automorphic representation of $\mathrm{GL}_{n_d}(\mathbb{A}_E)$ (for some positive integer $n_d$) which is conjugate selfdual of sign $(-1)^{n_d-1}$, and $\mathrm{Sym}^d\,\mathbb{C}^2$ is the $d+1$ dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{C})$. The A-parameters of $\pi$ and $\sigma$ are

$$\Pi=\Pi'\bigoplus(\mathbf{1}\otimes\mathrm{Sym}^1\,\mathbb{C}^2),\quad \Sigma=\mu^{-1}\Sigma'\bigoplus\mathbf{1} \tag{1.1}$$

respectively, where $\Pi'$ and $\Sigma'$ stand for the A-parameter of $\pi$ and $\sigma$ respectively (which are automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$), and $\mathbf{1}$ stands for the trivial representation of $\mathrm{GL}_1(\mathbb{A}_E)$.

We note that $\sigma_\infty$ is a discrete series representation, so it contributes only to the middle cohomology. Moreover Arthur's multiplicity formula implies that $\sigma_\infty$ is the only irreducible representation of $H_\infty$ which makes $\sigma_\mathrm{f}\otimes\sigma_\infty$ automorphic. Computation of the cohomology of $\sigma_\infty^\vee$ gives that

$$\mathrm{H}^*(Y)[(\sigma_\mathrm{f}^\vee)^{K_{H,\mathrm{f}}}]=\mathrm{H}^{r,r}(Y)[(\sigma_\mathrm{f}^\vee)^{K_{H,\mathrm{f}}}]=(\sigma_\mathrm{f}^\vee)^{K_{H,\mathrm{f}}}\otimes\mathrm{H}^{r,r}(\mathfrak{h},K_{H_\infty},\sigma_\infty^\vee).$$

The group $K_{H_\infty}$ is a fixed maximal compact subgroup of $H(F_\infty)$. The Lie algebra cohomology $\mathrm{H}^{r,r}(\mathfrak{h},K_{H_\infty},\sigma_\infty^\vee)$ on the right hand side is one dimensional.

Choose a number field $L$ over which $(\sigma_\mathrm{f}^\vee)^{K_{H,\mathrm{f}}}$ is defined. Let $\mathcal{H}_{H,L}$ be the Hecke algebra of bi-$K_{H,\mathrm{f}}$-invariant functions valued in $L$. Fix a prime $l$ and a place $\lambda$ of $L$ over $l$. Let $\overline{E}$ be the algebraic closure of $E$. Then $\mathrm{H}^{2r}(Y_{\overline{E}},L_\lambda)[(\sigma_\mathrm{f}^\vee)^{K_{H,\mathrm{f}}}]$ admits an action of $\mathcal{H}_{H,L}\times\Gamma_E$. The Shimura varieties $X$ and $Y$ have canonical models over $E$, which we denote again by $X$ and $Y$ (in this subsection only). The conjecture of Kottwitz, cf. [Kot90, Section 10], implies that as a module over $\mathcal{H}_{H,L} \times \Gamma_E$, we have

$$\mathrm{H}^{2r}(Y_{\overline{E}}, L_\lambda)[(\sigma_{\mathrm{f}}^\vee)^{K_{H,\mathrm{f}}}] = (\sigma_{\mathrm{f}}^\vee)^{K_{H,\mathrm{f}}} \boxtimes L_\lambda(-r)$$

where $L_\lambda(-r)$ stands for the Tate twist.

Similar considerations applies to $\pi$ (we may need to enlarge $L$ suitably). Let $\mathcal{H}_{G,L}$ be the Hecke algebra of bi-$K_{G,\mathrm{f}}$ invariant functions valued in $L$. Then as representations of $\mathcal{H}_{G,L} \times \Gamma_E$ we have

$$\mathrm{H}^{2(r+1)}(X_{\overline{E}}, L_\lambda)[(\pi_{\mathrm{f}}^\vee)^{K_{G,\mathrm{f}}}] = (\pi_{\mathrm{f}}^\vee)^{K_{G,\mathrm{f}}} \boxtimes L_\lambda(-r-1).$$

It follows that as representations of $(\mathcal{H}_{G,L} \times \mathcal{H}_{H,L}) \times \Gamma_E$, we have

$$\mathrm{H}^{2(n+1)}(X_{\overline{E}} \times Y_{\overline{E}}, L_\lambda(n+1))[(\pi_{\mathrm{f}}^\vee)^{K_{G,\mathrm{f}}} \otimes (\sigma_{\mathrm{f}}^\vee)^{K_{H,\mathrm{f}}}] = ((\pi_{\mathrm{f}}^\vee)^{K_{G,\mathrm{f}}} \otimes (\sigma_{\mathrm{f}}^\vee)^{K_{H,\mathrm{f}}}) \otimes \mathbf{1}$$

where $\mathbf{1}$ stands for the trivial representation of $\Gamma_E$.

Tate conjecture then predicts that the space $\mathrm{H}^{2(n+1)}(X_{\overline{E}} \times Y_{\overline{E}}, L_\lambda(n+1))[(\pi_{\mathrm{f}}^\vee)^{K_{G,\mathrm{f}}} \otimes (\sigma_{\mathrm{f}}^\vee)^{K_{H,\mathrm{f}}}]$ is generated by the cohomology class of an algebraic cycle. An obvious candidate is of course the cycle class of $Y$. Theorem 1.1 indeed says that this obvious candidate generates the space if and only if the conditions in the theorem holds. In particular our main theorem confirms Tate conjecture in the current context when those conditions hold.

1.3. **What do Gan, Gross and Prasad say?** We now explain the strategy of the proof of Theorem 1.1, and connect it to the nontempered GGP conjecture. Assume that the cycle class of $Y$ in $\mathrm{H}^{n+1,n+1}(X \times Y)[(\pi_{\mathrm{f}}^\vee)^{K_{G,\mathrm{f}}} \otimes (\sigma_{\mathrm{f}}^\vee)^{K_{H,\mathrm{f}}}]$ is not trivial, and we want to prove the central $L$-value is not zero. By Poincare duality, the cycle class of $Y$ being nontrivial means that we can construct an $(n,n)$-form $\alpha$ on $X \times Y$ such that

$$\int_Y \alpha \neq 0.$$

Note that by our assumption $\mathrm{H}^{n,n}(X \times Y)[\pi_{\mathrm{f}}^{K_{G,\mathrm{f}}} \otimes \sigma_{\mathrm{f}}^{K_{H,\mathrm{f}}}] = \mathrm{H}^{r,r}(X)[\pi_{\mathrm{f}}^{K_{G,\mathrm{f}}}] \otimes \mathrm{H}^{r,r}(Y)[\sigma_{\mathrm{f}}^{K_{H,\mathrm{f}}}]$. We construct an $(r,r)$-form on $X$ via Kudla–Millson's theta series valued in differential forms. Indeed one can show that all elements in $\mathrm{H}^{r,r}(X)[\pi_{\mathrm{f}}]$ are of this form. We pick any differential form in $\mathrm{H}^{r,r}(Y)[\sigma_{\mathrm{f}}]$. The resulting integral, after some preliminary reduction steps, reduces to

$$\int_{[H]} \theta_{\phi_{n+1}}^{f'}(h)\varphi(h)\mathrm{d}h, \tag{1.2}$$

where $\varphi \in \sigma$, $f' \in \pi'$ are automorphic forms on $[H]$ and $[H']$ respectively, $\phi_{n+1} \in \mathcal{S}(V(\mathbb{A})^r)$ is a Schwartz function, and $\theta_{\phi_{n+1}}^{f'}$ stands for the theta lift. Our assumption implies that $\theta_{\phi_{n+1}}^{f'}$ is a nontempered (cuspidal) automorphic form on $[G]$, and the integral (1.2) is precisely the one appearing in the global nontempered GGP conjecture, cf. [GGP20, Section 9]. Though we do not state it explicitly as a theorem, computing this integral indeed proves a case of the nontempered GGP conjecture.

The integral (1.2) is computed using the seesaw method, and is reduced to the Fourier–Jacobi period integral for $\pi'\otimes\sigma'$. Its connection to the central $L$-functions is the content of the (tempered) GGP conjecture for $H'\times H'$, and this conjecture is recently proved in [BLX]. This establishes the forward implication. The backward implication can be proved by reversing this argument.

The idea of the proof as outlined above is very simple, and is carried out in Section 5. Originally my goal was to write to a short note elaborating the above arguments. However we need to deal with various technical issues, which makes this paper much longer than what I expected. These issues are taken care of in Sections 2–4. We review theta lifts and doubling zeta integrals in Section 2 and 3 respectively, and establish necessary representation-theoretic results in Section 4. Most of these materials are either well understood or expected by the experts. Experienced readers are recommended to skip these sections on their first reading.

1.4. **Notation and Convention.** Throughout this paper, $E/F$ will stand for either a CM field extension as above, or its completion at a place. In the later case $F$ is either a nonarchimedean local field of characteristic zero, or $F=\mathbb{R}$. In either case we write $|\cdot|$ for the absolute value on $F$ or $\mathbb{A}$, and put $|z|_E=|N_{E/F}z|$ where $z\in\mathbb{A}_E$ or $E$.

Assume $F$ is a number field. For all objects which are products of objects over all places in a suitable sense, e.g. automorphic representations, characters, $L$-functions, adelic groups etc., we write $A=\otimes_v A_v$, $A_{\mathrm{f}}=\otimes_{v\nmid\infty}A_v$ and $A_\infty=\otimes_{v|\infty}A_v$.

Assume that $F$ is a local field of characteristic zero and $V$ a finite dimensional vector space over $F$. We denote by $\mathcal{S}(V)$ the space of Schwartz functions on $V$. If $F=\mathbb{R}$ these are the usual Schwartz functions on $V$. If $F$ is nonarchimedean these are locally constant functions on $V$ with compact support.

Let $G$ be an algebraic group over $F$. If $F$ is a number field we put $[G]=G(F)\backslash G(\mathbb{A})$, and if $v$ is a place we write $G_v=G(F_v)$. If $F$ is a local field of characteristic zero, we simply write $G$ for its group of $F$-points $G(F)$. When $F$ is a nonarchimedean local field of characteristic zero, a representation of $G$ means a smooth representation. When $F=\mathbb{R}$, a representation of $G$ means a smooth Frechet representation of moderate growth.

If $F=\mathbb{R}$ and $G$ an algebraic group over $F$, we often use the lower case gothic letter $\mathfrak{g}$ to denote its Lie algebra. Let $K$ be a maximal compact subgroup of $G$. We also work with $(\mathfrak{g},K)$-modules. A Harish-Chandra module is an admissible $(\mathfrak{g},K)$-module of finite length. The notation extends to the case when $F$ is a number field, in which case $\mathfrak{g}$ stands for the Lie algebra of $G_\infty$.

We almost work exclusively with unitarizable representations in this manuscript. Whenever we have a unitarizable representation we denote by $\langle -,-\rangle$ a fixed (hermitian) inner product on it. The convention is that it is linear in the first variable and anti-linear in the second variable.

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## 2. Weil representations and Theta lifts

We fix some notation and conventions on Weil representations and theta lifts in this section. We do not pursue generality of the setup but only work with what we need.

2.1. **Weil representations.** Let us first consider the local situation. We fix a place $v$ of $F$ and suppress from all notation. Then $F$ stands for a local field of characteristic zero. Recall that we fixed a nontrivial additive character $\psi$ of $F$. If $F = \mathbb{R}$, then our $\psi$ agrees with the choices made in [Pau98, Pau00]. It also agrees with the choice made in [Ato20], cf. [Ato20, p. 32].

Let $r$ be an integer and $W'$ be a split skew-hermitian space of dimension $2r$. Let $V$ be a hermitian space of dimension $n$. We fix a character $\chi : E^\times \to \mathbb{C}^\times$ such that $\chi|_{F^\times} = \eta^n$ and use $\chi$ to split the metaplectic cover over $\mathrm{U}(W') \times \mathrm{U}(V)$, cf. [Kud94]. There is a Weil representation of $\mathrm{U}(W') \times \mathrm{U}(V)$ which is realized on $\mathcal{S}(V^r)$. The explicit formulae for the Weil representation can be found in many references, e.g. *loc. cit.*. This realization of the what is usually referred to as the Schrodinger model in the literature. We denote the Weil representation often by $\omega_V$. Other relevant data are omitted from the notation as they are all fixed throughout our discussions.

2.2. **The Fock model.** We now assume that $F = \mathbb{R}$. We recall some basics about the Fock models for the (infinitesimal) Weil representation of $\mathrm{U}(W') \times \mathrm{U}(V)$. Many calculations later in this paper will be done in this model. A rather detailed description of the Fock model is given in [FH21, Appendix]. We just summarize what we need.

Assume that the signature of $V$ is $(p, q)$, and $p + q = n$. We choose basis $w_1, \ldots, w_{2r}$ of $W'$ and $v_1, \ldots, v_n$ of $V$, such that the skew-hermitian form and the hermitian form respectively are represented by the matrices

$$\begin{pmatrix} \sqrt{-1}1_r & \\ & -\sqrt{-1}1_r \end{pmatrix}, \quad \begin{pmatrix} 1_p & \\ & -1_q \end{pmatrix}$$

Let $W'_{\mathbb{C}} = W' \otimes_{\mathbb{R}} \mathbb{C}$, and $\mathbb{W} = V \otimes_{\mathbb{C}} W'_{\mathbb{C}}$. Define elements in $W'_{\mathbb{C}}$

$$w'_i = w_i \otimes 1 + \sqrt{-1} w_i \otimes \sqrt{-1}, \quad w''_i = w_i \otimes 1 - \sqrt{-1} w_i \otimes \sqrt{-1}, \quad 1 \leq i \leq r,$$

and (note the sign change)

$$w'_i = w_i \otimes 1 - \sqrt{-1} w_i \otimes \sqrt{-1}, \quad w''_i = w_i \otimes 1 + \sqrt{-1} w_i \otimes \sqrt{-1}, \quad r + 1 \leq i \leq 2r.$$

Let $\mathbb{W}''$ be the subspace of $\mathbb{W}$ spanned by

$$v_a \otimes w''_j, \quad 1 \leq a \leq p, \ 1 \leq j \leq 2r$$

and

$$v_a \otimes w'_j, \quad p + 1 \leq a \leq n, \ 1 \leq j \leq 2r.$$

The Fock model of the (infinitesimal) Weil representation, is the space of polynomials in $\mathbb{W}''$, on which the Lie algebra $\mathfrak{u}(W')_{\mathbb{C}} \times \mathfrak{u}(V)_{\mathbb{C}}$ acts. We denote by $Z = (z_{aj})_{1\leq a\leq n, 1\leq j\leq 2r}$ the $n \times 2r$ matrix of variables and make the partition

$$Z = \begin{pmatrix} Z^{++} & Z^{+-} \\ Z^{-+} & Z^{--} \end{pmatrix}$$

where $Z^{++}$ is of size $p \times r$. The action of $\mathfrak{u}(W')_{\mathbb{C}} \times \mathfrak{u}(V)_{\mathbb{C}}$ is given by some explicit formulae which are documented in [FH21, Lemma B.1, B.2].

We denote by $K \simeq \mathrm{U}(p) \times \mathrm{U}(q)$ and $K' \simeq \mathrm{U}(r) \times \mathrm{U}(r)$ maximal compact subgroups of $\mathrm{U}(V)$ and $\mathrm{U}(W')$ respectively. The action of $\mathfrak{k}'_{\mathbb{C}} \times \mathfrak{k}_{\mathbb{C}}$ exponentiates to an action of $\widetilde{K'} \times K$ where $\widetilde{K'}$ stands for a double covers of $K'$. The Fock model is a representation of

$$(\mathfrak{u}(W')_{\mathbb{C}}, \widetilde{K'}) \times (\mathfrak{u}(V)_{\mathbb{C}}, K).$$

Note that there is no cover of $K$ because of our particular situation that $W'$ is a split skew-hermitian space of dimension $2r$. The double cover of $K'$ can be described as follows. Let $\nu$ be an integer with the same parity as $p - q$, then a $\det^{\frac{\nu}{2}}$-cover of $\mathrm{U}(W')$ stands for

$$\widetilde{\mathrm{U}(W')} = \{(g, z) \in \mathrm{U}(W') \times \mathbb{C}^\times \mid (\det g)^\nu = z^2\},$$

and $\widetilde{K'}$ stands for the subgroup where $g \in K'$. There is a genuine character

$$\det{}^{\frac{\nu}{2}} : \widetilde{\mathrm{U}(W')} \to \mathbb{C}^\times, \quad \det{}^{\frac{\nu}{2}}(g, z) = z.$$

Different choices of $\nu$ give isomorphic groups. If $p - q$ (hence $\nu$) is even, then $\widetilde{\mathrm{U}(W')}$ is isomorphic to $\mathrm{U}(W') \times \{\pm 1\}$.

The action of $\widetilde{K'}$ is given by

$$((k'_1, k'_2), \epsilon) \cdot p(Z) = \epsilon^{-1} (\det k'_1)^{\frac{p-q+\nu}{2}} (\det k'_2)^{-\frac{p-q-\nu}{2}} p \begin{pmatrix} Z^{++} k'_1 & Z^{+-} \overline{k'_2} \\ Z^{-+} \overline{k'_1} & Z^{--} k'_2 \end{pmatrix}.$$

Note that the derivative of this action is independent of the choice $\nu$ (the formulae given in [FH21, Lemma B.2]). The action of $K$ is

$$(k_1, k_2) \cdot p(Z) = p \begin{pmatrix} k_1^{-1} Z^{++} & \overline{k_1}^{-1} Z^{+-} \\ \overline{k_2}^{-1} Z^{-+} & k_2^{-1} Z^{--} \end{pmatrix}.$$

Note that there is no cover over $K$ because of our particular situation that $W'$ is split of dimension $2r$.

We have a natural homomorphism

$$\mathrm{U}(W') \times \mathrm{U}(V) \to \mathrm{Sp}(4rn).$$

Let $\chi : \mathbb{C}^\times \to \mathbb{C}^\times$ be the character given by $\chi(z) = \left(z/\sqrt{z\overline{z}}\right)^\nu$. One the one hand, this character gives rise to a splitting

$$\alpha_\chi : \mathrm{U}(W') \to \mathrm{Mp}(4rn)$$

where $\mathrm{Mp}(4rn)$ is the $\mathbb{C}^1$-metaplectic covering of $\mathrm{Sp}(4rn)$. On the other hand, we have a homomorphism

$$\widetilde{\alpha}_{W'} : \widetilde{\mathrm{U}(W')} \to \widetilde{\mathrm{Sp}}(4rn) \to \mathrm{Mp}(4rn),$$

where the image of $\widetilde{\mathrm{U}(W')}$ in $\widetilde{\mathrm{Sp}}(4rn)$ is the inverse image of $\mathrm{U}(W')$ in $\widetilde{\mathrm{Sp}}(4rn)$ and $\widetilde{\mathrm{Sp}}(4rn) \to \mathrm{Mp}(4rn)$ is the natural map. Let $\Omega$ be the oscillator representation of $\mathrm{Mp}(4rn)$. By [Ato20, Proposition 3.2] we have

$$\Omega \circ \widetilde{\alpha}_{W'} = (\Omega \circ \alpha_\chi) \otimes \det{}^{-\frac{\nu}{2}}.$$

So once we choose the character $\chi$ and hence the splitting, the group $K'$ acts by

$$(k_1', k_2') \cdot p(Z) = (\det k_1')^{\frac{p-q+\nu}{2}} (\det k_2')^{-\frac{p-q-\nu}{2}} p \begin{pmatrix} Z^{++} k_1' & Z^{+-} \overline{k_2'} \\ Z^{-+} \overline{k_1'} & Z^{--} k_2' \end{pmatrix}.$$

For the rest of this paper, by the Weil representation $\omega_V$ (or its Fock model), we always mean the representation of the unitary group (or the relevant $(\mathfrak{g}, K)$ module) via the splitting given by a fixed character $\mu$, not the double cover.

We now come to the (probably) more familiar Schrodinger model $\mathcal{S}(V^r)$ of the Weil representation. The coordinates on $V^r$ are given by a matrix $X = (x_{i,a})$, $1 \le i \le n$ and $1 \le a \le r$. The Gaussian function on $V^r$ is

$$\varphi_0(X) = e^{-\pi \operatorname{Tr} {}^t\overline{X} X}.$$

Let $\mathcal{S}(V^r)^\dagger \subset \mathcal{S}(V^r)$ be the subspace consisting of functions of the form

$$p(X, \overline{X}) \varphi_0(X),$$

where $p$ is a polynomial function. The $K$ action is given simply by

$$k \cdot (p\varphi_0)(X) = p(k^{-1}X, \overline{k^{-1}X}) \varphi_0(X).$$

The $K'$ action is given by

$$(k_1', k_2') \cdot (p\varphi_0)(X) = (\det k_1')^{\frac{p-q+\nu}{2}} (\det k_2')^{-\frac{p-q-\nu}{2}} p(X k_1', \overline{X k_2'}) \varphi_0(X).$$

There is an intertwining map between the Fock model and the Schrodinger model. Then there is an isomorphism $\beta : \mathcal{S}(V^r)^\dagger \to \mathcal{P}$, which sends $\varphi_0$ to 1, and satisfies the following relations. To simplify notation, we write $\partial_{a,j}$ for derivative with respect to $x_{a,j}$ and $\overline{\partial}_{a,j}$ for derivative with respect to $\overline{x_{a,j}}$.

$$\begin{aligned}
&\beta(x_{a,j} - \frac{1}{\pi}\overline{\partial}_{a,j})\beta^{-1} = -\frac{\sqrt{-1}}{\sqrt{2\pi}} z_{a,j}, \quad 1 \le a \le p, 1 \le j \le r \\
&\beta(\overline{x_{a,j}} - \frac{1}{\pi}\partial_{a,j})\beta^{-1} = -\frac{\sqrt{-1}}{\sqrt{2\pi}} z_{a,j}, \quad 1 \le a \le p, r+1 \le j \le 2r \\
&\beta(x_{a,j} - \frac{1}{\pi}\overline{\partial}_{a,j})\beta^{-1} = \frac{\sqrt{-1}}{\sqrt{2\pi}} z_{a,j}, \quad p+1 \le a \le n, 1 \le j \le r \\
&\beta(\overline{x_{a,j}} - \frac{1}{\pi}\partial_{a,j})\beta^{-1} = \frac{\sqrt{-1}}{\sqrt{2\pi}} z_{a,j}, \quad p+1 \le a \le n, r+1 \le j \le 2r
\end{aligned} \tag{2.1}$$

The Fock model of the dual of $\omega_V^\vee$, is given by the same space $\mathbb{C}[Z]$, with the same partition, and the character $\mu^{-1}$ is used to split the metaplectic cover. With this choice, the actions differ from $\omega_V$ by several signs, and in particular the group $K'$ acts on 1 by the character

$$(k_1', k_2') \mapsto (\det k_1')^{-\frac{p-q+\nu}{2}} (\det k_2')^{\frac{p-q-\nu}{2}}.$$

In terms of the isomorphism (2.1), the right hand sides of the first two relations have the plus sign, while the last two have the minus sign. Let us note that the complex conjugation $\phi \mapsto \overline{\phi}$ is a anti-$\mathbb{C}$-linear automorphism on $\mathcal{S}(V^r)$ (hence $\mathcal{S}(V^r)^\dagger$) that identifies $\omega_V^\vee$ with $\overline{\omega_V}$. Under the isomorphism $\beta$, it has the effect of swapping $z_{a,j}$ and $-z_{a,j+r}$, i.e. if $1 \le j \le r$, then

$$\overline{\beta(z_{a,j})} = -\beta(z_{a,j+r}), \quad \overline{\beta(z_{a,j+r})} = -\beta(z_{a,j}).$$

So we may define $\overline{z_{a,j}} = -z_{a,j+r}$ and $\overline{z_{a,j+r}} = -z_{a,j+r}$ if $1 \le j \le r$. The map $z_{j,a} \mapsto \overline{z_{j,a}}$ then gives an identification of $\omega_V^\vee$ with $\overline{\omega_V}$ for the Fock models. This notation is more intuitive and convenient for use: after all $z_{a,j}$ and $z_{a,j+r}$ (when $1 \le a \le p$) stand for the differential operators

$$\sqrt{-2}\pi(x_{a,j} - \frac{1}{\pi}\overline{\partial}_{a,j}), \quad \sqrt{-2}\pi(\overline{x_{a,j}} - \frac{1}{\pi}\partial_{a,j}).$$

2.3. **Theta lifts.** We start from the local situation, so we assume that $F$ is a local field of characteristic zero. Let $\pi'$ be an irreducible representation of $\mathrm{U}(W')$. We let $\Theta_V(\pi')$ be the maximal $\mathrm{U}(W')$-invariant quotient of $\omega \otimes \pi^\vee$ (Hausdorff quotient if $F = \mathbb{R}$). This is a finite length representation of $\mathrm{U}(W')$, and is called the full or big theta lift of $\pi'$ to $\mathrm{U}(V)$. Its maximal semisimple quotient $\theta_V(\pi)$ is irreducible, and is called the theta lift of $\pi$ (from $\mathrm{U}(W')$ to $\mathrm{U}(V)$), cf. [How89a, GT16]. The space $W'$ will be always fixed throughout this paper. We will drop the subscript $V$ when it is clear from the context. Of course the roles of $V$ and $W'$ can be switched, and we have the theta lifts from $\mathrm{U}(V)$ to $\mathrm{U}(W')$. We still use the notation $\Theta_V$ and $\theta_V$ to denote this theta lifts.

In many questions, we often need to know if $\Theta(\pi)$ is irreducible. This question is not completely settled at the moment. We only have some partial information.

**Lemma 2.1.** *We have the following assertions.*

(1) *If $F$ is nonarchimedean, $n - 2r = 0, \pm 1$, and $\pi$ is an irreducible tempered representation of $\mathrm{U}(V)$. Then $\Theta(\pi)$ is irreducible.*
(2) *If $F = \mathbb{R}$ and at least one of $\mathrm{U}(V)$ and $\mathrm{U}(W')$ is compact. Let $\pi$ be an irreducible representation of either $\mathrm{U}(V)$ or $\mathrm{U}(W')$. Then $\Theta(\pi)$ is irreducible.*
(3) *If $F = \mathbb{R}$, $n = 2r + 2$, $\pi'$ is a discrete series representation of $\mathrm{U}(W')$, and the signature of $V$ is $(n-1, 1)$, then $\Theta(\pi')$ is irreducible.*

In the last assertion we consider theta lifts from $\mathrm{U}(W')$ to $\mathrm{U}(V)$, just to make the notation consistent with what we need later.

*Proof.* The first assertion is [GI14, Appendix C]. Note that the "odd residue characteristic" assumption in [GI14] is only included there for the validity of Howe's duality conjecture, which is completely established now. The second assertion is due to Howe [How89b]. The last assertion requires a little more explanation and relies on the theory of doubling zeta integral which we recall in the next section. The proof of this last assertion will be given in the appendix. □

We now consider global Weil representations and theta series. Assume that $F$ is a number field. By taking tensor products of Weil representations of $\mathrm{U}(W')(F_v)\times\mathrm{U}(V)(F_v)$ at all places, we obtain a Weil representation $\omega=\otimes\omega_v$ of $\mathrm{U}(V)(\mathbb{A})\times\mathrm{U}(V')(\mathbb{A})$, realized on $\mathcal{S}(V(\mathbb{A})^r)$. Let $\phi\in\mathcal{S}(V(\mathbb{A})^r)$ we define the theta function on $\mathrm{U}(W')(\mathbb{A})\times\mathrm{U}(V)(\mathbb{A})$ by

$$\theta(h',g,\phi)=\sum_{x\in V(F)^r}\omega(h',g)\phi(x).$$

Let $\pi'$ be an irreducible cuspidal automorphic representation of $\mathrm{U}(W')(\mathbb{A})$. Its global theta lift to $\mathrm{U}(V)(\mathbb{A})$ is the automorphic representation of $\mathrm{U}(V)(\mathbb{A})$ generated by the functions of the form

$$\theta_\phi^{f'}(g)=\int_{[\mathrm{U}(W')]}\overline{f'(h')}\theta(h',g,\phi)\mathrm{d}h',\quad f'\in\pi',\quad \phi\in\mathcal{S}(V(\mathbb{A})^r).$$

We denote it by $\theta(\pi')$.

**Lemma 2.2.** *If $\theta(\pi')$ consists of square integrable automorphic forms, e.g. cuspidal automorphic forms, then it is irreducible and*

$$\theta(\pi')=\otimes_v\theta(\pi'_v).$$

*Proof.* This is a consequence of the Howe duality, cf. [KR94, Corollary 7.1.3]. □

## 3. Doublinig zeta integrals

3.1. **The setup.** Assume first that $F$ is a number field. We keep the notation from the previous section. In particular $W'$ is a split skew-hermitian space of dimension $2r$ and $V$ is a hermitian space of dimension $n$. To simplify notation we put $H'=\mathrm{U}(W')$ and $G=\mathrm{U}(V)$. Let $-W'$ be the skew-hermitian space with the vector space $W'$ and skew-hermitian form $-1$ times that of $W'$, and $W'^{\square}=W'+(-W')$ the "doubled" space. Put $H'^{\square}=\mathrm{U}(W'^{\square})$ and we have an embedding

$$i: H'\times H'\to H'^{\square}. \tag{3.1}$$

Here we identify $\mathrm{U}(-W')$ with $H'$ via the identity map (they are physically the same subgroup of $\mathrm{GL}(W')$).

Let $W'^{\triangle}=\{(x,x)\mid x\in W'\}$ be the diagonal subspace of $W'^{\square}$, and $P'=M'N'$ be the parabolic subgroup of $H'^{\square}$ stabilizing this subspace. Recall that we have fixed a character $\mu$. Let $I(s,\mu)$ be the degenerate principal series representation of $H'^{\square}(\mathbb{A})$. More precisely, choose a basis such that the skew-hermitian form of $H'^{\square}$ is given by $\begin{pmatrix} & 1_{2r}\\ -1_{2r} & \end{pmatrix}$, and $P'$ consists of (blocked) upper triangular matrices. The space of $I(s,\mu)$ consists of functions on $H'^{\square}(\mathbb{A})$ with the property that

$$\mathcal{F}_s\left(\begin{pmatrix} a & *\\ & {}^t\overline{a}^{-1}\end{pmatrix}g\right)=\mu(\det a)|\det a|_E^{s+r}\mathcal{F}_s(g),\quad \begin{pmatrix} a & *\\ & {}^t\overline{a}^{-1}\end{pmatrix}\in P'(\mathbb{A}_E),$$

and the group $H'^{\square}$ acts by right translation. Put $s_0 = \frac{n-2r}{2}$. Let $\omega^{\square}$ be the Weil representation for $H'^{\square}(\mathbb{A}) \times G(\mathbb{A})$ (recall that the character $\mu$ is used to split the metaplectic cover), and $\phi^{\square} \in \mathcal{S}(V^{2r}(\mathbb{A}))$ be a Schwartz function. Then

$$h'^{\square} \mapsto \mathcal{F}_{s_0}^{\phi^{\square}}(h'^{\square}) = \omega^{\square}(h')\phi^{\square}(0)$$

defines a section of $I(s_0, \mu)$, which we call the Siegel–Weil section. It defines a $H'^{\square} \times G$-equivariant map

$$\mathcal{S}(V^{2r}(\mathbb{A})) \to I(s_0, \mu)$$

where $G$ acts on the target trivially.

Restricting $\omega^{\square}$ to the group $H' \times H'$ via the embedding (3.1) we have an isomorphism

$$\omega_V \otimes \overline{\omega_V}\mu \to \omega^{\square}|_{H' \times H'}, \quad \phi_1 \otimes \phi_2 \mapsto (\phi_1 \otimes \overline{\phi_2})^{\dagger}. \tag{3.2}$$

The group $G$ acts on the left hand side diagonally. In terms of the Schrodinger model, this is given by a partial Fourier transform, and has the property that

$$\omega^{\square}(i(h', 1))(\phi_1 \otimes \overline{\phi_2})^{\dagger}(0) = \langle \omega_V(h')\phi_1, \phi_2 \rangle,$$

where $\langle -, - \rangle$ stands for the $L^2$-inner product on $\mathcal{S}(V^r(\mathbb{A}))$.

Let $\mathcal{F}_s$ be a section of $I(s, \mu)$, and $E(h'^{\square}, \mathcal{F}_s)$ the usual Siegel Eisenstein series. Let $\pi'$ be an irreducible cuspidal automorphic representation of $H'(\mathbb{A})$, and $f'_1, f'_2 \in \pi'$. The doubling zeta integral is defined to be

$$Z(f'_1, f'_2, \mathcal{F}_s) = \int_{H'(\mathbb{A}) \times H'(\mathbb{A})} \overline{f'_1}(h'_1) f'_2(h'_2) E(i(h'_1, h'_2), \mathcal{F}_s) \mathrm{d}h'_1 \mathrm{d}h'_2.$$

When $f'_1 = \otimes f'_{1,v}$, $f'_2 = \otimes f'_{2,v}$ and $\mathcal{F}_s = \otimes \mathcal{F}_{v,s}$, we have

$$Z(f'_1, f'_2, \mathcal{F}_s = L(s, \pi \times \mu) \prod Z_v^{\natural}(f'_{1,v}, f'_{2,v}, \mathcal{F}_{v,s}),$$

where $Z_v^{\natural}(f'_{1,v}, f'_{2,v}, \mathcal{F}_{v,s}) = d_v(s) L(s, \pi_v \otimes \mu_v)^{-1} Z_v(f'_{1,v}, f'_{2,v}, \mathcal{F}_{v,s})$, where $d_v$ is a product of abelian local $L$-factors that satisfies $d_v(s_0) \neq 0$, and $Z_v(f'_{1,v}, f'_{2,v}, \mathcal{F}_s)$ is the local doubling zeta integral given by

$$\int_{H'_v} \overline{\langle \pi'_v(h') f'_{1,v}, f'_{2,v} \rangle} \mathcal{F}_{v,s}(i(h, 1)) \mathrm{d}h.$$

We are particularly interested in the case where $\mathcal{F}_{v,s}$ is a Siegel–Weil section at $s = s_0$. Let $\phi_1 = \otimes \phi_{1,v}, \phi_2 = \otimes \phi_{2,v} \in \mathcal{S}(V^r(\mathbb{A}))$ be Schwartz functions, then the local zeta integral equals

$$Z_v(f'_{1,v}, f'_{2,v}, \mathcal{F}_{v,s_0}^{(\phi_{1,v} \otimes \overline{\phi_{2,v}})^{\dagger}}) = \int_{H'_v} \overline{\langle \pi'_v(h') f'_{1,v}, f'_{2,v} \rangle} \langle \omega_V(h') \phi_{1,v}, \phi_{2,v} \rangle \mathrm{d}h',$$

whenever the integral is convergent. This is the case when $\pi'_v$ is square-integrable and $s_0 \geq -\frac{1}{2}$, i.e. $n \geq 2r - 1$, cf. [LR05, Lemma 2.1].

We denote this last integral also by $Z_v(f'_{1,v}, f'_{2,v}, \phi_{1,v}, \phi_{2,v})$. It is not identically zero when $\theta_V(\pi_v) \neq 0$, at least when $\pi_v$ is tempered and $n \geq 2r$, cf. [GQT14, Proposition 11.5] ($v$ nonarchimedean) and [Ich22, Proposition 7.1] ($v$ archimedean).

The following lemma will be useful later.

**Lemma 3.1.** *Assume that $v$ is archimedean, $n \geq 2r$, $\pi_v$ is tempered, and $\theta_V(\pi_v) \neq 0$. Fix an $H'_v \times G_v$-equivariant linear map $p : \pi_v'^{\vee} \otimes \omega_{V,v} \to \theta_V(\pi'_v)$. Then there is an inner product on $\theta_V(\pi'_v)$ such that*

$$Z_v(f'_{1,v}, f'_{2,v}, \phi_{1,v}, \phi_{2,v}) = \left\langle p(\overline{f'_{1,v}} \otimes \phi_{1,v}), p(\overline{f'_{2,v}} \otimes \phi_{2,v}) \right\rangle.$$

*Proof.* Note that this defines a $H'_v \times H'_v$-invariant linear form linear form

$$Z_v : \overline{\pi'_v} \otimes \pi'_v \otimes \omega_{V,v} \otimes \overline{\omega_{V,v}} \to \mathbb{C}.$$

Therefore it factor through $\Theta_V(\pi'_v) \otimes \overline{\Theta_V(\pi'_v)}$ and gives a nonzero hermitian form on $\Theta_V(\pi'_v)$. Moreover it is semi-positive definite, cf. [He03], and hence defines an inner product on $\Theta_{V',V}(\pi')/K$ where $K$ is the kernel of the hermitian form. Therefore $\Theta_V(\pi')/K$ must be semisimple, and thus coincides with the $\theta_V(\pi')$. □

*Remark* 3.2. The lemma should hold when $v$ is nonarchimedean. I however am not able to find a reference which proves the semipositivity of $Z_v$.

3.2. **Some calculations at the archimedean places.** We study the doubling zeta integrals at the archimedean places. Results in this subsection should be well-known to the experts. For the lack of suitable references, we provide some details.

In this subsection, we fix an archimedean place $v$, and suppress it from all notation. In other words, we assume $F = \mathbb{R}$.

Recall that we have the group $H' = \mathrm{U}(W')$, its Weil representation $\omega_V$ and $\omega_V^{\vee}$, and the "doubled" group $H'^{\square}$, and its Weil representation $\omega^{\square}$. We have maximal compact subgroups $K' \simeq \mathrm{U}(r) \times \mathrm{U}(r)$ of $H'$ and $K'^{\square} \simeq \mathrm{U}(2r) \times \mathrm{U}(2r)$ of $H'^{\square}$ respectively. Let us begin by examine the isomorphism

$$\omega_V \otimes \overline{\omega_V}\mu \to \omega^{\square}|_{H' \times H'}. \tag{3.3}$$

In terms of the Schrodinger model, this is given by a partial Fourier transform as in (3.2). In terms of the Fock model, this is just the "concatenation" of the polynomials. More precisely, as explained in Subsection 2.2, the Fock model for $\omega_V$ is denoted by $\mathcal{P} = \mathbb{C}[Z]$ and the for $\omega_V^{\vee}$ is denoted by $\mathcal{P}^{\vee} = \mathbb{C}[W]$, where $Z$ and $W$ are $n \times 2r$ matrices of variables. Both $Z$ and $W$ take the partition

$$Z = \begin{pmatrix} Z^{++} & Z^{+-} \\ Z^{-+} & Z^{--} \end{pmatrix}, \quad W = \begin{pmatrix} W^{++} & W^{+-} \\ W^{-+} & W^{--} \end{pmatrix}.$$

Then the Fock model $\mathcal{P}^{\square}$ of $\omega^{\square}$ is given by $\mathbb{C}[U]$ where $U$ is an $n \times 4r$ matrices, and

$$U = \begin{pmatrix} U^{++} & U^{+-} \\ U^{-+} & U^{--} \end{pmatrix} = \begin{pmatrix} Z^{++} & W^{+-} & Z^{+-} & W^{++} \\ Z^{-+} & W^{--} & Z^{--} & W^{-+} \end{pmatrix},$$

and $U^{++} = (Z^{++}, W^{+-})$, $U^{+-} = (Z^{+-}, W^{++})$, $U^{-+} = (Z^{-+}, W^{--})$ and $U^{--} = (Z^{--}, W^{-+})$. If $p(Z)$ and $q(W)$ are polynomials in $\mathcal{P}$ and $\mathcal{P}^{\vee}$ respectively, then $p(Z)q(W)$ is the element in $\mathcal{P}^{\square}$ under the isomorphism (3.3).

Let $\mathcal{F}_s^\circ$ be the classical section $I(s,\mu)$, i.e.

$$\mathcal{F}_s^\circ(h) = (\det h)^{\frac{n+\nu}{2}} \det(C\sqrt{-1}+D)^{-n} |\det(C\sqrt{-1}+D)|_{\mathbb{C}}^{-s+\frac{n-2r}{2}}, \quad h = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

If $V$ is anisotropic and $\phi^\square = 1 \in \mathcal{P}^\square$, i.e. $\phi^\square$ is the standard Gaussian in the Schrodinger model $\mathcal{S}(V^n)$, then the Siegel–Weil section $\mathcal{F}_s^{\phi^\square}$ equals a nonzero constant multiple of the classical section.

Let $I(s,\mu)_{K'^\square}$ be the subspace of $K'^\square$-finite vectors. By [Lee94, Lemma 2.5], as representations of $K'^\square$ we have

$$I(s,\mu)_{K'^\square} \simeq \bigoplus_\rho \rho \boxtimes \rho^\vee \mu.$$

where $\rho$ ranges over all irreducible representations of $\mathrm{U}(2r)$. Then the classical section $\mathcal{F}_s^\circ$ lies in the direct summand $\det^{\frac{n+\nu}{2}} \boxtimes \det^{-\frac{n-\nu}{2}}$, i.e. $\rho = \det^{\frac{n+\nu}{2}}$.

In what follows, we calculate in the Fock model $\mathcal{P}$. Recall that there is an isomorphism $\beta : \mathcal{S}(V^r)^\dagger \to \mathcal{P}$. To simplify notation, we suppress $\beta$ from all notation. This means that an element $z \in \mathcal{P}$ is viewed as a Schwartz function in $\mathcal{S}(V^r)$ via the isomorphism $\beta$.

**Lemma 3.3.** *Assume that $n = 2r+1$, $\nu = 1$ and $V$ is of signature $(2r,1)$. Assume that $\sigma'$ is a discrete series representation of lowest $K'$-type $\tau' = \det^{r+1} \boxtimes \det^{-r}$. Let $\mathcal{P}_0 \subset \mathcal{P}$ be the subspace generated by the polynomials of the form*

$$\det Z_I^{++} \det Z_J^{+-}, \quad I,J \subset \{1,\dots,2r\}, \quad \#I = \#J = r.$$

*Then the doubling zeta integral is nonzero when restricted to $\mathcal{P}_0 \otimes \mathcal{P}_0^\vee \otimes \tau'^\vee \otimes \tau'$.*

*Proof.* Consider the element $\phi^\square \in \mathcal{P}^\square$ given by

$$\phi^\square = \det U^{++} \det U^{+-} \in \mathcal{P}^\square.$$

Recall that $U^{++} = (Z^{++}, W^{+-})$ and $U^{+-} = (Z^{+-}, W^{++})$. So Laplace expansion of determinants tells us that

$$\phi^\square = \sum_{I,J} (-1)^{\|I\|+\|J\|} \det Z_I^{++} \det W_{I^c}^{+-} \det Z_J^{+-} \det W_{J^c}^{++}$$

where $I,J$ are order $r$ subsets of $\{1,\cdots,2r\}$, $I^c, J^c$ stand for their complements respectively, and $\|I\|$, $\|J\|$ are the sums of elements in $I$ and $J$ respectively. So if $h' \in H'$, then

$$\mathcal{F}_{s_0}^{\phi^\square}(\iota(h',1)) = \sum_{I,J} (-1)^{\|I\|+\|J\|} \langle \omega(h') \det Z_I^{++} \det Z_J^{+-}, \det W_{J^c}^{++} \det W_{I^c}^{+-} \rangle. \tag{3.4}$$

Note that $\det Z_I^{++} \det Z_J^{+-}$ and $\det W_{J^c}^{++} \det W_{I^c}^{+-}$ belong to $\mathcal{P}_0$. To see that $\mathcal{F}_{s_0}^{\phi^\square}$ is not a zero section, we compute $\mathcal{F}_{s_0}^{\phi^\square}(1)$, which equals

$$\sum_{I,J} (-1)^{\|I\|+\|J\|} \langle \det Z_I^{++} \det Z_J^{+-}, \det W_{J^c}^{++} \det W_{I^c}^{+-} \rangle.$$

In the pairing, $\det Z_I^{++}$ pairs with $\det W_{J^c}^{++}$, and $\det Z_J^{+-}$ with $\det W_{I^c}^{+-}$. It follows that the terms with $I \neq J^c$ are all zero. So the sum simplifies to

$$(-1)^{1+\cdots+2r}\sum_I \langle \det Z_I^{++} \det Z_{I^c}^{+-}, \det W_I^{++} \det W_{I^c}^{+-}\rangle,$$

which is a sum of finitely many positive real numbers (up to the sign in the front), and hence nonzero.

By its very construction, $K'$ acts on $\phi^\square$ by the character $\det^{r+1} \boxtimes \det^{-r}$. So $\mathcal{F}_{s_0}^{\phi^\square}$ is a nonzero multiple of the classical section. It is well-known that $Z(f', f', \mathcal{F}_{s_0}^\circ) \neq 0$, cf. [Gar08]. By (3.4), $\mathcal{F}_{s_0}^{\phi^\square}(\iota(h', 1))$ is a (sum of) matrix coefficient(s) in $\mathcal{P}_0$. The lemma follows. $\square$

We will need a similar result when $n = 2r + 2$.

**Lemma 3.4.** *Assume that $n = 2r + 2$, $\nu = 0$ and $V$ is of signature $(2r + 1, 1)$. Let $U_i^{+\pm}$ be the matrix $U^{+\pm}$ withe $i$-th row removed. Put*

$$\phi^\square = \sum_{i=1}^{2r+1} \det U_i^{++} \det U_i^{+-}.$$

*Then $\mathcal{F}_{s_0}^{\phi^\square}$ is a nonzero multiple of the classical section. In particular, if $\sigma'$ is a discrete series representation of lowest $K'$-type $\tau' = \det^{r+1} \boxtimes \det^{-r-1}$, and $f' \in \sigma'$ in the lowest $K'$-type, then*

$$Z(f', f', \mathcal{F}_{s_0}^{\phi^\square}) \neq 0.$$

*Proof.* The proof is very similar to Lemma 3.3. In fact by analyzing the action of $K'^\square$ action, we see that $\mathcal{F}_{s_0}^{\phi^\square}$ is a constant multiple of the classical section, and we only need to check that $\mathcal{F}_{s_0}^{\phi^\square}$ is nonzero. For this we just need to compute $\phi^\square(0)$. The same computation as the previous lemma show that for any fixed $i$, the value of $\det U_i^{++} \det U_i^{+-}$ at 0 equals

$$(-1)^{1+\cdots+2r}\sum_I \langle \det Z_I^{++} \det Z_{I^c}^{+-}, \det W_I^{++} \det W_{I^c}^{+-}\rangle,$$

where $I$ ranges over all order $r$ subset of $\{1, \ldots, i-1, \widehat{i}, i+1, \ldots, 2r+1\}$, i.e. the set with $i$ removed. It follows that it is $(-1)^{1+\cdots+2r}$ times a positive number. The sign $(-1)^{1+\cdots+2r}$ is independent of $i$, so the sum over $i$ is again nonzero.

The nonvanishing of the doubling zeta integral again follows from [Gar08]. $\square$

## 4. Multiplicities

We come back to the setup of Theorem 1.1 in the Introduction. Recall that we have the split skew-hermitian space $W'$ of dimension $n = 2r$, and hermitian space $W$ and $V$ of signature $(n, 1)$ and $(n + 1, 1)$ respectively. We have the unitary groups $H' = \mathrm{U}(W')$, $H = \mathrm{U}(W)$ and $G = \mathrm{U}(V)$.

The goal of this section is to study the representation theory behind Theorem 1.1. We fix a place $v$ of $F$ and work with objects over $F_v$. The goal is to study the space

$$\mathrm{Hom}_{H_v}(\pi_v \widehat{\otimes} \sigma_v, \mathbb{C}).$$

This is the main subject of the local (nontempered) GGP conjecture. We prove various results in this direction.

We will drop the subscript $v$, so $F$ stands for a local field of characteristic zero, $G = G_v$ is a unitary group over $F$, etc.

We will be working with various theta lifts at the same time and we fix the following characters to split the metaplectic covers. We have fixed a character $\mu$ of $E^\times$, which is given by $\mu(z) = z/\sqrt{z\overline{z}}$ if $F = \mathbb{R}$.

- For $H' \times G$, we use the trivial character $\mathbf{1}$. The Weil representation is denoted by $\omega_V$. The theta lifts in both directions are denoted by $\theta_V$.
- For $H' \times H$, we use the character $\mu$. The Weil representation is denoted by $\omega_W$. The theta lifts in both directions are denoted by $\theta_W$.
- For $H' \times \mathrm{U}(L)$, we use the character $\mu$. Here $L = E$ stands for the one dimension hermitian space with the hermitian form given by the norm. The Weil representation is denoted by $\omega_L$.

With these choices of the characters, we can check by the explicit formulae for the Weil representation that

$$\omega_V|_{H' \times (H \times \mathrm{U}(L))} \simeq \omega_W \widehat{\otimes} \omega_L, \tag{4.1}$$

where $H'$ acts on the two factors on the right diagonally.

4.1. **Multiplicities at the nonarchimedean places.** We assume in this subsection that $F$ is nonarchimedean.

Recall that $\pi'$ and $\sigma'$ are irreducible tempered representations of $H'$. We also have irreducible representations $\pi = \theta_V(\pi')$ of $G$ and $\sigma = \theta_V(\sigma'^\vee)^\vee$. Note that they are all unitary (or more precisely unitarizable) representations.

**Lemma 4.1.** *If* $\mathrm{Hom}_H(\pi \otimes \sigma, \mathbb{C}) \neq 0$, *then* $\mathrm{Hom}_{H'}(\pi' \otimes \sigma' \otimes \overline{\omega_L}, \mathbb{C}) \neq 0$.

*Proof.* Since $\mathrm{Hom}_H(\pi \otimes \sigma, \mathbb{C}) \neq 0$, and $\Theta_{W',V}(\pi')$ maps surjectively onto $\pi$, we conclude that

$$\mathrm{Hom}_H(\Theta_V(\pi') \otimes \sigma, \mathbb{C}) \neq 0,$$

which is equivalent to

$$\mathrm{Hom}_{H \times H'}(\omega_V \otimes \pi'^\vee \otimes \sigma, \mathbb{C}) \neq 0.$$

By (4.1), this is equivalent to

$$\mathrm{Hom}_{H'}(\pi'^\vee \otimes \Theta_W(\sigma^\vee) \otimes \omega_L, \mathbb{C}) \neq 0.$$

By Lemma 2.1, we know that $\Theta_W(\sigma^\vee)$ is irreducible and hence is isomorphic to $\sigma^\vee$. This proves the lemma. □

*Remark* 4.2. In theory we can run the argument backwards and prove that the nonvanishing of the two Hom spaces are equivalent. We however are not able to do this because we do not yet know if $\Theta_V(\pi')$ is irreducible.

4.2. **Cohomological representations.** In this subsection and the next, we work at the place $v = \iota$. We write $\mathrm{U}(p,q)$ for a unitary group of signature $(p,q)$, which is a subgroup of $\mathrm{GL}_{p+q}(\mathbb{C})$ consisting of matrices satisfying

$$ {}^t\overline{g}\begin{pmatrix} 1_p & \\ & -1_q \end{pmatrix} g = \begin{pmatrix} 1_p & \\ & -1_q \end{pmatrix}. $$

We take the diagonal compact Cartan subgroup. We take the maximal compact subgroup to be the diagonal $\mathrm{U}(p) \times \mathrm{U}(q)$. The complexified Lie algebra of $\mathrm{U}(p,q)$ is naturally identified with $\mathfrak{gl}_{p+q}(\mathbb{C})$. The infinitesimal character of a representation is identified with a sequence of $p+q$ numbers up to permutation.

Irreducible representations of a compact group is parametrized by their highest weights. A Harish-Chandra parameter of a discrete series representation of $\mathrm{U}(p,q)$ is a strictly decreasing sequence of integers (if $p+q$ is odd) or half integers (if $p+q$ is even), together with a labeling of $p$ of them the $+$ sign, and $q$ of them the $-$ sign. A Harish-Chandra parater is equivalently two sequences of strictly decreasing integers or half-integers $(a_1, \ldots, a_p; b_1, \ldots, b_q)$, with the understanding that $a_i$'s are labelled $+$ while $b_i$'s are labelled $-$.

We choose bases of $V$, $W$, and $W'$ such that $\mathrm{U}(V)$, $\mathrm{U}(W)$ and $\mathrm{U}(W')$ are identified as group of matrices preserving the hermitian or skew-hermitian form given respectively by

$$ \begin{pmatrix} 1_{n+1} & \\ & -1 \end{pmatrix}, \quad \begin{pmatrix} 1_n & \\ & -1 \end{pmatrix}, \quad \begin{pmatrix} \sqrt{-1}1_r & \\ & -\sqrt{-1}1_r \end{pmatrix}. $$

So $G = \mathrm{U}(n+1,1)$, $H = \mathrm{U}(n,1)$ and $H' = \mathrm{U}(r,r)$. Recall that we have the discrete series representations $\pi'$ and $\sigma'$ of $H'$, and $\pi = \theta_V(\pi')$ of $G$ and $\sigma = \theta_V(\sigma'^\vee)^\vee$. The representations $\pi$ and $\sigma$ are assumed to be cohomological (with respect to the trivial representation). We now describe these representations more explicitly. All the descriptions below follow from Li's explicit description of theta lifts of cohomological representations, cf. [Li90].

We consider $\pi$ and $\pi'$ first. Since $\pi$ is cohomological with respect to the trivial representation, its infinitesimal character is the same as that of the trivial representations. Li's description gives the following result. The representation $\pi'$ is a discrete series representation of $H'$ whose Harish-Chandra parameter is given by

$$ \left(\frac{n+1}{2}, \ldots, \frac{3}{2}; -\frac{3}{2}, \ldots, -\frac{n+1}{2}\right). $$

The representation $\pi$ can be described using cohomological induction. Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ be the parabolic subalgebra of $\mathfrak{gl}_{n+2}(\mathbb{C})$ given by the coroot

$$ \left(\frac{n+1}{2}, \ldots, \frac{3}{2}, 0, -\frac{3}{2}, \ldots, -\frac{n+1}{2}, 0\right). $$

Then we have $\pi = A_{\mathfrak{q}} = A_{\mathfrak{q}}(0)$. The lowest $K$-type of $\pi'$ and $\pi$ are given respectively by

$$(r+1,\dots,r+1)\times(-(r+1),\dots,-(r+1)), \quad (\underbrace{1,\dots,1}_{r},0,\underbrace{-1,\dots,-1}_{r})\times(0).$$

They correspond in the joint harmonics (we will not use this fact though).

Now we consider $\sigma$ and $\sigma'$. In this case Li's description gives the following. The representation $\sigma'$ is discrete series representations of $H'$ given by the Harish-Chandra parameter

$$\left(\frac{n+1}{2},\dots,\frac{3}{2};-\frac{1}{2},\dots,-\frac{n-1}{2}\right).$$

The representation $\sigma$ is an irreducible discrete series representation of $H$ given by the Harish-Chandra parameter

$$(r, r-1,\dots,1,-1,-2,\dots,-r;0),$$

Their lowest $K$-types are given respectively by

$$(r+1,\dots,r+1)\times(-r,\dots,-r), \quad (\underbrace{1,\dots,1}_{r},\underbrace{-1,\dots,-1}_{r})\times(0).$$

They correspond in the joint harmonics (again we will not use this fact).

4.3. **Multiplicities in cohomological packets.** We consider a cohomological Arthur parameter

$$\left(\bigoplus_{\substack{-(n+1)\le i\le n+1,\ \text{odd}\\ i\neq\pm 1}}\mu^i\right)\bigoplus\left(\mathbf{1}\otimes\mathbb{C}^2\right).$$

Its corresponding A-packet is constructed by Adams and Johnson [AJ87]. The representations in this packet are given as follows.

- $\pi = A_{\mathfrak{q}}$ as in the previous subsection. This representation gives the corresponding $L$-packet within the A-packet.
- An extra $n$ discrete series representations. In terms of their lowest $K$-type, they are of the form

$$(\underbrace{1,\dots,1}_{a},\underbrace{-1,\dots,-1}_{n+1-a})\times(n+1-2a)$$

  where $a = 0,1,\dots,r-1,r+2,\dots,n+1$. In terms of Harish-Chandra parameters, they are of the form

$$\left(r+\frac{1}{2},r-\frac{1}{2},\dots,\frac{1}{2},-\frac{1}{2},\dots,-r-\frac{1}{2}\right),$$

  where one of them is labelled $-$ and the rest are $+$. The $-$ is not labelled on either $\frac{1}{2}$ or $-\frac{1}{2}$.

The (tempered) $A$-packet of $H$ in which $\sigma$ lies consists of $n+1$ discrete series representations. The Harish-Chandra parameters of these representations are of the form

$$(r, r-1, \ldots, 1, 0, -1, \ldots, -(r-1), -r),$$

where one of them is labelled $-$ and the others are labelled $+$. The representation $\sigma$ is the one with the $-$ sign on 0.

**Lemma 4.3.** *Let $\pi_1$ and $\sigma_1$ be in the above packets for $G$ and $H$ respectively. Then*

$$\mathrm{Hom}_H(\pi_1 \widehat{\otimes} \sigma_1, \mathbb{C}) \neq 0$$

*if and only if $\pi_1 = \pi$ and $\sigma_1 = \sigma$.*

*Proof.* By (tempered) local GGP conjecture for Bessel models, cf. [He17, Xue23], we know that $\mathrm{Hom}_H(\pi_1 \widehat{\otimes} \sigma_1, \mathbb{C}) = 0$ if $\pi_1 \neq \pi$. It remains only to prove that

$$\mathrm{Hom}_H(\pi \widehat{\otimes} \sigma_1, \mathbb{C}) \neq 0$$

if and only if $\sigma_1 = \sigma$.

By Lemma 2.1, we have $\Theta_V(\pi')$ is irreducible and hence isomorphic to $\pi$. Therefore as in the proof of Lemma 4.1 we have

$$\mathrm{Hom}_H(\pi \widehat{\otimes} \sigma_1, \mathbb{C}) \neq 0 \tag{4.2}$$

is equivalent to

$$\mathrm{Hom}_{H'}(\pi'^{\vee} \widehat{\otimes} \Theta_W(\sigma_1^{\vee}) \widehat{\otimes} \omega_L, \mathbb{C}) \neq 0. \tag{4.3}$$

This in particular implies that $\Theta_W(\sigma_1^{\vee}) \neq 0$ when (4.2) holds. But according to Li's explicit description of theta lifts for discrete series representations, $\Theta_W(\sigma_1^{\vee}) \neq 0$ only when $\sigma_1 = \sigma$. This shows that

$$\mathrm{Hom}_H(\pi \widehat{\otimes} \sigma_1, \mathbb{C}) = 0$$

if $\sigma_1 \neq \sigma$.

Finally if $\sigma_1 = \sigma$, then the (tempered) local GGP for Fourier–Jacobi models [Xue24] implies that

$$\mathrm{Hom}_{H'}(\pi' \otimes \sigma' \otimes \overline{\omega_L}, \mathbb{C}) \neq 0.$$

Since $\Theta_W(\sigma^{\vee})$ maps surjectively onto $\sigma'^{\vee} = \theta_W(\sigma^{\vee})$, we conclude that (4.3) holds (with $\sigma_1$ replaced by $\sigma$). Therefore (4.2) holds when $\sigma_1 = \sigma$. This proves the lemma. $\square$

## 5. Periods

We prove the main theorem in this section. The idea is very simple and has been outlined in Subsection 1.3. We first review Kudla–Millson forms, and then construct a differential form $\alpha \in \mathrm{H}^{n,n}(X)[\pi_{\mathrm{f}} \otimes \sigma_{\mathrm{f}}]$ using it. Then we compute the integral $\int_Y \alpha$ by the seesaw diagram and reduce it to the (tempered) global GGP conjecture for Fourier–Jacobi periods on $H' \times H'$ which is recently established in [BLX].

Throughout this section, $F$ is a number field. We again need to work with various theta lifts at the same time. Recall that we have fixed a character $\mu$ of $E^\times \backslash \mathbb{A}_E^\times$ whose archimedean components are of the form $z \mapsto z/\sqrt{z\overline{z}}$. The notation and the choices of the characters to split the metaplectic covers are the same as in Section 3.

Recall also that we use gothic letters to denote the Lie algebras of the corresponding groups, e.g. if $G$ is an algebraic group over $F$, we denote by $\mathfrak{g}$ the Lie algebra of $G_\infty$ and $\mathfrak{g}_{\mathbb{C}}$ its complexification. If $v$ is an archimedean place, then $\mathfrak{g}_v$ and $\mathfrak{g}_{v,\mathbb{C}}$ stand for the corresponding objects for $G_v$.

5.1. **Kudla–Millson forms.** By our assumption, we have

$$G_\infty = \mathrm{U}(n+1,1) \times \mathrm{U}(n+2)^{d-1}.$$

The notation from Subsection 4.2 applies to $G_\infty$. In particular we choose coordinates ${}^t(x_1, \dots, x_{n+2})$ on $V_v$ for each $v \mid \infty$ such that the matrix representing the hermitian form on $V_\iota$ is $\begin{pmatrix} 1_n & \\ & -1 \end{pmatrix}$ and on other $V_v$'s are the identity matrix. The coordinates of $V_v^r$ is denoted by a $(n+2) \times r$ matrix $X$. We do not try to distinguish the coordinates at different places by adding more subscripts, as this will be clear from the context. We take the (diagonal) maximal compact subgroup

$$K_{G,\infty} = (\mathrm{U}(n+1) \times \mathrm{U}(1)) \times \mathrm{U}(n+2)^{d-1},$$

and we have the Cartan decomposition $\mathfrak{g} = \mathfrak{p}_G + \mathfrak{k}_G$. Note that $\mathfrak{p}_G = \mathfrak{p}_{G,\iota}$ as $G_v$ is compact if $v \neq \iota$. We have $\mathfrak{p}_{G,\mathbb{C}} = \mathfrak{p}_G^+ + \mathfrak{p}_G^-$ where $\mathfrak{p}_G^\pm$ are identified with the holomorphic and antiholomorphic tangent spaces of the symmetric domain $D_V$. The tangent spaces $\mathfrak{p}_G^\pm$ is identified with the space spanned by ${}^t(x_1, \cdots, x_{n+1})$ (resp. $(\overline{x_1}, \cdots, \overline{x_{n+1}})$). Let $\xi_i$ (resp. $\overline{\xi_i}$) be the linear form which sends ${}^t(x_1, \cdots, x_{n+1})$ to $x_i$ (resp. $(\overline{x_1}, \cdots, \overline{x_{n+1}})$ to $\overline{x_i}$). They give bases of $\mathfrak{p}_G^{\pm,\vee}$. Put $\wedge^{r,r}\mathfrak{p}_G^\vee = \wedge^r \mathfrak{p}_G^{+,\vee} \otimes \wedge^r \mathfrak{p}_G^{-,\vee}$.

For each place $v \mid \infty$, recall that we have the Schrodinger model $\mathcal{S}(V_v^r)$ and the Fock model $\mathcal{P}_v$ of the Weil representation $\omega_V$. We also have the subspace $\mathcal{S}(V_v^r)^\dagger \subset \mathcal{S}(V_v^r)$ and an isomorphism $\beta_v : \mathcal{S}(V_v^r)^\dagger \to \mathcal{P}_v$.

We do most of the calculations below in terms of the Fock model $\mathcal{P}_v$. Recall from Subsection 2.2 that $\mathcal{P}_v = \mathbb{C}[Z]$ where $Z$ is a $(n+2) \times n$ matrix of variables, and has the partition

$$Z = \begin{pmatrix} Z^{++} & Z^{+-} \\ Z^{-+} & Z^{--} \end{pmatrix},$$

where $Z^{++}$ is of size $(n+1)\times r$. To simplify notation, we identify $\mathcal{S}(V_v^r)^\dagger$ with $\mathcal{P}_v$ via $\beta$, and suppress the isomorphism $\beta$ from all notation, i.e. when we say $z\in\mathbb{C}[Z]$ is a Schwartz function, it really means the Schwartz function $\beta^{-1}(z)\in\mathcal{S}(V_v^r)^\dagger$.

Kudla and Millson constructed an explicit differential form on $D_V$ valued in $\mathcal{S}(V_\iota^r)$. Following Kulda and Millson [KM86], we consider a form

$$\sum_{I,J}\det Z_I^{++}\det Z_J^{+-}(\overline{\xi_I}\wedge\xi_J)\in\Big(\mathcal{S}(V_\iota^r)^\dagger\otimes\wedge^{r,r}\mathfrak{p}_G^\vee\Big)^{K_{G,\iota}} \tag{5.1}$$

where $I$ and $J$ are order $r$ subsets of $\{1,\cdots,n+1\}$, and $Z_I^{++}$ stands for the submatrix of $Z^{++}$ which keeps only the $i$-th rows when $i\in I$, and $\xi_I=\wedge_{i\in I}\xi_i$. Similar for $Z_J^{+-}$ and $\xi_J$.

Take $\Phi=\otimes\Phi_v\in\mathcal{S}(V^r(\mathbb{A}))\otimes\wedge^{r,r}\mathfrak{p}_G^\vee$, and assume that $\Phi_\iota$ is the Schwartz form (5.1) and

$$\Phi_v=e^{-\pi\operatorname{Tr}{}^t\overline{X}X}\in\mathcal{S}(V_v^r)^\dagger$$

is the Gaussian function for any infinite place $v\neq\iota$. Define a theta form

$$\Theta(h',g,\Phi)=\sum_{x\in V^r(F)}(\omega_V(h',g)\Phi)(x)\in\mathcal{A}(H')\otimes(\mathcal{A}(G)\otimes\wedge^{r,r}\mathfrak{p}_G^\vee)^{K_{G,\iota}}.$$

Here $\mathcal{A}(H')$ and $\mathcal{A}(G)$ stand for the spaces of automorphic forms on $H'(\mathbb{A})$ and $G(\mathbb{A})$ respectively.

Let $f'=\otimes f'_v\in\pi'$, where $f'_v$ is a nonzero vector in the lowest $K_{H',v}$-type of $\pi'_v$ (which is one dimensional) if $v\mid\infty$. Let

$$\Theta_\Phi^{f'}(g)=\int_{[H']}\overline{f'(h')}\Theta(h',g,\Phi)\mathrm{d}h'\in(\pi\otimes\wedge^{r,r}\mathfrak{p}_G^\vee)^{K_{G,\iota}}.$$

Then $\Theta_\Phi^{f'}$ defines a class in $\mathrm{H}^{r,r}(X)[\pi_\mathrm{f}]$.

**Lemma 5.1.** *As $f'$ and $\Phi$ vary (with $f'_\infty$ and $\Phi_\infty$ fixed as above), the elements $\Theta_\Phi^{f'}$'s generate*

$$\mathrm{H}^{r,r}(X)[\pi_\mathrm{f}]=\pi_\mathrm{f}\otimes\mathrm{H}^{r,r}(\mathfrak{g}_\mathbb{C},K_{G,\infty},\pi_\infty).$$

*Proof.* This is probably well-known to the experts. For the convenience of the reader, we provide a brief explanation.

Since the group $G$ is anisotropic (we use the assumption $d>1$ here), Lemma 2.2 applies. Moreover $\pi_\iota$ is the representation $A_\mathfrak{q}$ described in Subsection 4.2, and $\pi_v$ is the trivial representation if $v\mid\infty$ and $v\neq\iota$. By [VZ84, Theorem 3.3], the cohomology group $\mathrm{H}^{r,r}(\mathfrak{g}_\mathbb{C},K_{G,\infty},\pi_\infty)$ is one dimensional, so the right hand side is an irreducible $G(\mathbb{A}_\mathrm{f})$-module. Therefore we just need to know that $\Theta_\Phi^{f'}$ is not identically zero.

The nonvanishing of $\Theta_\Phi^{f'}$ follows from a "geometric" version of the Rallis inner product formula, cf. [Li, Theorem 3.7.1], which in turn is a direct consequence of a "geometric" Siegel–Weil formula, or sometimes referred to as a volume formula, cf. [Li, Theorem 3.6.1] and [Dun, Section 2.2] for an exposition of the proof. To show that $\Theta_\Phi^{f'}$ is nonzero, we compute

$$\int_X\Theta_\Phi^{f'}\wedge\overline{\Theta_\Phi^{f'}}\wedge c_1(\mathcal{L})$$

where $c_1(\mathcal{L}) = \sum_{1\le i\le n+1} \xi_i \wedge \overline{\xi_i} \in \mathrm{H}^{1,1}(X)$ stands for the first Chern class of the Hodge bundle on $X$. By the definition of $\Theta_\Phi^{f'}$, this reduces to the integral

$$\int_{[H']^2} \overline{f'(h_1')} f'(h_2') \left( \int_X \Theta(h_1', 1, \Phi) \wedge \overline{\Theta(h_2', 1, \Phi)} \wedge c_1(\mathcal{L}) \right) \mathrm{d}h_1' \mathrm{d}h_2'. \tag{5.2}$$

We now follow the notation of Section 3. Note that the notation is slightly different, where $n$ there is actually $n+2$ here, and $p$, $q$ there are $n+1$ and $1$ respectively here. We consider the Weil representation $\omega^\square$ of $H'^\square(\mathbb{A}) \times G(\mathbb{A})$, which is realized on $\mathbb{C}[U]$. Since $\Phi_\iota \wedge \overline{\Phi_\iota}$ equals the Kudla–Millson form for the group $H_\iota'^\square \times G_\iota$, cf. [KM86, Section 6, (6.3)], we know that

$$\Theta(h_1', 1, \Phi) \wedge \overline{\Theta(h_2', 1, \Phi)} = \Theta((h_1', h_2'), 1, \Phi^\square)$$

where

$$\Phi^\square = \otimes \Phi_v^\square \in (\mathcal{S}(V^n(\mathbb{A})))^\dagger \otimes \wedge^{n,n} \mathfrak{p}_G^\vee)^{K_{G,\iota}}$$

is given as follows. For $v \nmid \infty$, $\Phi_v^\square = (\Phi_v \otimes \overline{\Phi_v})^\dagger \in \mathcal{S}(V_v^n)$. For $v \mid \infty$ but $v \neq \iota$, $\Phi_v^\square$ is the standard Gaussian in $\mathcal{S}(V_\iota^n)$. For $v = \iota$, $\Phi_\iota^\square \in (\mathcal{S}(V_\iota^n)^\dagger \otimes \wedge^{n,n}\mathfrak{p}_G^\vee)^{K_{G,\iota}}$ is the Kudla–Millson form, given by

$$\sum_{1\le i,j\le n+1} \det U_i^{++} \det U_j^{+-} \overline{\xi^i} \wedge \xi^j$$

where $U_i^{+\pm}$ is the matrix $U^{+\pm}$ with $i$-th row removed (recall that $U^{+\pm}$ is a $(n+1)\times n$ matrix), and $\xi^i = \xi_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \xi_{n+1}$.

The Siegel–Weil formula explained in [Dun, Section 2.2] implies that the inner integral (5.2) equals (up to some nonzero constant depending on the measures)

$$\sum_{i=1}^{n+1} E\left(\iota(h_1', h_2'), \mathcal{F}_{s_0}^{\Phi_i^\square}\right)$$

where $\Phi_i^\square = \otimes \Phi_{i,v}$ is the Schwartz function in $\mathcal{S}(V^n(\mathbb{A}))^\dagger$ given by $\Phi_v^\square$ if $v \neq \iota$, and by $\Phi_{i,\iota}^\square = \det U_i^{++} \det U_i^{+-}$ if $v = \iota$.

The integral (5.2) then reduces to a doubling zeta integral

$$\sum_{i=1}^{n+1} Z(f', f', \mathcal{F}_{s_0}^{\Phi_i^\square}).$$

It equals $L(\frac{3}{2}, \pi')$ (which is nonzero) times the product of local doubling zeta integrals. If $v \nmid \infty$, the local doubling zeta integral is given by $Z(f_v', f_v', \Phi_v, \Phi_v)$ which is not identically zero by [GQT14, Proposition 11.5]. If $v \mid \infty$ and $v \neq \iota$, the local doubling zeta integral equals

$$Z(f_v', f_v', \mathcal{F}_{v,s_0}^\circ),$$

where $\mathcal{F}_{v,s_0}^\circ$ stands for the classical section. If $v = \iota$, the local doubling zeta integral equals

$$\sum_{i=1}^{n+1} Z(f_\iota', f_\iota', \mathcal{F}_{s_0}^{\Phi_{i,\iota}^\square}),$$

which is nonzero by Lemma 3.4. □

5.2. **Construction of differential forms.** We now work on $Y$. The notation we fixed on $G$ also applies to $H$. We choose the coordinates on $V_\iota$ such that $W_\iota$ consists of vectors ${}^t(x_1, \cdots, x_{n+2})$ with $x_{r+1} = 0$. We have the Cartan decomposition $\mathfrak{h} = \mathfrak{p}_H + \mathfrak{k}_H$ and $\mathfrak{p}_H^\pm$ is identified with holomorphic and anti-holomorphic tangent space of $D_W$. We construct a cohomology class in $\mathrm{H}^{r,r}(\mathfrak{h}_\mathbb{C}, K_{H,\infty}, \sigma_\infty)$. First if $v \neq \iota$, then $\sigma_v$ is the trivial representation and we take $\varphi_v = 1 \in \sigma_v$. Now consider the place $\iota$. Let $\tau$ be the lowest $K_{H,\iota}$-type of $\sigma_\iota$, its highest weight being $(1, \cdots, 1, -1, \cdots, -1) \times (0)$. Then $\tau^\vee$ appears in $\wedge^{r,r}\mathfrak{p}_H^\vee$ with multiplicity one, and this copy of $\tau^\vee$ is generated by $\xi_1 \wedge \cdots \wedge \xi_r \wedge \overline{\xi_{r+2}} \wedge \cdots \wedge \overline{\xi_{n+1}}$. Fix a basis of $v_\alpha$ of $\tau$, a dual basis $v_\alpha^\vee$ of $\tau^\vee$. Let $\xi_\alpha$ be the elements corresponding to $v_\alpha^\vee \in \tau^\vee$ viewed as elements in $\wedge^{r,r}\mathfrak{p}_H^\vee$. Put

$$\underline{\varphi}_\iota = \sum_\alpha v_\alpha \otimes \xi_\alpha \in (\sigma_\iota \otimes \wedge^{r,r}\mathfrak{p}_H^\vee)^{K_{H,\iota}}. \tag{5.3}$$

Take a $\Psi_v \in \sigma_v$ for each $v \nmid \infty$, $\Psi_\iota = \underline{\varphi}_\iota$ as above and $\Psi_v = 1 \in \sigma_v$ for all $v \mid \infty$ and $v \neq \iota$. Put

$$\Psi = \otimes_v \Psi_v \in (\sigma \otimes \wedge^{r,r}\mathfrak{p}_H^\vee)^{K_{H,\infty}} = \mathrm{H}^{r,r}(Y)[\sigma_\mathrm{f}].$$

Put

$$\Omega = \Theta_\Phi^{f'} \wedge \Psi \in \mathrm{H}^{n,n}(X \times Y)[\pi_\mathrm{f} \otimes \sigma_\mathrm{f}].$$

Our goal is to compute

$$\int_Y \Omega,$$

which will yield a proof of Theorem 1.1. The next lemma relates this integral to a period integral of automorphic forms. Let us introduce more notation before stating it. Recall that we have the Fock model $\mathcal{P} = \mathbb{C}[Z]$ of $\omega_{V_\iota}$ at the place $\iota$. The Fock model at of $\omega_{W_\iota}$ is nothing but that of $\omega_{V_\iota}$ with the $(r+1)$-th row removed. Let $\mathcal{P}_0 \subset \mathcal{P}$ be the subspace spanned by polynomials of the form $\det Z_I^{++} \det Z_J^{+-}$ where $I$ and $J$ are order $r$ subset of $\{1, \ldots, r, r+2, \ldots, n+1\}$. This subspace, as a representation of $K_{H,\iota}$, is isomorphic to $\wedge^{r,r}\mathfrak{p}_H$. Let us note that $\det Z_I^{++} \det Z_J^{+-}$ and $\overline{\xi_I} \wedge \xi_J$ form dual basis in $\mathcal{P}_0$ and $\wedge^{r,r}\mathfrak{p}_H^\vee$. The representation $\tau^\vee$ (which is isomorphic to $\tau$) is contained in this $\mathcal{P}_0$ with multiplicity one, cf. [PRRV67, Corollary 1 to Theorem 2.1] or [Sun17, Lemma 2.13] (not original but maybe easier to find reference). Let $\mathcal{P}_{00}$ be the subspace of $\mathcal{P}_0$ isomorphic to $\tau^\vee$. So the basis $v_\alpha^\vee$ gives an basis in $\mathcal{P}_{00}$, which we still denote by $v_\alpha^\vee$.

**Lemma 5.2.** *Assume that for any $v \nmid \infty$, the function $\Phi_v \in \mathcal{S}(V_v^r)$ takes the form $\Phi_{W,v} \otimes \Phi_{L,v}$ where $\Phi_{W,v} \in \mathcal{S}(W_v^r)$ and $\Phi_{L,v} \in \mathcal{S}(L_v^r)$, cf. the decomposition (4.1). Then*

$$\int_Y \Omega = \sum_\alpha \int_{[H]} \theta^{f'}_{\phi_{W,\alpha} \otimes \phi_L}(h) \varphi_\alpha(h) \mathrm{d}h.$$

*Here on the right hand side*

- *the sum ranges over the basis $v_\alpha$;*
- *$\phi_{W,\alpha} = (\otimes_{v \neq \iota} \phi_{W,v}) \otimes \phi_{W,\alpha}$, where $\phi_{W,v} = \Phi_{W,v}$ if $v \neq \iota$ and $\phi_{W,\alpha} \in \mathcal{S}(W_\iota^r)^\dagger$ is the Schwartz function corresponding to the element $v_\alpha^\vee$ in the Fock model of $\omega_{W_\iota}$;*

- $\phi_L = \otimes_v \phi_{L,v}$, $\phi_{L,v} = \Phi_{L,v}$ *if* $v \nmid \infty$ *and* $\phi_v \in \mathcal{S}(L_v^r)$ *is the Gaussian function, i.e.* $\phi_v = e^{-\pi \sum_a z_{n+1,a}\overline{z_{n+1,a}}}$, *if* $v \mid \infty$;
- $\varphi_\alpha = (\otimes_{v \neq \iota} \varphi_v) \otimes v_\alpha \in \sigma$.

*Proof.* Fix any $K_{H,\iota}$-invariant pairing on $\wedge^{r,r}\mathfrak{p}_H^\vee$. Up to some nonzero constant depending on the measures and the choice of the inner product on $\wedge^r \mathfrak{p}_H^+$, we conclude that

$$\int_Y \Omega = \int_Y (\Theta_\Phi^{f'})|_Y \wedge \Psi = \int_{[H]} \langle \mathrm{pr}(\Theta_\Phi^{f'})(h), \Psi(h)\rangle \mathrm{d}h.$$

The notation $\mathrm{pr}(\Theta_\Phi^{f'}(h))$ means the following. First we have $\Theta_\Phi^{f'} \in \pi \otimes \wedge^{r,r}\mathfrak{p}_G^\vee$. Note that $\mathfrak{p}_H$ is naturally a subspace of $\mathfrak{p}_G$ given by $x_{r+1} = 0$, and hence there is a natural projection map $\mathfrak{p}_G^\vee \to \mathfrak{p}_H^\vee$ which sends $\xi_{n+1}$ to 0. Then $\mathrm{pr}(\Theta_\Phi^{f'}) \in \pi \otimes \wedge^{r,r}\mathfrak{p}_H^\vee$ stands for the image of $\Theta_\Phi^{f'}$ under this map.

Let $\mathrm{pr}(\Phi_\iota)$ be the image of $\Phi_\iota \in \mathcal{S}(V^r)^\dagger \otimes \wedge^{r,r}\mathfrak{p}_G^\vee$ in $\mathcal{S}(V^r)^\dagger \otimes \wedge^{r,r}\mathfrak{p}_H^\vee$ under the natural projection $\mathfrak{p}_G^\vee \to \mathfrak{p}_H^\vee$. Then

$$\langle \mathrm{pr}(\Phi_\iota), \Psi_\iota\rangle = \sum_\alpha v_\alpha^\vee \otimes v_\alpha \in \mathcal{S}(V_\iota^r)^\dagger \otimes \sigma_\iota. \tag{5.4}$$

To see (5.4), we note that by the explicit form of $\Phi_\iota$, we have

$$\mathrm{pr}(\Phi_\iota) = \sum_{I,J} \det Z_I^{++} \det Z_J^{+-} (\overline{\xi_I} \wedge \xi_J) \in \mathcal{S}(V_\iota^r)^\dagger \otimes \wedge^{r,r}\mathfrak{p}_H^\vee,$$

where $I$ and $J$ range over order $r$ subsets of $\{1, \cdots, r, r+2, \cdots, n+1\}$. Then

$$\langle \mathrm{pr}(\Phi_\iota), \Psi_\iota\rangle = \sum_{I,J}\sum_\alpha \det Z_I^{++} \det Z_J^{+-} \otimes v_\alpha \langle \overline{\xi_I} \wedge \xi_J, \xi_\alpha\rangle.$$

We observe that $\xi_I$'s are weight vectors in $\wedge^r \mathfrak{p}_H^{+,\vee}$, and different $I$'s give different weights. Then $\overline{\xi_I} \wedge \xi_J$ where $I$ and $J$ are order $r$ subsets of $\{1, \cdots, r, r+2, \cdots, n+1\}$ form an orthogonal basis of $\wedge^{r,r}\mathfrak{p}_H^\vee$. The desired equality (5.4) follows.

It follows from (5.4) that

$$\langle \mathrm{pr}(\Theta_\Phi^{f'})(h), \Psi(h)\rangle = \sum_\alpha \theta^{f'}_{\phi_{W,\alpha}\otimes\phi_L}(h)\varphi_\alpha(h),$$

where the notation is explained in the lemma. This is what we want to prove. □

5.3. **Proof of Theorem 1.1.** We now put what we have together. First note that by [GGP20, Theorem 9.7], the $L$-function

$$\frac{L(s, \pi \otimes \sigma)}{L(s+\frac{1}{2}, \pi, \mathrm{Ad})L(s+\frac{1}{2}, \sigma, \mathrm{Ad})}$$

is holomorphic at $s = \frac{1}{2}$, and its value at $s = \frac{1}{2}$ differs from $L(\frac{1}{2}, \pi' \otimes \sigma' \otimes \mu^{-1})$ by a nonzero constant. So we may replace this condition on the $L$-functions by $L(\frac{1}{2}, \pi' \otimes \sigma' \otimes \mu^{-1}) \neq 0$.

Assume that the class of $Y$ in $\mathrm{H}^{n+1,n+1}(X \times Y)[\pi_{\mathrm{f}}^\vee \otimes \sigma_{\mathrm{f}}^\vee]$ is not zero. By Poincare duality, there is a differential form $\alpha \in \mathrm{H}^{n,n}(X \times Y)[\pi_{\mathrm{f}} \otimes \sigma_{\mathrm{f}}]$ such that

$$\int_Y \alpha \neq 0.$$

By Matsushima's formula, we have

$$\mathrm{H}^{n,n}(X\times Y)[\pi_{\mathrm{f}}\otimes\sigma_{\mathrm{f}}]=\bigoplus_{\pi_{1,\infty},\sigma_{1,\infty}}(\pi_{\mathrm{f}}\otimes\sigma_{\mathrm{f}})^{K_{G,\mathrm{f}}\times K_{H,\mathrm{f}}}\otimes\mathrm{H}^{n,n}(\mathfrak{g}_{\mathbb{C}}\times\mathfrak{h}_{\mathbb{C}},K_{G,\infty}\times K_{H,\infty},\pi_{1,\infty}\otimes\sigma_{1,\infty}),$$

where $\pi_{1,\infty}$ and $\sigma_{1,\infty}$ ranges over all irreducible representations of $G_\infty$ and $H_\infty$ such that $\pi_{\mathrm{f}}\otimes\pi_{1,\infty}$ and $\sigma_{\mathrm{f}}\otimes\sigma_{1,\infty}$ are automorphic. Then $\pi_{1,\infty}$ and $\sigma_{1,\infty}$ lie in the local A-packets described in Subsection 4.3. If $v\mid\infty$ and $v\neq\iota$, then $\pi_v$ and $\sigma_v$ are trivial representations, as $G_v$ and $H_v$ are compact. The Arthur's multiplicity formula (for tempered packets, or packets of unitary groups of rational rank at most one, cf. [Mok15, KMSW, CZ]) then implies that $\pi_{1,\iota}=\pi_\iota$ and $\sigma_{1,\iota}=\sigma_\iota$. Therefore

$$\alpha\in(\pi_{\mathrm{f}}\otimes\sigma_{\mathrm{f}})^{K_{G,\mathrm{f}}\times K_{H,\mathrm{f}}}\otimes\left(\mathrm{H}^{r,r}(\mathfrak{g}_{\mathbb{C}},K_{G,\infty},\pi_\infty)\otimes\mathrm{H}^{r,r}(\mathfrak{h}_{\mathbb{C}},K_{H,\infty},\sigma_\infty)\right),$$

Then $\int_Y\alpha\neq0$ implies that

$$\mathrm{Hom}_{H(\mathbb{A}_{\mathrm{f}})}(\pi_{\mathrm{f}}\otimes\sigma_{\mathrm{f}},\mathbb{C})\neq0.$$

We also proved in Lemma 4.3 that

$$\mathrm{Hom}_{H_\iota}(\pi_\iota\,\widehat{\otimes}\,\sigma_\iota,\mathbb{C})\neq0.$$

Of course if $v\mid\infty$ and $v\neq\iota$ we have the same assertion, as $\pi_v$ and $\sigma_v$ are trivial representations.

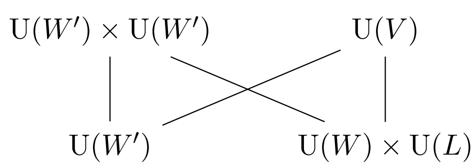
According to Lemma 5.1, the forms $\Theta_{\Phi}^{f'}$ generates $\pi_{\mathrm{f}}\otimes\mathrm{H}^{r,r}(\mathfrak{g}_{\mathbb{C}},K_{G,\infty},\pi_\infty)$, so we can choose $f',\Phi$ and $\Psi$ as in the previous two subsections, such that

$$\int_Y(\Theta_{\Phi}^{f'}|_Y)\wedge\Psi \tag{5.5}$$

is not identically zero. By Lemma 5.2 this equals

$$\sum_\alpha\int_{[H]}\theta^{f'}_{\phi_{W,\alpha}\otimes\phi_L}(h)\varphi_\alpha(h)\mathrm{d}h,$$

where the notation on the right hand side is explained in Lemma 5.2. We now make use of the seesaw diagram




and conclude that this integral equals

$$\sum_\alpha\int_{[H']}\overline{f'(h')}\theta^{\overline{\varphi_\alpha}}_{\phi_{W,\alpha}}(h')\theta_L(h',\phi_1)\mathrm{d}h'.$$

By assumption $\theta^{\overline{\varphi_\alpha}}_{\phi_{W,\alpha}}(h')\in\sigma'^\vee$. We now invoke the main theorem of [BLX], i.e. the (tempered) GGP conjecture for Fourier–Jacobi periods on unitary groups, and conclude that

$$L(\frac{1}{2},\pi'\otimes\sigma'\otimes\mu^{-1})\neq0.$$

This shows one direction of implication.

To prove the other direction, we need some additional lemmas.

**Lemma 5.3.** *Let $v$ be an infinite place of $F$. We have*

$$\mathrm{Hom}_{H'}(\pi'_v \widehat{\otimes} \sigma'_v \widehat{\otimes} \overline{\omega_{L,v}}, \mathbb{C}) \neq 0,$$

*and any nonzero $l$ in this space does not vanishes when restricted to the lowest $K_{H',v}$-types.*

*Proof.* The explicit descriptions of $\pi'_v$ and $\sigma'_v$ are given in Subsection 4.2. They are irreducible discrete series representations who lowest $K_{H',v}$-types are $\det^{r+1} \boxtimes \det^{-r-1}$ and $\det^{r+1} \boxtimes \det^{-r}$ respectively. The nonvanishing of the Hom space follows from the local GGP conjecture for Fourier–Jacobi models, cf. [Xue24].

To see that $l$ does not vanish when restricted to the lowest $K_{H'}$-types, we make use of the fact that

$$l \otimes \bar{l} \in \mathrm{Hom}_{H'}(\pi' \widehat{\otimes} \sigma' \widehat{\otimes} \overline{\omega_L}, \mathbb{C}) \otimes \overline{\mathrm{Hom}_{H'}(\pi' \widehat{\otimes} \sigma' \widehat{\otimes} \overline{\omega_L}, \mathbb{C})}$$

can be realized using integration of matrix coefficients, cf. [Xue24, Theorem 3.2]. This means that to see that $l$ is nonvanishing, we just need to show that

$$\int_{H'} \langle \pi'(h')f', f' \rangle \langle \sigma'(h')\varphi', \varphi' \rangle \overline{\langle \omega_L(h')\phi_L, \phi_L \rangle} \mathrm{d}h' \neq 0,$$

where $f' \in \pi'$ and $\varphi' \in \sigma'$ are in the lowest $K_{H'}$-type, and $\phi_L \in \mathcal{S}(L^r)$ is the Gaussian function. This follows directly from Sun's positivity of matrix coefficients, cf. [Sun09, Theorem 1.5]. Indeed using the the Cartan decomposition of $H'$, we see that the above integral equals

$$\int_{A^+} \mu(a) \langle \pi'(a)f', f' \rangle \langle \sigma'(a)\varphi', \varphi' \rangle \overline{\langle \omega_L(a)\phi_L, \phi_L \rangle} \mathrm{d}a, \tag{5.6}$$

where $A^+$ stands for the subgroup of $H' = \mathrm{U}(r, r)$ given by

$$\mathrm{diag}[a_1, \cdots, a_r, a_1^{-1}, \cdots, a_r^{-1}], \quad a_1, \ldots, a_r \in \mathbb{R}_{>0},$$

and $\mu(a)$ is a positive function on $A^+$. Sun's result then implies that

$$\langle \pi'(a)f', f' \rangle > 0, \quad \langle \sigma'(a)\varphi', \varphi' \rangle > 0.$$

Direct computation also shows that

$$\langle \omega_L(a)\phi_L, \phi_L \rangle > 0.$$

As a result, the integral (5.6) is positive. □

**Lemma 5.4.** *Fix a nonzero linear form $p : \mathcal{P} \otimes \sigma_\iota \to \sigma_\iota'^{\vee}$. Then $p$ restricts to a nonzero pairing between $\mathcal{P}_{00}$ and $\tau$. In particular, if we take the basis $v_\alpha^\vee$ of $\mathcal{P}_{00}$ and the basis $v_\alpha$ in $\tau$, then*

$$\sum_\alpha p(v_\alpha^\vee \otimes v_\alpha) \neq 0.$$

*Proof.* Fix a nonzero map $p' : \omega_{V_\iota} \otimes \sigma'_\iota \to \sigma_\iota^\vee$. We first note that $\mathcal{P}_0 \otimes \tau'$ maps into $\tau^\vee$ under $p'$. Indeed $\tau'$ is the only irreducible representation of $K_{G,\iota}$ that $\mathcal{P}_0$ and $\sigma_\iota$ shares. By lemma 3.1 and Lemma 3.3, we know that the map $p'$ is nonzero when restricted to $\mathcal{P}_0 \otimes \tau'_\iota$, and hence we have a nonzero homomorphism

$$p' : \mathcal{P}_0 \otimes \tau' \to \tau^\vee.$$

This implies that the natural homomorphism

$$\omega_V \otimes \sigma'_\iota \otimes \sigma_\iota \to \mathbb{C}$$

is nonzero when restricted to $\mathcal{P}_0 \otimes \tau' \otimes \tau$, which further implies that $p$ is nonzero when restricted to $\mathcal{P}_0 \otimes \tau$. But $\tau^\vee$ appears with multiplicity one in $\mathcal{P}_0$, i.e. $\mathcal{P}_{00}$, so $p$ restricts to a nontrivial pairing between $\mathcal{P}_{00}$ and $\tau$. □

We now prove the other direction of implication. Assume that $L(\frac{1}{2}, \pi' \otimes \sigma' \otimes \mu^{-1}) \neq 0$ and $\mathrm{Hom}_{H(\mathbb{A}_\mathrm{f})}(\pi_\mathrm{f} \otimes \sigma_\mathrm{f}, \mathbb{C}) \neq 0$. By Lemma 4.1, we know that

$$\mathrm{Hom}_{H(\mathbb{A}_\mathrm{f})}(\pi'_\mathrm{f} \otimes \sigma'_\mathrm{f} \otimes \overline{\omega_{L,\mathrm{f}}}, \mathbb{C}) \neq 0.$$

So we conclude that there is an $f' \in \pi'$ and $\varphi' \in \sigma'$ and a $\phi_1 \in \mathcal{S}(L(\mathbb{A})^r)$ such that

$$\int_{[H']} f'(h')\varphi'(h')\theta_L(h', \phi_1)\mathrm{d}h' \neq 0.$$

By Lemma 5.3, $f'_\infty$ and $\varphi'_\infty$ can be taken in the lowest $K_{H',\infty}$-type of $\pi'_\infty$ and $\sigma'_\infty$ respectively, and $\phi_{L,v} \in \mathcal{S}(V_v^r)^\dagger$ is the Gaussian function for each $v \mid \infty$. By Lemma 5.4, this implies that $\varphi'$ can be taken to be of the form $\sum_\alpha \overline{\theta^{\varphi_\alpha}_{\phi_{W,\alpha}}(h')}$ (the notation being as in Lemma 5.2). Now we can reverse the previous seesaw argument and conclude that we can construct $\Theta^{f'}_\Phi$ and $\Psi$ such that (5.5) is nonzero. Thus the cycle class of $Y$ in $\mathrm{H}^{n+1,n+1}(X \times Y)[\pi_\mathrm{f}^\vee \otimes \sigma_\mathrm{f}^\vee]$ is not zero.

This completes the proof of Theorem 1.1.

## Appendix A. Irreducibility of some full theta lifts

We prove the third assertion in Lemma 2.1 in this subsection. We indeed prove it a slightly general form, i.e. We do not need to impose assumptions on the signature of $W'$. This assumption has been in place throughout this paper, but it is irrelevant to this lemma.

For the ease of the readers, we repeat the setup. We work with $F = \mathbb{R}$. Let $W'$ and $V$ be skew-hermitian space of dimension $n'$ and hermitian space of dimension $n$ respectively. The signature of $V$ is $(n-1, 1)$. Put $H' = \mathrm{U}(W')$ and $G = \mathrm{U}(V)$. Denote by $\omega_V$ the Weil representation of $H' \times G$. This Weil representation depends on several choices of the characters, which we do not mention here as they are not very relevant to our discussion. We just assume that we have fixed these choices. Let $\pi'$ be an irreducible representation of $H'$, and $\Theta(\pi')$ be the maximal Hausdorff $H'$-invariant quotient of $\omega \widehat{\otimes} \pi'^\vee$. Denote by $\theta(\pi')$ the maximal semisimple quotient of $\Theta(\pi')$, which is irreducible.

**Lemma A.1.** *Assume $\pi'$ is an irreducible discrete series representation of $H'$ and $n = n' + 2$. Then $\Theta(\pi')$ is irreducible and hence equals $\theta(\pi')$.*

*Proof.* Fix an orthogonal decomposition $V = V_0 + L$ where $L$ is a negative line and $V_0$ is a positive definite hermitian space of dimension $n$. Let $K = \mathrm{U}(V_0)$. If $\rho$ is an irreducible representation of $K$, then the multiplicity of $\rho$ in $\Theta(\pi')|_K$ is either zero or one, cf. [SZ12]. We will show that $\rho$ has the same multiplicity in $\Theta(\pi')|_K$ and $\theta(\pi')|_K$, and thus proving $\Theta(\pi') = \theta(\pi')$.

We denote by $\omega_{V_0}$ and $\omega_L$ the Weil representation of $H' \times K$ and $H' \times \mathrm{U}(L)$ respectively. Again there are several choice of the characters involved, but we fix one choice, such that we have

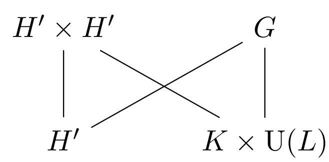
$$\omega_V|_{H' \times (K \times \mathrm{U}(L))} \simeq \omega_{V_0} \widehat{\otimes} \omega_L.$$

The Weil representations $\omega_V$, $\omega_{V_0}$ and $\omega_L$ are realized on some Schwartz spaces $\mathcal{S}_V$, $\mathcal{S}_{V_0}$ and $\mathcal{S}_L$ respectively such that we have $\mathcal{S}_V = \mathcal{S}_{V_0} \widehat{\otimes} \mathcal{S}_L$. We denote by $\Theta_0$ the theta lifts between the groups $H'$ and $K$. Since $K$ is compact, we know that $\Theta_0(\rho)$ is irreducible (possibly zero) for all irreducible representations $\rho$ of $K$.

First we consider $\Theta(\pi')$. Let

$$M(\rho) = \dim \mathrm{Hom}_K(\Theta(\pi')|_K, \rho).$$

We make use of the following seesaw diagram


,

which gives

$$M(\rho) = \dim \mathrm{Hom}_{H'}(\Theta_0(\rho) \widehat{\otimes} \pi'^{\vee} \widehat{\otimes} \omega_L, \mathbb{C}).$$

The representation $\Theta_0(\rho)$ is irreducible since $K$ is compact.

Because $\Theta_0(\rho)$ is irreducible, by [LS13] we have $M(\rho) \leq 1$ and $M(\rho) = 1$ if and only if $\mathrm{Hom}_{H'}(\Theta_0(\rho) \widehat{\otimes} \pi'^{\vee} \widehat{\otimes} \omega_L, \mathbb{C}) \neq 0$.

We now study the multiplicity of $\rho$ in $\theta(\pi')$ using the explicit intertwining map. Put $\pi = \theta(\pi')$. We need the theory of doubling zeta integrals which we recalled in Subsection 3.1. Consider the integral

$$\int_{\mathrm{U}(W')} \overline{\langle \pi'(h')\varphi'_1, \varphi'_2 \rangle} \langle \omega_V(h', g)\phi_1, \phi_2 \rangle \mathrm{d}h',$$

as a function of $g \in \mathrm{U}(V)$. Here $\varphi_1, \varphi_2 \in \pi'$, and $\phi_1, \phi_2 \in \mathcal{S}(V^r)$. By Lemma 3.1, it equals

$$\langle \pi(g)p(\overline{\varphi'_1} \otimes \phi), p(\overline{\varphi'_2} \otimes \phi_2) \rangle.$$

where $p : \pi'^{\vee} \widehat{\otimes} \omega_V \to \pi$ is the canonical $H'$-invariant map.

We come back to the study of the the multiplicity of $\rho$ in $\pi$. Put

$$m(\rho) = \dim \mathrm{Hom}_K(\pi|_K, \rho).$$

As $\pi$ is irreducible, we have $m(\rho) \leq 1$ by [SZ12]. Since $K$ is compact, we know that $m(\rho) = 1$ if and only if

$$\int_K \langle \pi(k) f, f \rangle \overline{\langle \rho(k) v, v \rangle} \mathrm{d}k$$

is not identically zero, where $f \in \pi$ and $v \in \rho$. From the discussion above on the doubling zeta integrals, we conclude that $m(\rho) = 1$ if and only if we can find $\varphi \in \pi'$ and $\phi \in \mathcal{S}_V$ such that

$$\int_K \int_{H'} \overline{\langle \pi'(h')\varphi', \varphi' \rangle} \langle \omega_V(h', k)\phi, \phi \rangle \overline{\langle \rho(k) v, v \rangle} \mathrm{d}h' \mathrm{d}k \neq 0.$$

Since $\mathcal{S}_V = \mathcal{S}_{V_0} \widehat{\otimes} \mathcal{S}_L$, we can take $\phi \in \mathcal{S}_V$ to be of the form $\phi_0 \otimes \phi_L$, where $\phi_0 \in \mathcal{S}_{V_0}$ and $\phi_L \in \mathcal{S}_L$.

Since $K$ is compact and $\pi'$ is a discrete series representation, we know that the double integral is absolutely convergent, and hence we can change the order of integration, and conclude that

$$\int_{H'} \overline{\langle \pi'(h')\varphi', \varphi'^{\vee} \rangle} \left( \int_K \langle \omega_W(h', k)\phi_W, \phi_W \rangle \overline{\langle \rho(k) v, v \rangle} \mathrm{d}k \right) \langle \omega_L(h')\phi_L, \phi_L \rangle \mathrm{d}h' \neq 0.$$

As in the case of $\pi$, the inner integral, as a function of $h' \in H'$, is a matrix coefficient of $\Theta_0(\rho)$. In conclusion, we have that $m(\rho) = 1$ if and only if $\Theta_0(\rho)$ is nonzero (and irreducible because $K$ is compact), and we can $f' \in \Theta_0(\rho)$ such that

$$\int_{H'} \overline{\langle \pi'(h')\varphi', \varphi' \rangle} \langle \theta(\rho)(h') f', f' \rangle \langle \omega_L(h')\phi_L, \phi_L \rangle \mathrm{d}h' \neq 0. \tag{A.1}$$

Let us note that $\Theta_0(\rho)$ is a discrete series representation of $H'$, and is in particular tempered. That $\Theta_0(\rho)$ is a discrete series representation is proved in the same way as [GI14, Proposition 16.1] (only the nonarchimedean case is treated there, but the proof in the archimedean case goes exactly the same). So by [Xue24, Theorem 3.2], we know that the integral (A.1) is not identically zero if and only if $\mathrm{Hom}_{H'}(\Theta_0(\rho) \widehat{\otimes} \pi'^{\vee} \widehat{\otimes} \omega_L, \mathbb{C}) \neq 0$. So finally we conclude that both $M(\rho)$ and $m(\rho)$ are either 0 or 1, and $M(\rho) = 1$ if and only if $m(\rho) = 1$. This implies that $\Theta(\pi') = \theta(\pi')$ and proves the lemma. ☐

## Appendix B. Kottwitz's conjecture

The goal of this appendix is to provide an explicit form of Kottwitz's conjecture for unitary Shimura varieties, in particular the Shimura varieties we encounter in this manuscript. The notation is independent from the rest of the manuscript.

B.1. **Shimura varieties.** Let $F$ be a degree $d$ totally real field and $E$ a CM extension. For simplicity we will assume that $d > 1$. Let $\mathcal{V}$ be a hermitian space of dimension $n$ over $E$, and the signature of $\mathcal{V} \otimes \mathbb{C}$ is

$$(n-1, 1) \times (n, 0)^{d-1}.$$

The unique archimedean place over which $\mathcal{V}$ is not anisotropic is denoted by $\iota$. We choose an orthonormal basis of $\mathcal{V}$ such that at the archimedean places the hermitian form of $\mathcal{V}$ is represented by the matrices

$$\begin{pmatrix} 1_{n-1} & \\ & -1 \end{pmatrix}, 1_n, \dots, 1_n.$$

Let $G = \mathrm{U}(\mathcal{V})$ be the corresponding unitary group, which is an algebraic group over $F$. Let $T$ be a diagonal torus in $G$. Let

$$h : \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{C}^\times \to G(\mathbb{R}), \quad z \mapsto \left( \begin{pmatrix} z/\overline{z} 1_{n-1} & \\ & \overline{z}/z 1 \end{pmatrix}, 1_n, \dots, 1_n \right)$$

be the usual homomorphism defining unitary Shimura varieties. Associated to this is cocharacter $\mu_h : \mathbb{C}^\times \to T(\mathbb{C})$, defined over the reflex field $E$, given by

$$z \mapsto \begin{pmatrix} 1_{n-1} & \\ & z \end{pmatrix}.$$

Let $D$ be the space of $G(\mathbb{R})$ conjugacy classes of the homomorphisms $\mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{C}^\times \to G(\mathbb{R})$ containing $h$, and $K \subset G(\mathbb{A}_\mathrm{f})$ an open compact subgroup. Define

$$X = G(\mathbb{Q}) \backslash (D \times G(\mathbb{A}_\mathrm{f})/K)$$

to be the Shimura variety attached to $G$ in the usual way. It has a canonical model over $E$, where $E$ is viewed as a subfield of $\mathbb{C}$ via the embedding $\iota$.

Let $\pi = \pi_\mathrm{f} \otimes \pi_\infty$ be an irreducible cuspidal automorphic representation of $G(\mathbb{A})$, cohomological with respect to the trivial representation, and $\pi_\mathrm{f}^K \neq 0$. Choose a large number field $L$ over which $\pi_\mathrm{f}^K$ as a module over the Hecke algebra $\mathcal{H}_{G,L}$ is defined. Here $\mathcal{H}_{G,L}$ stands for the convolution algebra of bi-$K$-invariant functions on $G(\mathbb{A}_\mathrm{f})$ valued in $L$. Let $\lambda$ be a finite place of $L$. We consider the cohomology

$$\mathrm{H}^*(X_{\overline{E}}, L_\lambda) = \sum_i (-1)^i \, \mathrm{H}^i(X_{\overline{E}}, L_\lambda)$$

on which both Hecke algebra $\mathcal{H}_{G,L}$ and the Galois group $\Gamma_E$ act. The conjecture of Kottwitz describes the $\pi_\mathrm{f}$-isotypic component it. Kottwitz's original statement of the conjecture is quite involved. This is the goal of appendix is to make it explicit for the particular unitary Shimura variety at hand.

B.2. **Arthur's conjecture.** The description given in Kottwitz conjecture relies on Arthur's (conjectural) endoscopic classification of automorphic representations of $G(\mathbb{A})$. Significant progress has been made towards Arthur's conjecture for $G$, cf. [CZ, Mok15, KMSW], and it is reasonable to expect this conjecture will be fully established in the near future. We review Arthur's conjecture in this subsection. For more details we refer the readers to [GGP20].

By an (elliptic) A-parameter $\psi$ of $G$ we mean a formal sum of form

$$\bigoplus_{1 \le i \le r} \Pi_i \otimes \mathrm{Sym}^{k_i - 1} \mathbb{C}^2,$$

where $\Pi_i$ is an irreducible automorphic representation of $\mathrm{GL}_{n_i}(\mathbb{A}_E)$, conjugate self-dual of signature $(-1)^{k_i+n}$, and the (formal) direct summands are distinct. Define an elementary abelian group $A_\psi = (\mathbb{Z}/2\mathbb{Z})^r$, which comes with a natural basis $a_1, \ldots, a_r$, and each $a_i$ correspond to the (formal) direct summand $\Pi_i \otimes \mathrm{Sym}^{k_i-1} \mathbb{C}^2$.

Let $v$ be a place of $F$. Each $\Pi_{i,v}$ is an irreducible representation of $\mathrm{GL}_{n_i}(E_v)$, and by the local Langlands correspondence gives rise to an $n_i$-dimensional representation $\psi_{i,v}$ of $W_{E_v}$, again conjugate self-dual of a suitable sign. Let

$$\psi_v = \bigoplus_{1 \le i \le r} \psi_{i,v} \otimes \mathrm{Sym}^{k_i-1} \mathbb{C}^2$$

be the representation of $W_{E_v} \times \mathrm{SL}_2(\mathbb{C})$. This is the local A-parameter at the place $v$. There is a elementary abelian 2-group $A_{\psi_v}$, which is a counterpart of the global component group $A_\psi$. Attached to this local A-parameter is a finite set of irreducible representations of $G(F_v)$, which we call a local A-packet and denote by $\Phi_{\psi_v}$, and a map

$$\Phi_{\psi_v} \to \widehat{A_{\psi_v}} = \mathrm{Hom}(A_{\psi_v}, \mathbb{Z}/2\mathbb{Z}), \quad \pi_v \mapsto \eta_{\pi_v}.$$

While the set $\Phi_{\psi_v}$ is canonical, the map $\eta_v$ is not and it depends on the normalization of transfer factors or alternatively some choice of the Whittaker data. For our purpose we do not need to make this explicit.

There is a canonical homomorphism $A_\psi \to A_{\psi_v}$, and hence a diagonal homomorphism

$$A_\psi \to \prod_v A_{\psi_v}.$$

Let $\pi = \otimes_v \pi_v$ be an irreducible admissible representation of $G(\mathbb{A})$, and $\pi_v \in \Phi_{\psi_v}$. Then Arthur's conjecture, or more precisely Arthur's multiplicity formula, claims that $\pi$ is automorphic if and only if the product character

$$\prod_v \eta_{\pi_v}$$

is trivial when restricted to $A_\psi$.

B.3. **Adams–Johnson packets.** At the archimedean place $\iota$, the packet $\Phi_{\psi_\iota}$ is constructed explicitly by Adam and Johnson using cohomological inductions, cf. [AJ87]. We review this construction in this subsection.

Because of the assumption of being cohomological (with respect to the trivial representation), the parameter $\psi_\iota$ must be of the form

$$\bigoplus_{1 \le i \le r} \bigoplus_{1 \le j \le n_i} \xi_{m_{ij}} \otimes \mathrm{Sym}^{k_i-1} \mathbb{C}^2,$$

where $m_{ij} \in \mathbb{Z}$ and $\xi_{m_{ij}}$ is the character of $\mathbb{C}^\times$ given by $z \mapsto (z/\sqrt{z\bar{z}})^{m_{ij}}$. The integers $m_{ij}$'s are distinct, and $m_{ij}$ has the same parity with $k_i$. We note that $\xi_{m_{i1}} \oplus \cdots \oplus \xi_{m_{i,n_i}}$ is the local Langlands parameter of $\Pi_\iota$.

The parameter $\psi_\iota$, or rather the sequence of integers

$$(\underbrace{m_{11},\dots,m_{11}}_{k_1},\underbrace{m_{12},\dots,m_{12}}_{k_1},\dots,\underbrace{m_{21},\dots,m_{21}}_{k_2},\dots,\underbrace{m_{r,n_r},\dots,m_{r,n_r}}_{k_r}) \tag{B.1}$$

defines a parabolic subgroup $Q$ of $\mathrm{GL}_n(\mathbb{C})$ with the Levi component $L$ isomorphic to $\mathrm{GL}_{k_1}(\mathbb{C})^{n_1}\times\cdots\times\mathrm{GL}_{k_r}(\mathbb{C})^{n_r}$. Note that $Q$ is not necessarily defined over $\mathbb{R}$ while $L$ is, and indeed $L_0 = G_\iota \cap L$ is a real form.

Recall that $G_\iota(\mathbb{C})$ is isomorphic to $\mathrm{GL}_n(\mathbb{C})$ and $T_\iota(\mathbb{C})$ is the diagonal torus. Let $W(G,T)\simeq\mathfrak{S}_n$, $W(L,T)\simeq(\mathfrak{S}_{k_1})^{n_1}\times\cdots\times(\mathfrak{S}_{k_r})^{n_r}$ be the absolute Weyl groups, and $W_{\mathbb{R}}(G,T)\simeq\mathfrak{S}_{n-1}$ the relative Weyl group. The packet $\Phi_{\psi_\iota}$ is in one-to-one correspondence with the double coset

$$W(L,T)\backslash W(G,T)/W_{\mathbb{R}}(G,T).$$

Let $S$ be a set of representatives, and $\mathfrak{q}$ be the Lie algebra of $Q$, which is a $\theta$-stable parabolic of $G$. Then the packet $\Phi_{\psi_\iota}$ is indeed given by the cohomological inductions

$$\Phi_{\psi_\iota} = \{A_{w^{-1}\mathfrak{q}w} \mid w\in S\}.$$

To each representation $\pi_\iota$ in $\Phi_{\psi_\iota}$, Kottwitz associated a character $\lambda_{\pi_\iota}$ of $A_{\psi_\iota}$ which we now explain. The group $\widehat{G}$ is isomorphic to $\mathrm{GL}_n(\mathbb{C})$, and let $\widehat{T}$ be the diagonal torus. The parameter $\psi_\iota$, or rather the sequence (B.1) defines a parabolic subgroup $\widehat{P}$ of $\widehat{G}$ with the Levi component $\widehat{M}$. The local component group $A_{\psi_\iota}$ is defined to be the subgroup of order two elements in the center of $\widehat{M}$ and is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{n_1+\cdots+n_r}$. Then $A_{\psi_\iota}$ is viewed as a subgroup of $\widehat{T}$. We identify $W(\widehat{G},\widehat{T})$ as $W(G,T)$ and view elements in $S$ as elements in $W(\widehat{G},\widehat{T})$.

Recall that from the definition of the Shimura variety, we have the cocharacter $\mu_h$ of $T$, or equivalently a character of $\widehat{T}$. This character is simply given by

$$\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n\end{pmatrix}\mapsto t_n.$$

Let $\pi_\iota = A_{w^{-1}\mathfrak{q}w}$ be a representation in $\Phi_{\psi_\iota}$, then the character $\lambda_{\pi_\iota}$ is given by

$$s\mapsto \lambda_{\pi_\iota}(s) = \mu_h(w^{-1}sw),\quad s\in A_{\psi_\iota}.$$

B.4. **Kottwitz's conjecture.** Recall that we have the global component group $A_\psi\simeq(\mathbb{Z}/2\mathbb{Z})^r$, and an embedding $A_\psi\to A_{\psi_\iota}$ given by

$$(x_1,x_2,\cdots,x_r)\mapsto(\underbrace{x_1,\dots,x_1}_{n_1},\underbrace{x_2,\dots,x_2}_{n_2},\dots,\underbrace{x_r,\dots,x_r}_{n_r}).$$

Let $\chi_i$ be the character of the group $A_\psi$ given by $(x_1,x_2,\cdots,x_r)\to x_i$. Let $\nu$ be the restriction of $\lambda_{\pi_\iota}$ to $A_\psi$. Then there is a unique $i_0\in\{1,\cdots,r\}$ such that $\nu=\chi_{i_0}$ (this makes use of the fact that $\mu_h$ has that particularly simple form).

The irreducible cuspidal automorphic representations $\Pi_{i_0}$ give rise to an $n_{i_0}$-dimensional irreducible representations $V_{i_0}$ of $\Gamma_E$. After enlarging $L$ suitably, we may assume that $V_{i_0}$ comes from a representation defined over $L$. After tensoring with $L_\lambda$ we obtain a representation defined over $L_\lambda$, which we still denote by $V_{i_0}$. Write $V_{i_0}(a) = V_{i_0} \otimes L_\lambda(a)$ where $L_\lambda(a)$ stands for the Tate twist. Kottwitz conjecture in this case states that

$$\mathrm{H}^*(X, L_\lambda)[\pi_{\mathrm{f}}] = \pm \pi_{\mathrm{f}} \boxtimes \left( \bigoplus_{\substack{-(k_{i_0}-1) \leq a \leq k_{i_0}-1 \\ a \equiv (k_{i_0}-1) \mathrm{mod} 2}} V_{i_0}(a-(n-1)) \right),$$

as $\mathcal{H}_{G,L} \times \Gamma_E$ modules.

Write the $\Gamma_E$ module in the above expression as $V(\nu)$. Fix an embedding $L_\lambda \to \mathbb{C}$. The space $V(\nu) \otimes_{L_\lambda} \mathbb{C}$, as a vector space, is isomorphic to

$$\bigoplus_{\pi'_\infty} \mathrm{H}^*(\mathfrak{g}_\infty, K_\infty, \pi'_\infty)$$

where $\pi'_\infty$ ranges over all elements in the local Arthur packet given by $\psi_\infty$ such that $\pi_{\mathrm{f}} \otimes \pi'_\infty$ is automorphic. One implication is that when $\pi'_\infty$ ranges over the local Arthur packet, the degrees of all nonzero cohomologies have the same parity.

As another consequence, different direct summands in $V(\nu)$ have different weights (at least assuming Ramanujan conjecture), so they contribute to different cohomological degrees. In particular, the $\pi_{\mathrm{f}}$-isotypic part in each cohomology with a fixed degree is irreducible, and hence semisimple and completely determined by the Frobenius trace.

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Department of Mathematics, The University of Arizona, Tucson, AZ, 85721
*Email address*: xuehang@arizona.edu